Limits – Limits at Infinity

So far, while studying limits we have examined how a function behaves as x approaches some finite value, e.g. $x \to c$.

$$\lim_{x \to c}\{f(x)\}$$

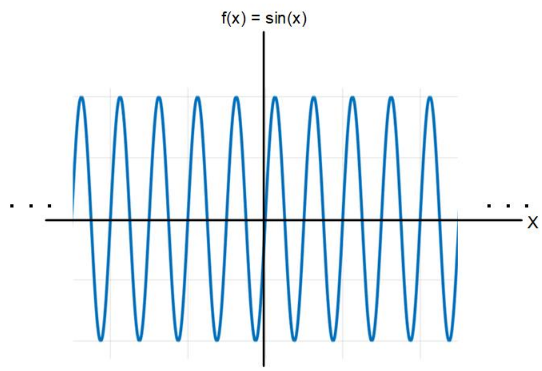
It is also useful to examine how a function behaves as x approaches either negative or positive infinity.

$\lim_{x \to -\infty}\{f(x)\}$	$\lim_{x \to \infty}\{f(x)\}$

Three possible scenarios for these limits are listed below.

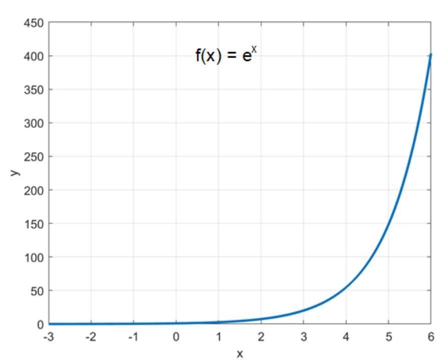
1. The function may not approach any one value, i.e. the limit does not exist. As an example, sinusoid functions oscillate indefinitely in both directions and therefore the limits do not exist.

$$\lim_{x \to \infty}\{\sin(x)\} = \lim_{x \to -\infty}\{\sin(x)\} = DNE$$



2. The function may approach negative or positive infinity. As an example, exponential function approaches positive infinity as x approaches infinity.

$$\lim_{x \to \infty}\{e^x\} = \infty$$



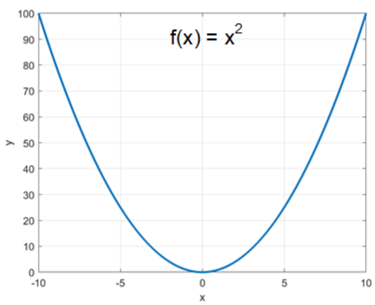
3. The function may approach a finite value. The exponential function can again be used for this case where we see that the function approaches zero as x approaches negative infinity.

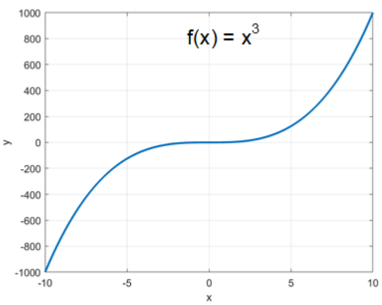
$$\lim_{x \to -\infty}\{e^x\} = 0$$

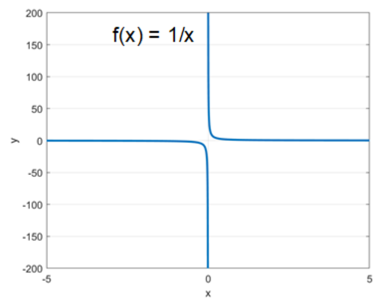
Certain classes of functions have limits at infinity that can be written in general terms. The first class of functions we describe are power functions of the form $f(x) = x^n$, where n is a positive integer.

Limits at Infinity of Power Functions	
For all positive integers, n	
$\lim_{x\to\infty}\{x^n\} = \infty$	$\lim_{x\to\infty}\{x^{-n}\} = 0$
$\lim_{x\to-\infty}\{x^n\} = \begin{cases}\infty & if\ n\ is\ even\\ -\infty & if\ n\ is\ odd\end{cases}$	$\lim_{x\to-\infty}\{x^{-n}\} = 0$

Some examples demonstrating the theorem above are shown below.



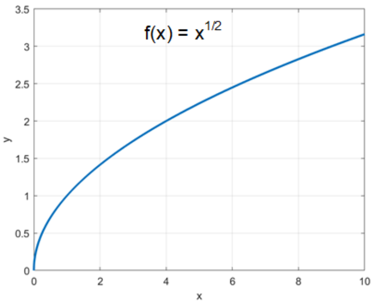


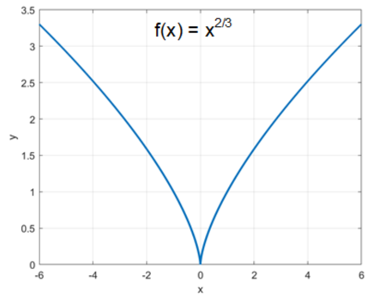


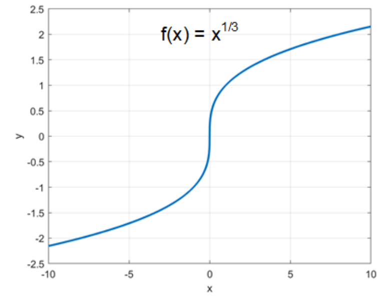
Another class of functions where we can apply general rules for are radical functions of the form $f(x) = \sqrt[q]{x^p} = x^{p/q}$, where p and q are positive integers.

Limits at Infinity of Radical Functions		
For all positive integers, p and q		
<u>q is even e.g. $x^{1/2}, x^{1/4}$</u>	<u>p is even, q is odd e.g. $x^{2/3}, x^{2/5}$</u>	<u>p is odd, q is odd e.g. $x^{1/3}, x^{1/5}$</u>
$\lim_{x\to\infty}\{x^{p/q}\} = \infty$	$\lim_{x\to\infty}\{x^{p/q}\} = \infty$	$\lim_{x\to\infty}\{x^{p/q}\} = \infty$
$\lim_{x\to-\infty}\{x^{p/q}\} = DNE$	$\lim_{x\to-\infty}\{x^{p/q}\} = \infty$	$\lim_{x\to-\infty}\{x^{p/q}\} = -\infty$

Some examples demonstrating the theorem above are shown below.







A final class of functions we will discuss are rational functions. Recall a rational function is an algebraic fraction that has a polynomial in both the numerator and denominator. Before we introduce general rules for this class of functions let's illustrate with an example and attempt to evaluate the following infinite limit.

$$\lim_{x\to\infty}\left\{\frac{20x^2-3x}{3x^5-4x^2+5}\right\}$$

Applying the quotient law is not valid since it gives an indeterminate form. Previously in this situation we attempted to algebraically rewrite the function. For rational functions our algebraic method will be to divide the numerator and denominator by the highest power of x in the denominator as illustrated below.

$$\begin{aligned} f(x) &= \frac{20x^2-3x}{3x^5-4x^2+5}\cdot\left(\frac{1/x^5}{1/x^5}\right) \\ &= \frac{\frac{20}{x^3}-\frac{3}{x^4}}{3-\frac{4}{x^3}+\frac{5}{x^5}} \end{aligned}$$

The limit can now be evaluated using basic limit laws and direct substitution.

$$\begin{aligned} \lim_{x\to\infty}\left\{\frac{\frac{20}{x^3}-\frac{3}{x^4}}{3-\frac{4}{x^3}+\frac{5}{x^5}}\right\} &= \frac{\left(\lim_{x\to\infty}\left\{\frac{20}{x^3}\right\}\right)-\left(\lim_{x\to\infty}\left\{\frac{3}{x^4}\right\}\right)}{\left(\lim_{x\to\infty}\{3\}\right)-\left(\lim_{x\to\infty}\left\{\frac{4}{x^3}\right\}\right)+\left(\lim_{x\to\infty}\left\{\frac{5}{x^5}\right\}\right)} \\ &= \frac{\left(\lim_{x\to\infty}\left\{\frac{20}{\infty}\right\}\right)-\left(\lim_{x\to\infty}\left\{\frac{3}{\infty}\right\}\right)}{\left(\lim_{x\to\infty}\{3\}\right)-\left(\lim_{x\to\infty}\left\{\frac{4}{\infty}\right\}\right)+\left(\lim_{x\to\infty}\left\{\frac{5}{\infty}\right\}\right)} \\ &= \frac{(0)-(0)}{(3)-(0)+(0)} \\ &= \frac{0}{3} \\ &= 0 \end{aligned}$$

We can now describe a general procedure as follows:

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}$$

Where $a_n \neq 0$ and $b_m \neq 0$.

Dividing the numerator and denominator by x^m we have

$$f(x) = \frac{\frac{a_n x^n}{x^m} + \frac{a_{n-1}x^{n-1}}{x^m} + \cdots + \frac{a_0}{x^m}}{\frac{b_m \cancel{x^m}}{\cancel{x^m}} + \frac{b_{m-1}x^{m-1}}{x^m} + \cdots + \frac{b_0}{x^m}}$$

Next, we can factor out $\left(\frac{x^n}{x^m}\right)$ from the numerator

$$f(x) = \left(\frac{x^n}{x^m}\right)\left(\frac{a_n + a_{n-1}x^{-1} + \cdots + a_0 x^{-n}}{b_m + b_{m-1}x^{-1} + \cdots + b_0 x^{-m}}\right)$$

Evaluating this form using either positive or negative infinity, using basic limit laws as we did for the above example, we have

$$\lim_{x\to\pm\infty}\{f(x)\} = \left(\lim_{x\to\pm\infty}\left\{\frac{x^n}{x^m}\right\}\right)\cdot\left(\frac{\lim_{x\to\pm\infty}\{a_n + a_{n-1}x^{-1} + \cdots + a_0 x^{-n}\}}{\lim_{x\to\mp\infty}\{b_m + b_{m-1}x^{-1} + \cdots + b_0 x^{-m}\}}\right)$$

$$= \left(\lim_{x\to\pm\infty}\left\{\frac{x^n}{x^m}\right\}\right)\cdot\left(\frac{\lim_{x\to\pm\infty}\{a_n\} + \lim_{x\to\pm\infty}\{a_{n-1}x^{-1}\} + \cdots + \lim_{x\to\infty}\{a_0 x^{-n}\}}{\lim_{x\to\pm\infty}\{b_m\} + \lim_{x\to\pm\infty}\{b_{m-1}x^{-1}\} + \cdots + \lim_{x\to\infty}\{b_0 x^{-m}\}}\right)$$

$$= \left(\lim_{x\to\pm\infty}\left\{\frac{x^n}{x^m}\right\}\right)\cdot\left(\frac{a_n + 0 + \cdots + 0}{b_m + 0 + \cdots + 0}\right)$$

$$= \left(\frac{a_n}{b_m}\right)\left(\lim_{x\to\pm\infty}\left\{\frac{x^n}{x^m}\right\}\right)$$

Which leaves us to analyze the behavior of the much simpler function x^n/x^m. The evaluation of the limit will depend on the specific values of both n and m.

The three different cases are listed below.

1. $n = m$

$$\lim_{x\to\infty}\left\{\frac{x^n}{x^n}\right\} = \lim_{x\to\infty}\{1\} = 1 \qquad \lim_{x\to-\infty}\left\{\frac{x^n}{x^n}\right\} = \lim_{x\to-\infty}\{1\} = 1$$

2. $n < m$, i.e. $m - n > 0$

$$\lim_{x\to\infty}\left\{\frac{1}{x^{m-n}}\right\} = \lim_{x\to\infty}\left\{\frac{1}{\infty}\right\} = 0 \qquad \lim_{x\to-\infty}\left\{\frac{1}{x^{m-n}}\right\} = \lim_{x\to-\infty}\left\{\frac{1}{\pm\infty}\right\} = 0$$

3. $n > m$, i.e. $n - m > 0$

$n - m$: even

$$\lim_{x\to\infty}\left\{\frac{x^{n-m}}{1}\right\} = \lim_{x\to\infty}\{x^{n-m}\} = \infty \qquad \lim_{x\to-\infty}\left\{\frac{x^{n-m}}{1}\right\} = \lim_{x\to-\infty}\{x^{n-m}\} = \infty$$

$n - m$: odd

$$\lim_{x\to\infty}\left\{\frac{x^{n-m}}{1}\right\} = \lim_{x\to\infty}\{x^{n-m}\} = \infty \qquad \lim_{x\to-\infty}\left\{\frac{x^{n-m}}{1}\right\} = \lim_{x\to-\infty}\{x^{n-m}\} = -\infty$$

The following summarizes the results from above.

Limits at Infinity of Rational Functions
The limits at infinity of a rational function depend ONLY on the leading terms of the numerator and denominator. If $a_n, b_m \neq 0$, then $$\lim_{x\to\pm\infty}\left\{\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}\right\} = \left(\frac{a_n}{b_m}\right)\lim_{x\to\pm\infty}\left\{\frac{x^n}{x^m}\right\}$$ With the following cases that depend on the relative values of n and m 1. $n = m$: $$\left(\frac{a_n}{b_m}\right)\lim_{x\to\pm\infty}\left\{\frac{x^n}{x^n}\right\} = \frac{a_n}{b_m}$$ 2. $n < m$: $$\left(\frac{a_n}{b_m}\right)\lim_{x\to\pm\infty}\left\{\frac{1}{x^{m-n}}\right\} = 0$$ 3. $n > m$, $n - m$: Even $$\left(\frac{a_n}{b_m}\right)\lim_{x\to\pm\infty}\{x^{n-m}\} = \infty$$ $n - m$: Odd $$\left(\frac{a_n}{b_m}\right)\lim_{x\to\pm\infty}\{x^{n-m}\} = \pm\infty$$

Finally, we end with a few examples.

Examples: Evaluate the following limits.

a.) $\lim_{x\to\infty}\left\{\frac{3x^2+20x}{4x^2+91}\right\}$ b.) $\lim_{x\to\infty}\left\{\frac{3x^2+20x}{2x^4+3x^3-29}\right\}$ c.) $\lim_{x\to\infty}\left\{\frac{9x^2-2}{6-29x}\right\}$

d.) $\lim_{x\to-\infty}\left\{\frac{3x^3-10}{x+4}\right\}$ e.) $\lim_{x\to-\infty}\left\{\frac{12x+25}{\sqrt{16x^2+100x+500}}\right\}$ f.) $\lim_{x\to-\infty}\left\{\frac{8x^2+7x^{1/3}}{\sqrt{16x^4+6}}\right\}$

Solutions:

a.) $\lim_{x\to\infty}\left\{\frac{3x^2+20x}{4x^2+91}\right\}=\left(\frac{3}{4}\right)\lim_{x\to\infty}\left\{\frac{x^2}{x^2}\right\}=\left(\frac{3}{4}\right)(1)=\frac{3}{4}$

b.) $\lim_{x\to\infty}\left\{\frac{3x^2+20x}{2x^4+3x^3-29}\right\}=\left(\frac{3}{2}\right)\lim_{x\to\infty}\left\{\frac{x^2}{x^4}\right\}=\left(\frac{3}{2}\right)\lim_{x\to\infty}\left\{\frac{1}{x^2}\right\}=\left(\frac{3}{2}\right)(0)=0$

c.) $\lim_{x\to\infty}\left\{\frac{9x^2-2}{6-29x}\right\}=\left(\frac{9}{-29}\right)\lim_{x\to\infty}\left\{\frac{x^2}{x}\right\}=\left(\frac{9}{-29}\right)\lim_{x\to\infty}\{x\}=\infty$

d.) $\lim_{x\to-\infty}\left\{\frac{3x^3-10}{x+4}\right\}=\left(\frac{3}{1}\right)\lim_{x\to-\infty}\left\{\frac{x^3}{x}\right\}=\left(\frac{3}{2}\right)\lim_{x\to-\infty}\{x^2\}=\infty$

e.) $\lim_{x\to-\infty}\left\{\frac{12x+25}{\sqrt{16x^2+100x+500}}\right\}=\left(\frac{12}{\sqrt{16}}\right)\lim_{x\to-\infty}\left\{\frac{x}{\sqrt{x^2}}\right\}=\left(\frac{12}{4}\right)\lim_{x\to-\infty}\{-1\}=-3$

f.) $\lim_{x\to-\infty}\left\{\frac{8x^2+7x^{1/3}}{\sqrt{16x^4+6}}\right\}=\left(\frac{8}{\sqrt{16}}\right)\lim_{x\to-\infty}\left\{\frac{x^2}{\sqrt{x^4}}\right\}=\left(\frac{8}{4}\right)\lim_{x\to-\infty}\{1\}=2$

Note the last two limits contain radicals, and therefore are not strictly rational functions. However, we can treat the terms inside the radical the same as we would if the radical wasn't there and ignore all lower terms. One important point is to leave the radical until the last step so you can properly determine the sign of the answer. This is shown in problem e.) where we essentially have the $n=m$ case, and therefore would expect a positive value for the limit. However, by leaving the radical we see that the denominator will remain positive while the numerator will be negative.

Final Summary for Limits – Limits at Infinity

Limits at Infinity of Power Functions
For all positive integers, n $\lim_{x\to\infty}\{x^n\} = \infty$ $\lim_{x\to\infty}\{x^{-n}\} = 0$ $\lim_{x\to-\infty}\{x^n\} = \begin{cases} \infty \ if\ n\ is\ even \\ -\infty \ if\ n\ is\ odd \end{cases}$ $\lim_{x\to-\infty}\{x^{-n}\} = 0$

Limits at Infinity of Radical Functions

For all positive integers, p and q

q is even e.g. $x^{1/2}, x^{1/4}$	p is even, q is odd e.g. $x^{2/3}, x^{2/5}$	p is odd, q is odd e.g. $x^{1/3}, x^{1/5}$
$\lim_{x\to\infty}\{x^{p/q}\} = \infty$	$\lim_{x\to\infty}\{x^{p/q}\} = \infty$	$\lim_{x\to\infty}\{x^{p/q}\} = \infty$
$\lim_{x\to-\infty}\{x^{p/q}\} = DNE$	$\lim_{x\to-\infty}\{x^{p/q}\} = \infty$	$\lim_{x\to-\infty}\{x^{p/q}\} = -\infty$

Limits at Infinity of Rational Functions

The limits at infinity of a rational function depend ONLY on the leading terms of the numerator and denominator.

If $a_n, b_m \neq 0$, then

$$\lim_{x\to\pm\infty}\left\{\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}\right\} = \left(\frac{a_n}{b_m}\right)\lim_{x\to\pm\infty}\left\{\frac{x^n}{x^m}\right\}$$

With the following cases that depend on the relative values of n and m

1. $n = m$:

$$\lim_{x\to\pm\infty}\left\{\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}\right\} = \frac{a_n}{b_m}$$

2. $n < m$:

$$\lim_{x\to\pm\infty}\left\{\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}\right\} = 0$$

3. $n > m$,

   $n - m$: Even

$$\lim_{x\to\pm\infty}\left\{\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}\right\} = \infty$$

   $n - m$: Odd

$$\lim_{x\to\pm\infty}\left\{\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}\right\} = \pm\infty$$

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