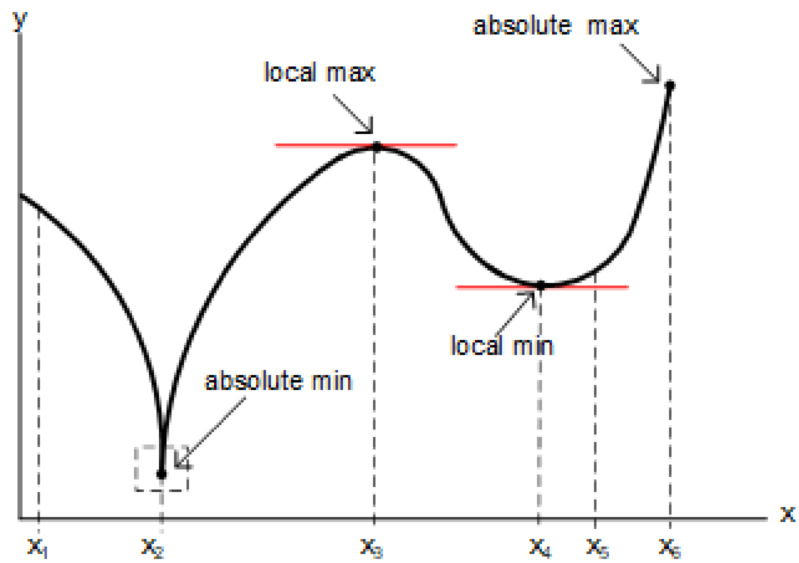
Derivative Applications – Extreme Values

There are numerous instances where one would be interested in finding the minimum or maximum value of a function. For example, a manufacturer may want to find the dimensions of a container with a given volume that will minimize the cost of the material. Another example is where a store owner is trying to set a price for a certain product so that profits are maximized. We refer to the minimum and maximum of functions as extreme values or extrema, (singular: extremum). The process of finding extreme values of a given function is referred to as *optimization*. When searching for extreme values of a function, we are generally interested in looking over a certain interval. Using the example from above regarding maximizing profits, we would obviously not consider negative prices, and would likely limit our search to some maximum price point. To begin this topic let's take a look at the function below, which contains various types of extreme values.



The various peaks and valleys of the function are referred to as l*ocal* maximum and minimum values, respectively. These are locations where the function obtains an extreme value over a small window. Using our knowledge of the derivative we should notice that the extreme values at x_3 and x_4 occur at locations where the slope of a tangent line, (i.e. the derivative), is zero. However, x_2 is also an extreme value that occurs at a location of a cusp, where the derivative does not exist. Another thing to notice is the fact that the local maximum and minimum values may not be the largest and smallest values the function obtains over a given interval. To distinguish this, we use the term *absolute* maximum and minimum values. These absolute extreme values may occur at one of the local extreme values or at the endpoints of the interval under consideration. For example, if we consider the interval as $[x_1, x_6]$, the local absolute maximum value occurs at the endpoint of the interval, i.e., x_6. However, if we consider the interval as $[x_1, x_5]$, then the absolute maximum occurs at x_3, which is also a local maximum. These observations can be summarized formally with various definitions and theorems, which we introduce below.

We start by defining absolute extreme values. In plain terms we can say that the absolute minimum and maximum values are the smallest and largest values a function obtains over a given interval.

Absolute Extreme Values Definition
Consider the function, $f(x)$, over an interval, I, and let $a \in I$. We say that $f(a)$ is the: • **Absolute minimum** of f on I if $f(a) \leq f(x)$ for all $x \in I$. • **Absolute maximum** of f on I if $f(a) \geq f(x)$ for all $x \in I$.

The next question we can ask is: "Does every function have a minimum or maximum?". The answer is *no*, which can be seen with the relatively simple function $f(x) = x$. This function increases without bound as $x \to \infty$ and decreases without bound as $x \to -\infty$. Of course, in this case if we restrict ourselves to a certain *closed* interval, we will indeed find both a minimum and maximum value occurring at the interval end points. Fortunately, a general theorem does exist that guarantees the existence of extreme values. In plain terms it states that every *continuous* function will obtain both a minimum and maximum value over any *closed* interval.

Absolute Extreme Value Theorem
If a function, $f(x)$, is continuous over a closed interval, $[a, b]$, then $f(x)$ has both a minimum and maximum on $[a, b]$.

As the above definition and theorem dealt with *absolute* extreme values, let's now define *local* extreme values.

Local Extreme Values Definition
Consider the function, $f(x)$. We say that $f(a)$ is a: • **Local minimum** that occurs at $x = a$ if $f(a)$ is the minimum value of f on some small interval containing a. • **Local maximum** that occurs at $x = a$ if $f(a)$ is the maximum value of f on some small interval containing a.

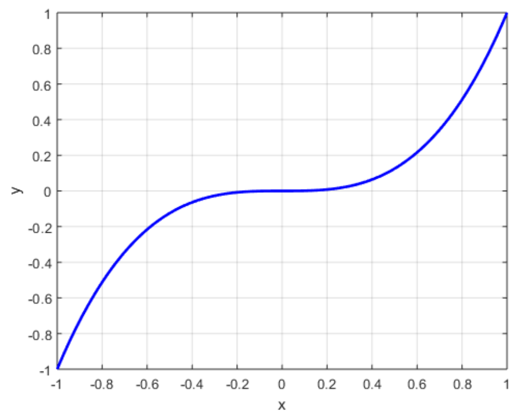
We can now ask the question: "How do we find local extreme values?". Recall, in the introductory figure we saw that local extreme values occur at locations where the derivative is zero or does not exists. With this in mind we first define so-called critical points.

Critical Points Definition
A number c in the domain of $f(x)$ is called a **critical point** if either of the following are true: • $f'(c) = 0$ • $f'(c)$ does not exist.

As the next theorem shows, the definition of critical points is critical for finding local extreme values.

Fermat's Theorem of Local Extrema
If $f(c)$ is a local minimum or maximum, then c is a critical point of f. Note: This theorem does not claim that all critical points are local extreme values, but rather that all local extreme values are critical points.

Regarding the note in Fermat's theorem, we can use the example of the function $f(x) = x^3$. The derivative of $f(x)$ is $3x^2$, which is zero at $x = 0$, and hence $x = 0$ is a critical point. However, as we can see in the figure below, $f(0)$ is neither a minimum nor a maximum.



In the next section we will cover how to identify whether a critical point is indeed a *local* maximum or minimum. In this section, we will instead use the critical points, together with the endpoints, to find the ***absolute*** extreme values over a closed interval. We describe a general procedure below.

Finding Absolute Extreme Values
To find the absolute extreme values of a function, $f(x)$, over a closed interval, $[a, b]$ we: 1. Find critical points, $x = \{c_1, c_2, \ldots c_N\}$, of $f(x)$ over the closed interval, $[a, b]$. 2. Evaluate $f(x)$ at the critical points, $\{f(c_1), f(c_2), \ldots f(c_N)\}$, *and* at the endpoints, $\{f(a), f(b)\}$. 3. The absolute minimum and maximum values are the smallest and largest among the evaluated values from step 2.

We will end this section with some examples. In each example we also plot the function for illustration purposes.

Example 1:

Find the absolute extreme values of the function: $f(x) = 2x^3 - 15x^2 + 24x + 7$ on $[0,6]$.

We follow the steps from above.

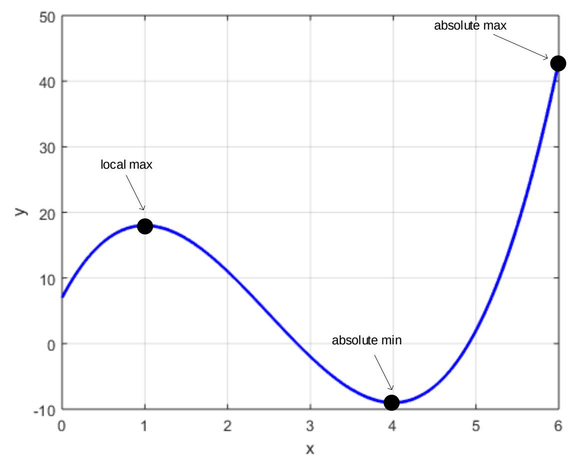
1. As the function is differentiable everywhere, we find the critical points by solving $f'(x) = 0$

$$f'(x) = 6x^2 - 30x + 24 = 0$$
$$x^2 - 5x + 4 = 0$$
$$(x-4)(x-1) = 0$$

The critical points are: $c_1 = 4$ and $c_2 = 1$.

2. Next, we evaluate the function at the critical points and the endpoints to determine the absolute extrema.

$\boldsymbol{f(4)}$	**-9**	**Absolute Minimum**
$f(1)$	18	
$f(0)$	7	
$\boldsymbol{f(6)}$	**43**	**Absolute Maximum**



The absolute minimum value of $f(x)$ on $[0,6]$ is -9, and it occurs at the critical point, $x = 4$.

The absolute maximum value of $f(x)$ on $[0,6]$ is 43, and it occurs at the endpoint, $x = 6$.

Example 2:

Find the absolute extreme values of the function: $f(x) = 1 - (x-1)^{2/3}$ on $[-1,2]$.

Again, we start by taking the derivative.

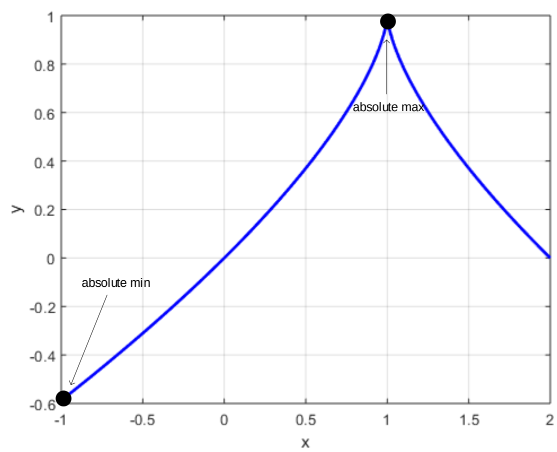
$$f'(x) = -\frac{2}{3(x-1)^{1/3}}$$

Then recall that critical values occur when $f'(x)$ is either zero or not defined. In this case the derivative in non-zero for all values of x, however it is undefined at the vertical asymptote, which is found by setting the denominator to zero.

$$3(x-1)^{1/3} = 0$$
$$x - 1 = 0$$
$$x = 1$$

Next, we evaluate the function at the critical point, $c_1 = 1$, and the endpoints to determine the absolute extrema.

$\boldsymbol{f(1)}$	**1**	**Absolute Maximum**
$\boldsymbol{f(-1)}$	$\boldsymbol{\approx -0.59}$	**Absolute Minimum**
$f(2)$	0	



Example 3:

Find the absolute extreme values of the function: $f(x) = sin(x) + cos^2(x)$ on $[0,2\pi]$.

Solution:

$$f'(x) = cos(x) - 2\,cos(x)\,sin(x) = 0$$
$$cos(x)\,(1 - 2\,sin(x)) = 0$$

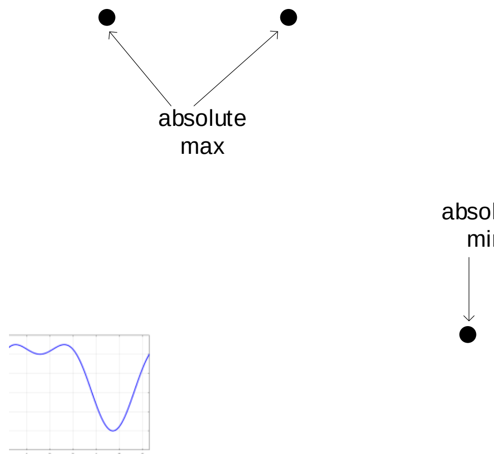
Setting both factors to zero we have:

$cos(x) = 0$, which results in critical points of: $c_1 = \frac{\pi}{2}$ and $c_2 = \frac{3\pi}{2}$

$sin(x) = \frac{1}{2}$, which results in critical points of: $c_3 = \frac{\pi}{6}$ and $c_4 = \frac{5\pi}{6}$

We then evaluate all critical points and endpoints.

$f\left(\frac{\pi}{2}\right)$	1	
$f\left(\frac{3\pi}{2}\right)$	-1	**Absolute Minimum**
$f\left(\frac{\pi}{6}\right)$	$\frac{5}{4}$	**Absolute Maximum**
$f\left(\frac{5\pi}{6}\right)$	$\frac{5}{4}$	**Absolute Maximum**
$f(0)$	1	
$f(2\pi)$	1	



The absolute minimum value of $f(x)$ on $[0,2\pi]$ is -1, and it occurs at the critical point, $x = \frac{3\pi}{2}$.

There are two absolute maximum values for $f(x)$ on $[0,2\pi]$. The value is $\frac{5}{4}$ and it occurs at the critical points, $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

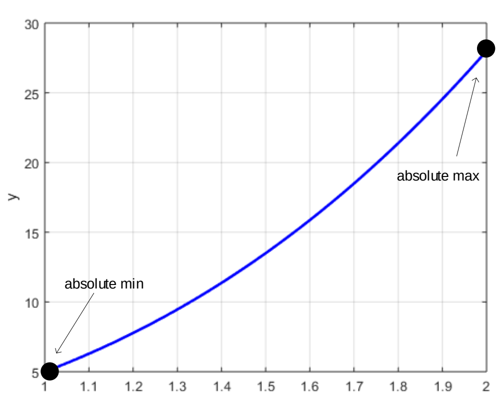
Example 4:

Find the absolute extreme values of the function: $f(x) = 2x^3 + 3x^2$ on $[1,2]$.

$$f'(x) = \cancel{6}x^2 + \cancel{6}x = 0$$
$$x(x+1) = 0$$

Therefore, we have the following two critical points: $c_1 = 0$ and $c_2 = -1$, neither of which are in the given interval. In this case we need only evaluate the endpoints.

$$f(1) = 2 + 3 = 5, \text{ and } f(2) = 2 \cdot 2^3 + 3 \cdot 2^2 = 28$$



The absolute minimum value of $f(x)$ on $[1,2]$ is 5, and it occurs at the endpoint, $x = 1$.

The absolute maximum value of $f(x)$ on $[1,2]$ is 28, and it occurs at the endpoint, $x = 2$.

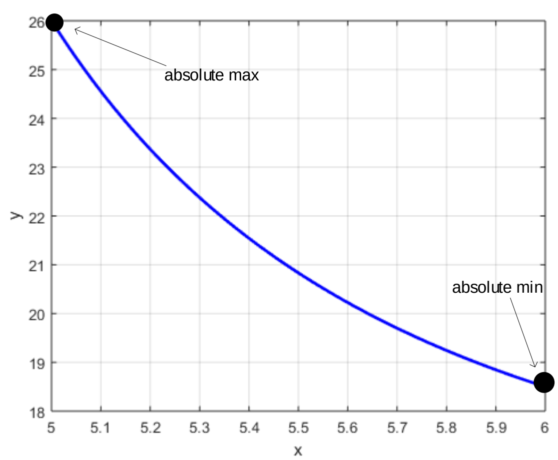
Example 5:

Find the absolute extreme values of the function: $f(x) = \frac{x^2+1}{x-4}$ on $[5,6]$.

$$f'(x) = \frac{2x(x-4) - (x^2+1)}{(x-4)^2}$$
$$= \frac{x^2 - 8x - 1}{(x-4)^2}$$

Using the quadratic formula on the numerator to find where $f'(x) = 0$, we find two critical points: $c_1 \cong 8.12$ and $c_2 \cong -0.12$, both of which are outside of the interval. The third critical point where $f'(x)$ is undefined occurs at $x = 4$, which is also not in the given interval. Therefore, similar to the previous example, both extreme values occur at the endpoints. Evaluating we have

$$f(5) = \frac{25+1}{5-4} = 26, \text{ and } f(6) = \frac{36+1}{6-4} = 18.5$$



The absolute minimum value of $f(x)$ on $[5,6]$ is 18.5, and it occurs at the endpoint, $x = 6$.

The absolute maximum value of $f(x)$ on $[5,6]$ is 26, and it occurs at the endpoint, $x = 5$.

Example 6:

Find the absolute extreme values of the function: $f(x) = tan(x) - 2x$ on $[0,1]$.

$$f'(x) = sec^2(x) - 2 = 0$$

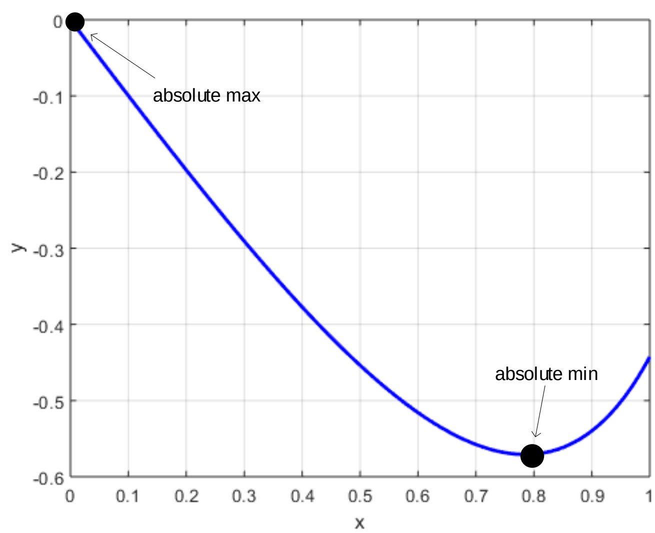
$$\frac{1}{cos^2(x)} = 2$$

$$cos(x) = \pm\frac{\sqrt{2}}{2}$$

Which results in only one critical point, for $\frac{\sqrt{2}}{2}$, over the given interval: $c_1 = \frac{\pi}{4}$.

Next, we evaluate this critical point along with the endpoints to find the extreme values.

$\boldsymbol{f\left(\frac{\pi}{4}\right)}$	$\boldsymbol{tan\left(\frac{\pi}{4}\right) - 2\left(\frac{\pi}{4}\right) = 1 - \frac{\pi}{2} \cong -0.57}$	**Absolute Minimum**
$\boldsymbol{f(0)}$	$\boldsymbol{tan(0) - 2(0) = 0}$	**Absolute Maximum**
$f(1)$	$tan(1) - 2(1) \cong -0.44$	



The absolute minimum value of $f(x)$ on $[0,1]$ is ≈ -0.57, and it occurs at the critical point, $x = \frac{\pi}{4}$.

The absolute maximum value of $f(x)$ on $[0,1]$ is 0, and it occurs at the endpoint, $x = 0$.

Example 7:

Find the absolute extreme values of the function: $f(x) = \frac{ln(x)}{x}$ on $[1,3]$.

$$f'(x) = \frac{1 - ln(x)}{x^2}$$

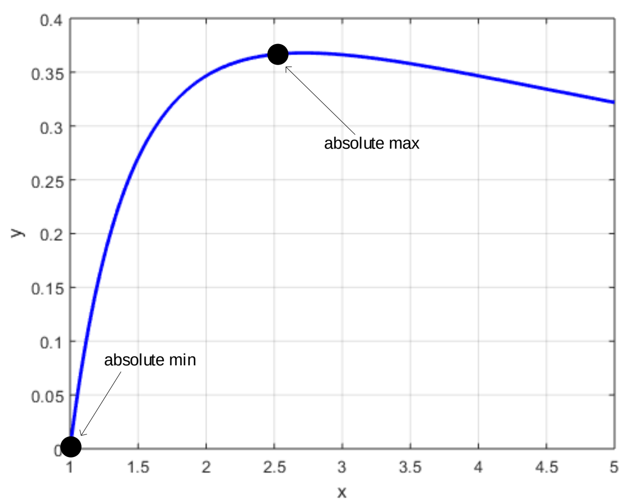
The first critical point occurs when $1 - ln(x) = 0$, therefore

$$ln(x) = 1$$
$$x = e^1$$

The second critical point occurs at $x = 0$, however this point is located outside of the given interval.

Finally, we evaluate the first critical point and the endpoints.

$\boldsymbol{f(e^1)}$	$\boldsymbol{\frac{ln(e^1)}{e^1} = e^{-1} \cong 0.367}$	**Absolute Maximum**
$\boldsymbol{f(1)}$	$\boldsymbol{\frac{ln(1)}{1} = 0}$	**Absolute Minimum**
$f(3)$	$\frac{ln(3)}{3} \cong 0.366$	



The absolute minimum value of $f(x)$ on $[1,3]$ is 0, and it occurs at the endpoint, $x = 1$.

The absolute maximum value of $f(x)$ on $[1,3]$ is e^{-1}, and it occurs at the endpoint, $x = e^1$.

Final Summary for Derivative Applications **– Extreme Values**

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