Limits – Algebraic Evaluation

When studying continuity, we learned that if a function is continuous at $x = c$, we can evaluate the limit using direct substitution.

$$\lim_{x \to c}\{f(x)\} = f(c)$$

However, if $f(c)$ is undefined we need to find alternative methods for evaluating the limit. When we were first introduced to limits, we learned that they can usually be evaluated numerically. This method allowed us to gain valuable insight, however as you may have noticed, the procedure is very time consuming. Fortunately, there are alternative methods. In some cases, we can rewrite $f(x)$ using algebra in a way that allows us to then use direct substitution. Asked to evaluate $\lim_{x \to c}\{f(x)\}$ we can proceed as follows. We first determine whether $f(c)$ evaluates to an indeterminant form. If it does, we can then attempt to transform $f(x)$ algebraically into a new expression where $f(c)$ is defined and the limit can be evaluated using direct substitution.

Indeterminant Form
We say that $f(x)$ has an indeterminate form at $x = c$ when $f(c)$ evaluates to one of the following forms: $\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty \cdot 0, \quad \infty - \infty$

When trying to rewrite $f(x)$ we hope to remove the term that makes the function indeterminate. It's not always easy to see how to proceed, however there are certain commonly used algebraic techniques that we review below.

1. Factor Polynomials

Rational expressions are sometimes presented in a way that they may evaluate to an indeterminate form. When this is the case, we can try to factor both the numerator and denominator in hopes that some factors will cancel. An example is shown below.

Evaluate the following limit:

$$\lim_{x \to 3}\left\{\frac{x^2 - 4x + 3}{x^2 + x - 12}\right\}$$

As shown below direct substitution results in an indeterminate form.

$$f(3) = \frac{3^2 - 4 \cdot 3 + 3}{3^2 + 3 - 12} = \frac{0}{0}$$

Therefore, we try to rewrite $f(x)$ by factoring the numerator and denominator.

$$f(x) = \frac{x^2 - 4x + 3}{x^2 + x - 12} = \frac{\cancel{(x-3)}(x-1)}{\cancel{(x-3)}(x+4)} = \frac{(x-1)}{(x+4)}$$

The limit can now be evaluated using direct substitution.

$$\lim_{x\to 3}\left\{\frac{x^2-4x+3}{x^2+x-12}\right\} = \lim_{x\to 3}\left\{\frac{(x-1)}{(x+4)}\right\} = \frac{(3-1)}{(3+4)} = \frac{2}{7}$$

2. Common Denominator

If two or more rational expressions that both evaluate to infinity are subtracted, we get the indeterminate, $\infty - \infty$. In this case we can try to find a common denominator to rewrite the expression. An example is shown below.

Evaluate the following limit:

$$\lim_{x\to 1}\left\{\frac{1}{x-1}-\frac{2}{x^2-1}\right\}$$

As shown below direct substitution results in an indeterminate form.

$$f(1) = \frac{1}{1-1}-\frac{2}{1-1} = \infty - \infty$$

Therefore, we try to rewrite $f(x)$ using a common denominator.

$$\begin{aligned} f(x) &= \frac{1}{x-1}-\frac{2}{x^2-1} \\ &= \frac{1}{(x-1)}-\frac{2}{(x-1)(x+1)} \\ &= \frac{(x+1)-2}{(x-1)(x+1)} \\ &= \frac{\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} \\ &= \frac{1}{(x+1)} \end{aligned}$$

The limit can now be evaluated using direct substitution.

$$\lim_{x\to 1}\left\{\frac{1}{x-1}-\frac{2}{x^2-1}\right\} = \lim_{x\to 1}\left\{\frac{1}{(x+1)}\right\} = \frac{1}{(1+1)} = \frac{1}{2}$$

3. Expand Factored Terms

In some cases, expanding factored terms may result in cancellation of key terms. An example is shown below.

Evaluate the following limit:

$$\lim_{x\to 0}\left\{\frac{(x+2)^2-4}{x}\right\}$$

As shown below direct substitution results in an indeterminate form.

$$f(0) = \frac{(0+2)^2 - 4}{0} = \frac{0}{0}$$

Therefore, we try to rewrite $f(x)$ by expanding the numerator.

$$\begin{aligned} f(x) &= \frac{(x+2)^2 - 4}{x} \\ &= \frac{x^2 + 4x + \cancel{4 - 4}}{x} \\ &= \frac{\cancel{x}(x+4)}{\cancel{x}} \\ &= x + 4 \end{aligned}$$

The limit can now be evaluated using direct substitution.

$$\lim_{x \to 0} \left\{ \frac{(x+2)^2 - 4}{x} \right\} = \lim_{x \to 0} \{x + 4\} = 0 + 4 = 4$$

4. Multiply by the Conjugate

In math a conjugate refers to the changing of signs between the two terms in a binomial. For example, the conjugate of $(3x^2 - 5x)$ is $(3x^2 + 5x)$. When the limit of an expression involving a binomial with the square root evaluates to an indeterminate we can try to rewrite the expression by multiplying the numerator and denominator by the conjugate of the binomial. An example is shown below.

Evaluate the following limit:

$$\lim_{x \to 4} \left\{ \frac{\sqrt{x} - 2}{x - 4} \right\}$$

As shown below direct substitution results in an indeterminate form.

$$f(0) = \frac{\sqrt{4} - 2}{4 - 4} = \frac{0}{0}$$

Therefore, we try to rewrite $f(x)$ by using the conjugate technique mentioned above. Note we do not expand the denominator.

$$\begin{aligned} f(x) &= \frac{\sqrt{x} - 2}{x - 4} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2} \\ &= \frac{\cancel{(x - 4)}}{\cancel{(x - 4)}(\sqrt{x} + 2)} \\ &= \frac{1}{\sqrt{x} + 2} \end{aligned}$$

The limit can now be evaluated using direct substitution.

$$\lim_{x\to 4}\left\{\frac{\sqrt{x}-2}{x-4}\right\} = \lim_{x\to 4}\left\{\frac{1}{\sqrt{x}+2}\right\} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

Before we move on and provide additional examples for practice, we mention an important aspect of evaluating limits algebraically. Given an indeterminate form, it is NOT always possible to algebraically manipulate the expression to allow for evaluation of the limit using direct substitution as we did in the above examples. In these cases, other tools are required, such as numerical or graphical techniques presented earlier. However, an alternative technique that can sometimes work, particularly with limits involving trigonometric functions, is called the squeeze theorem, which we introduce in the next section.

Before ending this lesson, let's review a few more examples using the algebraic techniques discussed above.

Examples:

Evaluate the following limits.

a.) $\lim_{x\to -1}\left\{\frac{x^2+2x+1}{x+1}\right\}$

b.) $\lim_{x\to 8}\left\{\frac{x^3-64x}{x-8}\right\}$

c.) $\lim_{x\to 0}\left\{\frac{\frac{1}{(x+2)^2}-\frac{1}{4}}{x}\right\}$

d.) $\lim_{x\to 4}\left\{\frac{(x+2)^2-9x}{x-4}\right\}$

e.) $\lim_{x\to 4}\left\{\frac{x-4}{\sqrt{x}-\sqrt{8-x}}\right\}$

f.) $\lim_{x\to 4}\left\{\frac{\sqrt{5-x}-1}{2-\sqrt{x}}\right\}$

g.) $\lim_{x\to 2}\left\{\frac{1}{4x-8}-\frac{1}{x^2-4}\right\}$

h.) $\lim_{\theta\to \frac{\pi}{4}}\left\{\frac{1}{\tan(\theta)-1}-\frac{2}{\tan^2(\theta)-1}\right\}$

a.) Substitution yields the indeterminate $0/0$, therefore we try to algebraically rewrite the expression before we attempt substitution.

$$f(x) = \frac{x^2+2x+1}{x+1}$$
$$f(x) = \frac{\cancel{(x+1)}(x+1)}{\cancel{x+1}}$$
$$f(x) = x+1$$

Substitution can now be used to evaluate the limit.

$$\lim_{x\to -1}\{x+1\} = \lim_{x\to -1}\{-1+1\} = 0$$

b.) Substitution yields the indeterminate $0/0$, therefore we try to algebraically rewrite the expression before we attempt substitution.

$$f(x) = \frac{x^3 - 64x}{x - 8}$$
$$f(x) = \frac{x(x^2 - 64)}{x - 8}$$
$$f(x) = \frac{x\cancel{(x-8)}(x + 8)}{\cancel{x - 8}}$$
$$f(x) = x(x + 8)$$

Substitution can now be used to evaluate the limit.

$$\lim_{x \to 8}\{x(x + 8)\} = \lim_{x \to 8}\{8(8 + 8)\} = 128$$

c.) Substitution yields the indeterminate $0/0$, therefore we try to algebraically rewrite the expression before we attempt substitution.

$$f(x) = \frac{\frac{1}{(x + 2)^2} - \frac{1}{4}}{x}$$
$$f(x) = \frac{1}{x(x + 2)^2} - \frac{1}{4x}$$
$$f(x) = \frac{4 - (x + 2)^2}{4x(x + 2)^2}$$
$$f(x) = \frac{4 - x^2 - 4x\cancel{- 4}}{4x(x + 2)^2}$$
$$f(x) = \frac{-\cancel{x}(x + 4)}{4x(x + 2)^2}$$
$$f(x) = \frac{-\cancel{x}(x + 4)}{4\cancel{x}(x + 2)^2}$$
$$f(x) = \frac{-(x + 4)}{4(x + 2)^2}$$

Substitution can now be used to evaluate the limit.

$$\lim_{x \to 0}\left\{\frac{-(x + 4)}{4(x + 2)^2}\right\} = \lim_{x \to 0}\left\{\frac{-(0 + 4)}{4(0 + 2)^2}\right\} = -\frac{1}{4}$$

d.) Substitution yields the indeterminate $0/0$, therefore we try to algebraically rewrite the expression before we attempt substitution.

$$f(x) = \frac{(x+2)^2 - 9x}{x-4}$$

$$f(x) = \frac{x^2 + 4x + 4 - 9x}{x-4}$$

$$f(x) = \frac{x^2 - 5x + 4}{x-4}$$

$$f(x) = \frac{\cancel{(x-4)}(x-1)}{\cancel{x-4}}$$

$$f(x) = (x-1)$$

Substitution can now be used to evaluate the limit.

$$\lim_{x \to 4}\{(x-1)\} = \lim_{x \to 4}\{4-1\} = 3$$

e.) Substitution yields the indeterminate $0/0$, therefore we try to algebraically rewrite the expression before we attempt substitution.

$$f(x) = \frac{x-4}{\sqrt{x} - \sqrt{8-x}}$$

$$f(x) = \frac{x-4}{\left(\sqrt{x} - \sqrt{8-x}\right)} \cdot \frac{\left(\sqrt{x} + \sqrt{8-x}\right)}{\left(\sqrt{x} + \sqrt{8-x}\right)}$$

$$f(x) = \frac{(x-4)\left(\sqrt{x} + \sqrt{8-x}\right)}{\left(x - (8-x)\right)}$$

$$f(x) = \frac{(x-4)\left(\sqrt{x} + \sqrt{8-x}\right)}{(2x-8)}$$

$$f(x) = \frac{\cancel{(x-4)}\left(\sqrt{x} + \sqrt{8-x}\right)}{2\cancel{(x-4)}}$$

$$f(x) = \frac{\left(\sqrt{x} + \sqrt{8-x}\right)}{2}$$

Substitution can now be used to evaluate the limit.

$$\lim_{x \to 4}\left\{\frac{\left(\sqrt{x} + \sqrt{8-x}\right)}{2}\right\} = \lim_{x \to 4}\left\{\frac{\left(\sqrt{4} + \sqrt{8-4}\right)}{2}\right\} = 2$$

f.) Substitution yields the indeterminate $0/0$, therefore we try to algebraically rewrite the expression before we attempt substitution. Note this function has a square root term in both the numerator and denominator. In this case we will attempt to perform the conjugate twice.

$$f(x) = \frac{\sqrt{5-x}-1}{2-\sqrt{x}}$$

$$f(x) = \frac{\left(\sqrt{5-x}-1\right)}{\left(2-\sqrt{x}\right)} \cdot \frac{\left(\sqrt{5-x}+1\right)}{\left(\sqrt{5-x}+1\right)}$$

$$f(x) = \frac{(5-x)-1}{\left(\sqrt{5-x}+1\right)\left(2-\sqrt{x}\right)}$$

$$f(x) = \frac{(4-x)}{\left(\sqrt{5-x}+1\right)\left(2-\sqrt{x}\right)} \cdot \frac{\left(2+\sqrt{x}\right)}{\left(2+\sqrt{x}\right)}$$

$$f(x) = \frac{\cancel{(4-x)}\left(2+\sqrt{x}\right)}{\left(\sqrt{5-x}+1\right)\cancel{(4-x)}}$$

$$f(x) = \frac{\left(2+\sqrt{x}\right)}{\left(\sqrt{5-x}+1\right)}$$

Substitution can now be used to evaluate the limit.

$$\lim_{x\to 4}\left\{\frac{\left(2+\sqrt{x}\right)}{\left(\sqrt{5-x}+1\right)}\right\} = \lim_{x\to 4}\left\{\frac{\left(2+\sqrt{4}\right)}{\left(\sqrt{5-4}+1\right)}\right\} = \frac{4}{2} = 2$$

g.) Substitution yields the indeterminate $\infty - \infty$, therefore we try to algebraically rewrite the expression before we attempt substitution.

$$f(x) = \frac{1}{4x-8} - \frac{1}{x^2-4}$$

$$f(x) = \frac{1}{4(x-2)} - \frac{1}{(x-2)(x+2)}$$

$$f(x) = \frac{(x+2)-4}{4(x-2)(x+2)}$$

$$f(x) = \frac{\cancel{(x-2)}}{4\cancel{(x-2)}(x+2)}$$

$$f(x) = \frac{1}{4(x+2)}$$

Substitution can now be used to evaluate the limit.

$$\lim_{x\to 2}\left\{\frac{1}{4(x+2)}\right\} = \lim_{x\to 2}\left\{\frac{1}{4(2+2)}\right\} = \frac{1}{16}$$

h.) Substitution yields the indeterminate $\infty - \infty$, therefore we try to algebraically rewrite the expression before we attempt substitution.

$$f(x) = \frac{1}{\tan(\theta) - 1} - \frac{2}{\tan^2(\theta) - 1}$$

$$f(x) = \frac{1}{\tan(\theta) - 1} - \frac{2}{(\tan(\theta) - 1)(\tan(\theta) + 1)}$$

$$f(x) = \frac{(\tan(\theta) + 1) - 2}{(\tan(\theta) - 1)(\tan(\theta) + 1)}$$

$$f(x) = \frac{\cancel{(\tan(\theta) - 1)}}{\cancel{(\tan(\theta) - 1)}(\tan(\theta) + 1)}$$

$$f(x) = \frac{1}{(\tan(\theta) + 1)}$$

Substitution can now be used to evaluate the limit.

$$\lim_{x \to \frac{\pi}{4}} \left\{ \left(\frac{1}{\tan(\theta) + 1} \right) \right\} = \lim_{x \to \frac{\pi}{4}} \left\{ \left(\frac{1}{\tan\left(\frac{\pi}{4}\right) + 1} \right) \right\} = \lim_{x \to \frac{\pi}{4}} \left\{ \frac{1}{1 + 1} \right\} = \frac{1}{2}$$

Final Summary for Limits – Algebraic Evaluation

Direct Substitution
If a function is continuous at $x = c$, we can evaluate the limit using direct substitution. $$\lim_{x\to c}\{f(x)\} = f(c)$$

Indeterminate Forms
We say that $f(x)$ has an indeterminate form at $x = c$ when $f(c)$ evaluates to one of the following forms: $$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty \cdot 0, \quad \infty - \infty$$

Algebraic Evaluation Techniques
To evaluate $\lim_{x\to c}\{f(x)\}$ when $f(c)$ is an indeterminate we can try to algebraically rewrite $f(x)$ so that $f(c)$ no longer evaluates to an indeterminate. **If possible**, the limit can then be evaluated using direct substitution. Some common algebraic techniques to rewrite $f(x)$ are as follows: 1. Factor all factorable polynomials 2. Combine terms using a common denominator where possible 3. Expand factored terms where possible 4. Multiply numerator and denominator by the conjugate of the binomial with a square root term. Note: Given an indeterminate form, it is NOT always possible to algebraically manipulate the expression to allow for evaluation of the limit using direct substitution. In these cases, we need to rely on other techniques.

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