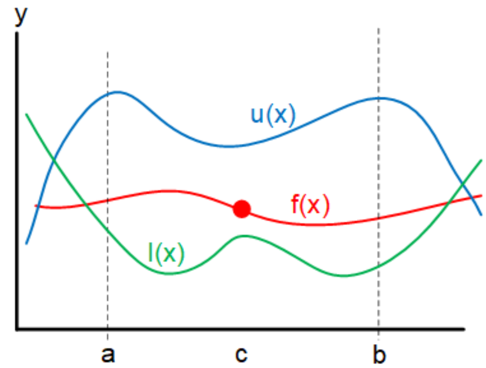
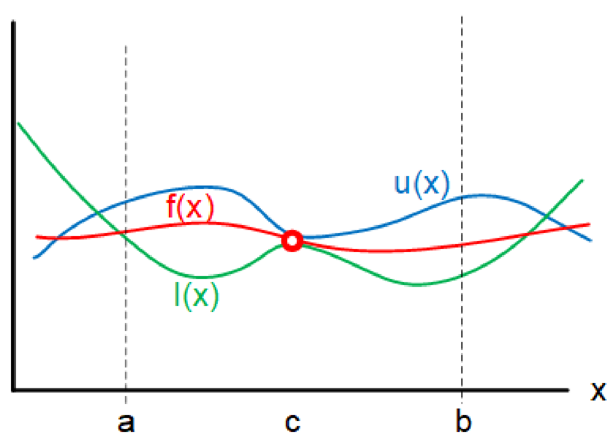
Limits – Trigonometric Limits

As mentioned in the previous section, it's not always possible to evaluate limits using algebraic techniques alone. This is especially true for trigonometric functions. In certain of these cases, we can use the results of the so-called Squeeze Theorem, which we introduce below, to help us evaluate these types of limits. To understand the squeeze theorem, we start with the idea of a function, $f(x)$, being "trapped" between an upper function, $u(x)$, and a lower function, $l(x)$, in an open interval, (a, b), as shown below.



The squeeze theorem extends the above picture so that the upper and lower functions get closer to one another at a point, c, to the extent that they "squeeze" the middle function at c. It is formally stated as follows:

Squeeze Theorem
If for all points in an open interval (a, b), excluding the point $x = c$, where $a < c < b$, the following is true: $$l(x) \leq f(x) \leq u(x) \quad \text{and} \quad \lim_{x \to c}\{l(x)\} = \lim_{x \to c}\{u(x)\} = L$$ Then: $$\lim_{x \to c}\{f(x)\} = L$$



$f(x)$ is squeezed between $l(x)$ and u(x) at the point $x = c$.

Let’s look at a particular example to see how the squeeze theorem is used.

Example 1: Prove the following limit using the squeeze theorem.

$$\lim_{x\to 0}\left\{x\sin\left(\frac{1}{x}\right)\right\}=0$$

We start with the fact that the sine function is restricted to values between -1 and 1.

$$-1\le\sin\left(\frac{1}{x}\right)\le 1$$

Multiplying this inequality through by x we have

$$-x\le x\sin\left(\frac{1}{x}\right)\le x$$

Setting $l(x)=-x$ and $u(x)=x$ we can directly apply the squeeze theorem as follows:

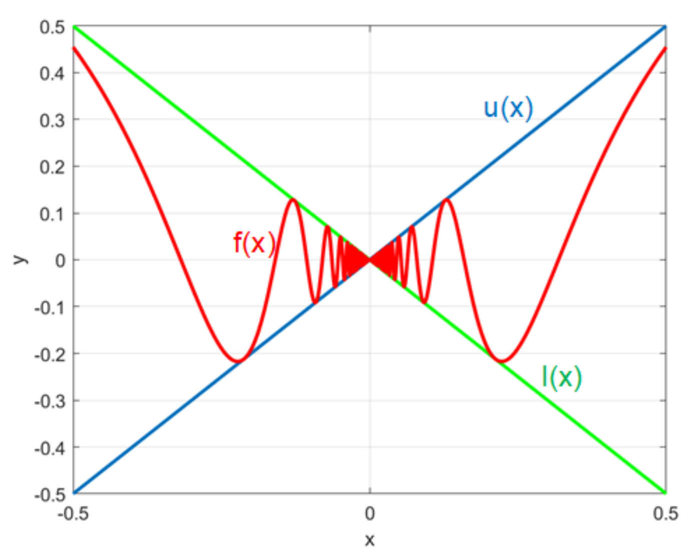
Since,

$$-x\le x\sin\left(\frac{1}{x}\right)\le x\quad\text{and}\quad\lim_{x\to 0}\{-x\}=\lim_{x\to 0}\{x\}=0$$

Then,

$$\lim_{x\to 0}\left\{x\sin\left(\frac{1}{x}\right)\right\}=0$$

The figure below illustrates the results from above by showing all three functions.



We will return to additional examples using the squeeze theorem, but first we state two important trigonometric limits. Although these can both be proved using the squeeze theorem, we state them below without proof. More importantly we show how to use these results to evaluate certain trigonometric limits without the need to directly apply the squeeze theorem.

Important Trigonometric Limits	
$\lim_{x\to 0}\left\{\frac{\sin(x)}{x}\right\} = \lim_{x\to 0}\left\{\frac{x}{\sin(x)}\right\} = 1$	$\lim_{x\to 0}\left\{\frac{1-\cos(x)}{x}\right\} = 0$

We now demonstrate how these results can be used to evaluate the two examples below.

a.) $\lim_{x\to 0}\left\{\frac{\sin(5x)}{x}\right\}$ b.) $\lim_{x\to 0}\left\{\frac{1-\cos(3x)}{x}\right\}$

a.) Setting $u = 5x$, and therefore $x = \frac{u}{5}$, we can rewrite and evaluate the limit as shown.

$$\lim_{u\to 0}\left\{\frac{\sin(u)}{u/5}\right\} = \left(\lim_{u\to 0}\{5\}\right)\cdot\left(\lim_{u\to 0}\left\{\frac{\sin(u)}{u}\right\}\right) = 5\cdot 1 = 5$$

Where, we used the fact that since $x \to 0$, so does $5x = u \to 0$.

b.) We use a similar substitution; $u = 3x$, and therefore $x = \frac{u}{3}$, and rewrite the limit.

$$\lim_{u\to 0}\left\{\frac{1-\cos(u)}{u/3}\right\} = \left(\lim_{u\to 0}\{3\}\right)\cdot\left(\lim_{u\to 0}\left\{\frac{1-\cos(u)}{u}\right\}\right) = 3\cdot 0 = 0$$

Below are additional examples using both the squeeze theorem and the methods in the previous two examples.

Examples:

Evaluate the following limits.

a.) $\lim_{x\to 0}\left\{x^2\cos\left(\frac{1}{x}\right)\right\}$

b.) $\lim_{x\to 1}\left\{(x-1)\sin\left(\frac{\pi}{x-1}\right)\right\}$

c.) $\lim_{x\to 2}\left\{(x^2-4)\cos\left(\frac{1}{x-2}\right)\right\}$

d.) $\lim_{x\to 0}\left\{\frac{\tan(x)}{x}\right\}$

e.) $\lim_{x\to 0}\left\{\frac{\tan(3x)}{\tan(2x)}\right\}$

f.) $\lim_{x\to 0}\left\{\frac{\sin(2x)\sin(3x)}{x^2}\right\}$

g.) $\lim_{x\to 0}\left\{\frac{\sin(2x)(1-\cos(x))}{x^2}\right\}$

h.) $\lim_{x\to 0}\left\{\frac{\sin(2x)\sin(3x)}{x\sin(5x)}\right\}$

a.) The squeeze theorem can be applied by starting with the fact that the range of the cosine function is $[-1, 1]$, and then proceeding as we did in the first example in the introduction.

$$-1 \le \cos\left(\frac{1}{x}\right) \le 1$$
$$-x^2 \le x^2\cos\left(\frac{1}{x}\right) \le x^2$$

And since,

$$-x^2 \le x^2\cos\left(\frac{1}{x}\right) \le x^2 \quad \text{and} \quad \lim_{x\to 0}\{-x^2\} = \lim_{x\to 0}\{x^2\} = 0$$

Then,

$$\lim_{x\to 0}\left\{x^2\cos\left(\frac{1}{x}\right)\right\} = 0$$

b.) Here we use a similar argument as we did in problem a.

$$-1 \le \sin\left(\frac{\pi}{x-1}\right) \le 1$$
$$-(x-1) \le (x-1)\sin\left(\frac{\pi}{x-1}\right) \le (x-1)$$

And since,

$$-(x-1) \le (x-1)\sin\left(\frac{\pi}{x-1}\right) \le (x-1) \quad \text{and} \quad \lim_{x\to 1}\{x-1\} = \lim_{x\to 1}\{1-x\} = 0$$

Then,

$$\lim_{x\to 1}\left\{(x-1)\sin\left(\frac{\pi}{x-1}\right)\right\} = 0$$

c.) We again use the same technique from the above two problems.

$$-1 \le \cos\left(\frac{1}{x-2}\right) \le 1$$
$$-(x^2-4) \le (x^2-4)\cos\left(\frac{1}{x-2}\right) \le (x^2-4)$$

And since,

$$-(x^2-4) \le (x^2-4)\cos\left(\frac{1}{x-2}\right) \le (x^2-4)$$
and
$$\lim_{x\to 2}\{-(x^2-4)\} = \lim_{x\to 2}\{(x^2-4)\} = 0$$

Then,

$$\lim_{x\to 2}\left\{(x^2-4)\cos\left(\frac{1}{x-2}\right)\right\} = 0$$

d.) In this case we first rewrite the tangent function and then use a combination of the limit laws and results of known limits.

$$\lim_{x\to 0}\left\{\frac{\tan(x)}{x}\right\} = \lim_{x\to 0}\left\{\frac{\sin(x)}{x\cos(x)}\right\} = \left(\lim_{x\to 0}\left\{\frac{1}{\cos(x)}\right\}\right)\left(\lim_{x\to 0}\left\{\frac{\sin(x)}{x}\right\}\right) = 1\cdot 1 = 1$$

e.) We start, as we did in the previous problem, by rewriting the tangent function. We also employ a 'trick', whereby we multiply and divide by the arguments of the trigonometric functions. Note, we will use similar techniques for the remaining problems also.

$$\begin{aligned}
\lim_{x\to 0}\left\{\frac{\tan(3x)}{\tan(2x)}\right\} &= \lim_{x\to 0}\left\{\frac{\sin(3x)\cos(2x)}{\cos(3x)\sin(2x)}\left(\frac{2x}{3x}\right)\left(\frac{3x}{2x}\right)\right\} \\
&= \lim_{x\to 0}\left\{\left(\frac{\sin(3x)}{3x}\right)\left(\frac{2x}{\sin(2x)}\right)\frac{3\cancel{x}\cos(2x)}{2\cancel{x}\cos(3x)}\right\} \\
&= \left(\lim_{x\to 0}\left\{\left(\frac{\sin(3x)}{3x}\right)\right\}\right)\left(\lim_{x\to 0}\left\{\left(\frac{2x}{\sin(2x)}\right)\right\}\right)\left(\lim_{x\to 0}\left\{\frac{3}{2}\right\}\right)\left(\lim_{x\to 0}\left\{\frac{\cos(2x)}{\cos(3x)}\right\}\right) \\
&= (1)(1)\left(\frac{3}{2}\right)(1) \\
&= \frac{3}{2}
\end{aligned}$$

f.)

$$\lim_{x\to 0}\left\{\frac{\sin(2x)\sin(3x)}{x^2}\right\} = \left(\lim_{x\to 0}\left\{\frac{\sin(2x)}{x}\right\}\right)\left(\lim_{x\to 0}\left\{\frac{\sin(3x)}{x}\right\}\right)$$

$$= \left(\lim_{x\to 0}\left\{\frac{\sin(2x\ )}{x}\cdot\left(\frac{2}{2}\right)\right\}\right)\left(\lim_{x\to 0}\left\{\frac{\sin(3x)}{x}\cdot\left(\frac{3}{3}\right)\right\}\right)$$

$$= \left(\lim_{x\to 0}\{2\}\right)\left(\lim_{x\to 0}\left\{\frac{\sin(2x\ )}{2x}\right\}\right)\left(\lim_{x\to 0}\{3\}\right)\left(\lim_{x\to 0}\left\{\frac{\sin(3x)}{x}\right\}\right)$$

$$= (2)(1)(3)(1)$$

$$= 6$$

g.)

$$\lim_{x\to 0}\left\{\frac{\sin(2x)\,(1-\cos(x))}{x^2}\right\} = \lim_{x\to 0}\left\{\frac{\sin(2x)}{x}\right\}\lim_{x\to 0}\left\{\frac{(1-\cos(x))}{x}\right\}$$

$$= \left(\lim_{x\to 0}\left\{2\,\frac{\sin(2x)}{2x}\right\}\right)\left(\lim_{x\to 0}\left\{\frac{(1-\cos(x))}{x}\right\}\right)$$

$$= (2)(0)$$

$$= 0$$

h.)

$$lim_{x\to 0}\left\{\frac{sin(2x)\,sin(3x)}{x\,sin(5x)}\right\} = lim_{x\to 0}\left\{\frac{sin(2x)\,sin(3x)}{x\,sin(5x)}\cdot\left(\frac{2}{2}\right)\left(\frac{3x}{3x}\right)\left(\frac{5x}{5x}\right)\right\}$$

$$= \left(lim_{x\to 0}\left\{2\,\frac{sin(2x)}{2x}\,3\cancel{x}\,\frac{sin(3x)}{3x}\,\frac{1}{5\cancel{x}}\,\frac{5x}{sin(5x)}\right\}\right)$$

$$= \left(lim_{x\to 0}\left\{2\,\frac{sin(2x)}{2x}\right\}\right)\left(3\,lim_{x\to 0}\left\{\frac{sin(3x)}{3x}\right\}\right)\left(\frac{1}{5}\,lim_{x\to 0}\left\{\frac{5x}{sin(5x)}\right\}\right)$$

$$= (2)(3)\left(\frac{1}{5}\right)$$

$$= \frac{6}{5}$$

Final Summary for Limits – Trigonometric Limits

Squeeze Theorem
If for all points in an open interval (a, b), excluding the point $x = c$, where $a < c < b$, the following is true: $$l(x) \leq f(x) \leq u(x) \quad \text{and} \quad \lim_{x \to c}\{l(x)\} = \lim_{x \to c}\{u(x)\} = L$$ Then: $$\lim_{x \to c}\{f(x)\} = L$$
Important Trigonometric Limits
$$\lim_{x \to 0}\left\{\frac{\sin(x)}{x}\right\} = \lim_{x \to 0}\left\{\frac{x}{\sin(x)}\right\} = 1 \qquad \lim_{x \to 0}\left\{\frac{1 - \cos(x)}{x}\right\} = 0$$

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