Differentiation – Introduction to the Derivative

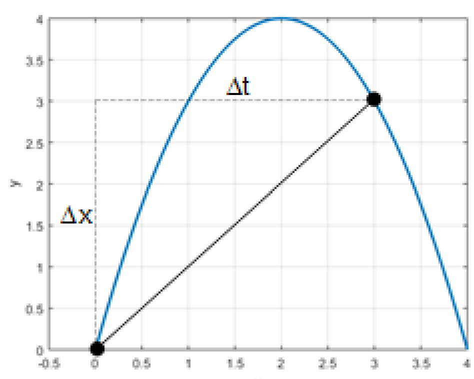
The two major areas of study in calculus are differential calculus, looking at how one quantity changes with respect to another, and integral calculus, dealing with the accumulation of one quantity as it varies with respect to another quantity. At the moment these descriptions may seem unclear, however they will become clearer as we progress with our studies. We begin our study of calculus with the first major area, i.e. differential calculus.

The measure we use to describe how one quantity changes with respect to another is called a *rate of change*. In calculus the *derivative* is the name we give to this measure and d*ifferentiation* is the process used to compute the derivative. Recall, we motivated the study of limits by looking at velocity, which is a rate of change that should be familiar to most of us. Let's return to the example of velocity and more precisely specify how the ordinary notion of rate of change relates to the derivative in calculus. We'll use the example below where the position of a particle, x [meters], changes with time, t [seconds], according to the following equation.

$$x(t) = -(t-2)^2 + 4, \qquad t \geq 0$$

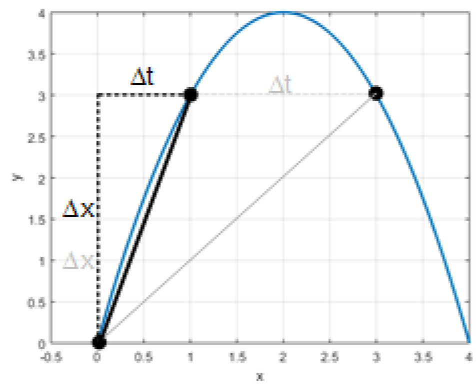
For the moment let's assume we observe the position of the particle only at $t_1 = 0$ and $t_2 = 3$. Given this information how would we compute the particles rate of change, i.e. velocity? In general, the velocity, (*rate of change of position with respect to time*), is computed as the change in position divided by the change in time, sometimes referred to as the difference quotient.

$$v = \frac{\Delta x}{\Delta t} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{x(3) - x(0)}{3 - 0} = \frac{3}{3} = 1\ m/s$$



From the above figure we see that the velocity is equivalent to the slope of a line connecting the two points. We further notice, however, that the particle did not follow this straight-line path. What we measured was the *average velocity* of the particle between these two points. We can get a better idea of the velocity at $t_1 = 0$ by bringing our second observation point, t_2, closer to our first, t_1. Let's recompute the velocity with $t_2 = 1$.

$$v = \frac{\Delta x}{\Delta t} = \frac{x(1) - x(0)}{1 - 0} = \frac{3}{1} = 3\ m/s$$

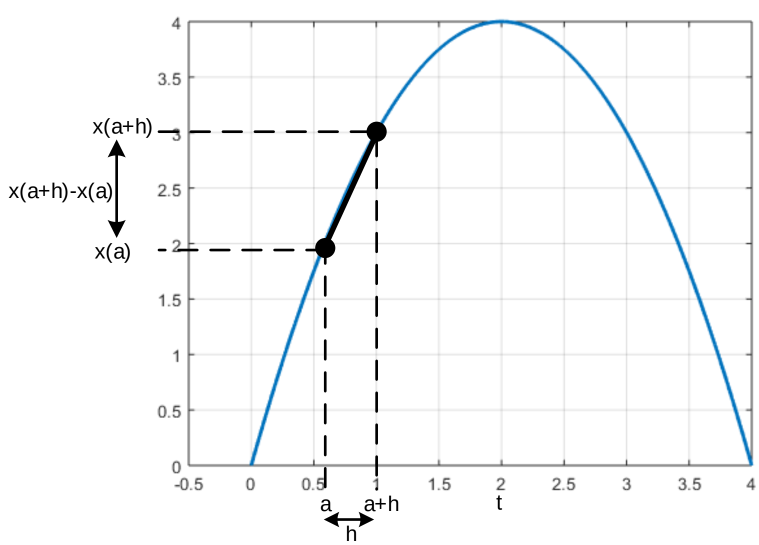


In this figure we notice that the slope of the line, and therefore the velocity, seems much closer to what the slope appears to be at $t_1 = 0$. However, this is still a measure of the average velocity of the particle between these new points. What we would like to know is the velocity at *exactly* the time $t_1 = 0$. We refer to this as the *instantaneous velocity*, and to compute this value we need bring the two observation points as close together as possible. Of course, if we make them equal, we obtain the indeterminate expression, $0/0$. Luckily, we have previously learned that we can use the notion of a limit to help us resolve this issue. With this in mind we can define the *instantaneous velocity, v_{inst},* as the limit of the difference quotient as follows:

$$v_{inst} = \lim_{\Delta t \to 0} \left\{ \frac{\Delta x}{\Delta t} \right\} = \lim_{t_2 \to t_1} \left\{ \frac{x(t_2) - x(t_1)}{t_2 - t_1} \right\}$$

In this form the limit is not easily evaluated, however we can rewrite the difference quotient using $t_1 = a$ and $t_2 = a + h$ as illustrated in the figure below. In this case the instantaneous velocity at time $t = a$ can then be expressed as:

$$v_{inst} = \lim_{h \to 0} \left\{ \frac{x(a+h) - x(a)}{h} \right\}$$



We can now attempt to evaluate this limit using $a = 0$.

$$\begin{aligned} v_{inst} &= \lim_{h\to 0}\left\{\frac{x(0+h)-x(0)}{h}\right\} \\ &= \lim_{h\to 0}\left\{\frac{(-(h-2)^2+4)-(-(0-2)^2+4)}{h}\right\} \\ &= \lim_{h\to 0}\left\{\frac{(-h^2+4h\cancel{-4+4})-(0)}{h}\right\} \\ &= \lim_{h\to 0}\left\{\frac{\cancel{h}(-h+4)}{\cancel{h}}\right\} \\ &= \lim_{h\to 0}\{(-h+4)\} \\ &= 0+4 \\ &= 4\ m/s \end{aligned}$$

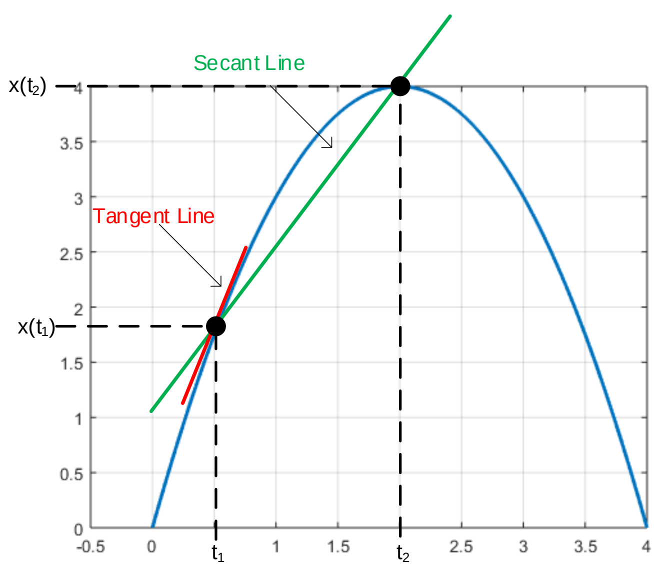
Let's summarize whet we have learned. We initially computed the *average velocity* using $t_1 = 0$ and $t_2 = 3$, which gave us a value of $1\ m/s$. We then moved t_2 closer to t_1 and computed an average velocity of $3\ m/s$. Finally, we moved t_2 as close as possible to t_1, i.e. let $h \to 0$, and found the *instantaneous velocity* at $t_1 = 0$ to be $4\, m/s$. Note that each time we are computing a slope of a line.

The *average velocity* from $t_1 = a$ to $t_2 = a + h$ is equivalent to the slope of the line that passes through the points $(a, x(a))$ and $(a + h\,, x(a + h\,))$. This line is referred to as the secant line.

$$Slope\ of\ the\ secant\ line = average\ rate\ of\ change = v_{avg} = \frac{x(a+h)-x(a)}{h}$$

And the *instantaneous velocity* at time $t_1 = a$ is equivalent to the slope of a line that passes through a single point $(a, x(a))$. This line is referred to as the tangent line.

$$Slope\ of\ the\ tangent\ line = instantaneous\ rate\ of\ change = v_{inst} = \lim_{h\to 0}\left\{\frac{x(a+h)-x(a)}{h}\right\}$$



We now interpret the concepts from above in the language of calculus. The instantaneous rate of change, defined by the limit of the difference quotient, is what we refer to in calculus as the *derivative.* Furthermore, the process of computing this value is called *differentiation*. It's important to remember, however, that there are two quantities involved when computing the derivative. In the examples above they were the position, x, and the time, t. Using these quantities as examples, it is more precise to refer to the derivative as *"the derivative of x with respect to t"*, and the process of differentiation as "*differentiating x with respect to t*".

The Definition of the Derivative
The derivative of a function, $f(x)$, with respect to x at a point $x = a$, which we refer to as $f'(a)$, is the limit of the difference quotient (if it exists): $$f'(a) = \lim_{h \to 0} \left\{ \frac{f(a+h) - f(a)}{h} \right\}$$ If the limit exists, then we say the function is *differentiable* at the point $x = a$. Note: The derivative at a point $x = a$ can also be interpreted as the slope of the tangent line to the graph of $f(x)$ at the point, $P = \big(a, f(a)\big)$.

Let's do some examples to get more familiar with the ideas presented above.

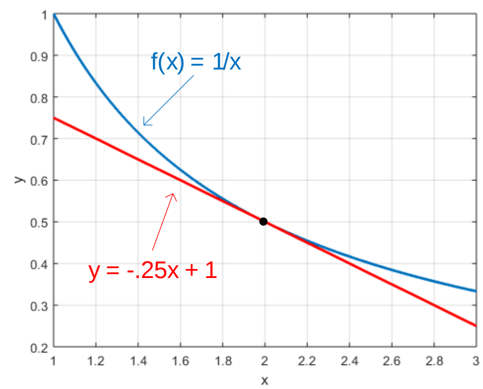
Example 1: Compute $f'(2)$, the derivative of $f(x)$ at the point $x = 2$, where $f(x) = \frac{1}{x}$. Then find the equation of the tangent line to $f(x)$ at $x = 2$.

Solution 1: Using the definition of the derivative from above with $a = 2$ we have:

$$\begin{aligned}
f'(2) &= \lim_{h \to 0} \left\{ \frac{f(2+h) - f(2)}{h} \right\} \\
&= \lim_{h \to 0} \left\{ \frac{\frac{1}{2+h} - \frac{1}{2}}{h} \right\} \\
&= \lim_{h \to 0} \left\{ \frac{\frac{2-(2+h)}{4+2h}}{h} \right\} \\
&= \lim_{h \to 0} \left\{ \frac{-\cancel{h}}{\cancel{h}(4+2h)} \right\} \\
&= \lim_{h \to 0} \left\{ \frac{-1}{(4+2h)} \right\} \\
&= \frac{-1}{(4+0)} \\
&= -\frac{1}{4}
\end{aligned}$$

Recall that the point slope formula can be used to find the equation of a line when we are given the slope and a point on the line. As we are told to find the tangent line of $f(x)$ at $x = 2$ our point is $(x_1, y_1) = \big(2, f(2)\big) = (2, 1/2)$. The slope of our tangent line, m, by definition is given by $f'(2) = -\frac{1}{4}$. The tangent line computation is shown below on the left, while the figure on the right shows both the original function and the tangent line to the function at $x = 2$.

$$
\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - \frac{1}{2} &= -\frac{1}{4}(x - 2) \\
y &= -\frac{1}{4}x + \frac{1}{2} + \frac{1}{2} \\
y &= -\frac{1}{4}x + 1
\end{aligned}
$$



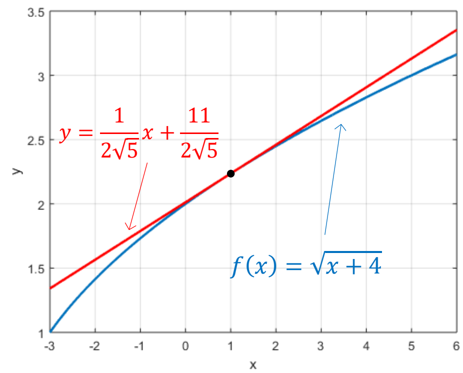
Example 2: Compute $f'(1)$, the derivative of $f(x)$ at the point $x = 1$, where $f(x) = \sqrt{x+4}$. Then find the equation of the tangent line to $f(x)$ at $x = 1$.

Solution 2: We proceed the same as we did above to compute $f'(1)$.

$$
\begin{aligned}
f'(1) &= \lim_{h\to 0}\left\{\frac{f(1+h) - f(1)}{h}\right\} \\
&= \lim_{h\to 0}\left\{\frac{\sqrt{1+h+4} - \sqrt{1+4}}{h}\right\} \\
&= \lim_{h\to 0}\left\{\left(\frac{\sqrt{5+h} - \sqrt{5}}{h}\right)\left(\frac{\sqrt{5+h} + \sqrt{5}}{\sqrt{5+h} + \sqrt{5}}\right)\right\} \\
&= \lim_{h\to 0}\left\{\left(\frac{\cancel{5} + \cancel{h} - \cancel{5}}{\cancel{h}(\sqrt{5+h} + \sqrt{5})}\right)\right\} \\
&= \lim_{h\to 0}\left\{\left(\frac{1}{(\sqrt{5+h} + \sqrt{5})}\right)\right\} \\
&= \frac{1}{(\sqrt{5+0} + \sqrt{5})} \\
&= \frac{1}{2\sqrt{5}}
\end{aligned}
$$

For the tangent line we also proceed as we did for example 1.

$$y - y_1 = m(x - x_1)$$
$$y - \sqrt{5} = \frac{1}{2\sqrt{5}}(x - 1)$$
$$y = \frac{1}{2\sqrt{5}}x + \frac{1}{2\sqrt{5}} + \sqrt{5}$$
$$y = \frac{1}{2\sqrt{5}}x + \frac{11}{2\sqrt{5}}$$



Example 3: Find the derivative of a generic linear function, $f(x) = mx + b$, and constant function, $g(x) = C$ at the point $x = a$.

Solution 3: Before we attempt to compute these derivatives, we recall the tangent line definition of the derivative from above: "*The derivative at a point $x = a$ can also be interpreted as the slope of the tangent line to the graph of $f(x)$ at the point, $P = (a, f(a))$*". Based on this definition we should see that the derivative of the linear function is the same for all x values, namely, m. Furthermore, as the graph of a constant function is a horizontal line the derivative is zero for all x. For illustrative purposes we prove that the derivative of a linear function is a constant value equal to the slope of the line, m.

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0}\left\{\frac{f(a+h) - f(a)}{h}\right\} \\
&= \lim_{h\to 0}\left\{\frac{m(a+h) + b - (ma+b)}{h}\right\} \\
&= \lim_{h\to 0}\left\{\frac{\cancel{ma} + m\cancel{h} + \cancel{b} - \cancel{ma} - \cancel{b}}{\cancel{h}}\right\} \\
&= \lim_{h\to 0}\{m\} \\
&= m
\end{aligned}
$$

The Derivative as a Function:

As we may have noticed, the derivative, $f'(a)$, will generally vary as a function of a. Therefore, if we consider a as an independent variable, e.g. x, then we can think of the derivative of a function, $f(x)$, resulting in a new function, $f'(x)$ or f'. This *function view* of the derivative will prove to be of much use as we continue our studies of calculus. With this in mind we can restate the definition of the derivative from above viewing the derivative as a function instead of a single value.

The Derivative as a Function
The derivative of a function, $f(x)$, with respect to x is another function, $f'(x)$, defined as: $$f'(x) = \lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} \right\}$$ The domain of $f'(x)$ consists of all values of x in the domain of $f(x)$ for which the limit above exists. We say that $f(x)$ is differentiable wherever $f'(x)$ exists.

As an illustration let's compute the derivative of $f(x) = x^2$, graph the function and its derivative, $f'(x)$, and then discuss some key observations.

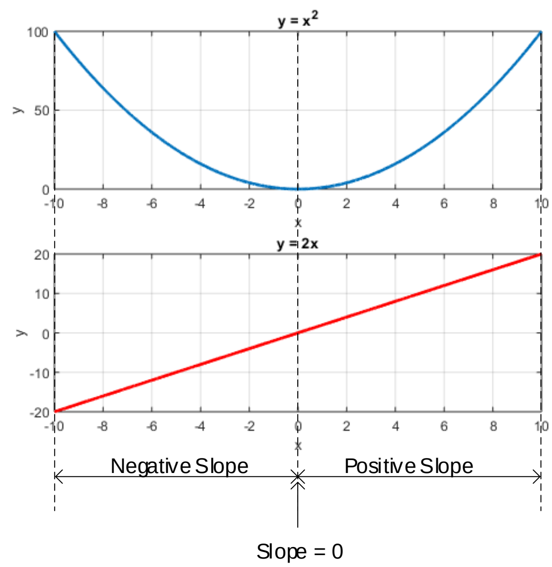
$$f'(x) = \lim_{h \to 0} \left\{ \frac{(x+h)^2 - x^2}{h} \right\}$$

$$f'(x) = \lim_{h \to 0} \left\{ \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} \right\}$$

$$f'(x) = \lim_{h \to 0} \left\{ \frac{\cancel{h}(2x+h)}{\cancel{h}} \right\}$$

$$f'(x) = \lim_{h \to 0} \{(2x+h)\}$$

$$f'(x) = 2x$$



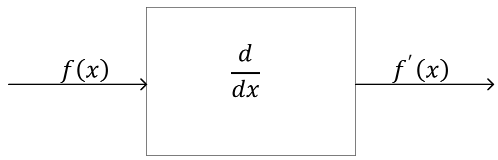
We see from the computation on the left that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. The graphs on the right-hand side also show the usefulness of obtaining the derivative as function. Based on the graph of $f(x)$ we can see that the slope is negative for $x < 0$, positive for $x > 0$, and exactly 0 and $x = 0$. This behavior is even more clearly shown in the graph of $f'(x)$. Being able to identify the intervals of negative and positive derivatives, as well as any points that have a zero slope, has many practical applications that we will introduce later in our studies.

Leibniz Notation for the Derivative:

The prime notation for the derivative, $f'(x)$, was introduced by the French mathematician Joseph Louis Lagrange. Even further if $y = f(x)$, we can write $y'(x)$ or y'. Another standard notation for the derivative was introduced by Leibniz and is shown below.

$$\frac{df(x)}{dx} \quad \text{or} \quad \frac{dy(x)}{dx}$$

Written in this form we can think of $\frac{d}{dx}$ as an operator that performs a function on $f(x)$ and outputs a new function, $f'(x)$.



To specify the value of the derivative for a fixed value of x, e.g. a, we write:

$$\left.\frac{df(x)}{dx}\right|_{x=a} \quad \text{or} \quad \left.\frac{dy(x)}{dx}\right|_{x=a}$$

It is also common to replace $f(x)$ with f and write.

$$\frac{df}{dx} \quad \text{or} \quad \frac{dy}{dx}$$

With this notation we can now think of dy and dx as infinitely small increments of y and x, just as Δy and Δx represent finite increments of y and x. This notion is illustrated below assuming again that $y = f(x)$.

$$\lim_{\Delta x \to 0} \left\{\frac{\Delta y}{\Delta x}\right\} = \frac{dy}{dx} = \lim_{\Delta x \to 0} \left\{\frac{f(x + \Delta x) - f(x)}{\Delta x}\right\}$$

As you may have already noticed from the few examples, the differentiation using the limit definition can be very time consuming. This is especially true for more complex functions. Fortunately, there are some basic rules of differentiation that can be established to help us compute derivatives without needing to explicitly evaluate the limit definition. We will begin to introduce these rules in the next section. We will end this introductory section with one final example.

Find the derivative of $f(x) = x^3 - 5x^2 + 6x$ and graph both $f(x)$ and $f'(x)$. Then compute $\left.\frac{df(x)}{dx}\right|_{x=a}$ for $a = 0.5, 1.5$.

$$f'(x) = \lim_{h\to 0}\left\{\frac{(x+h)^3 - 5(x+h)^2 + 6(x+h) \; - x^3 + 5x^2 - 6x}{h}\right\}$$

$$f'(x) = \lim_{h\to 0}\left\{\frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 \cancel{- 5x^2} - 10xh \; - 5h^2 \cancel{+ 6x} + 6h \cancel{- x^3} \cancel{+ 5x^2} \cancel{- 6x}}{h}\right\}$$

$$f'(x) = \lim_{h\to 0}\left\{\frac{\cancel{h}(3x^2 + 3xh + h^2 - 10x \; - 5h + 6)}{\cancel{h}}\right\}$$

$$f'(x) = \lim_{h\to 0}\{(3x^2 + 3xh + h^2 - 10x \; - 5h + 6)\}$$

$$f'(x) = (3x^2 + 3x\cdot 0 + 0^2 - 10x \; - 5\cdot 0 + 6)$$

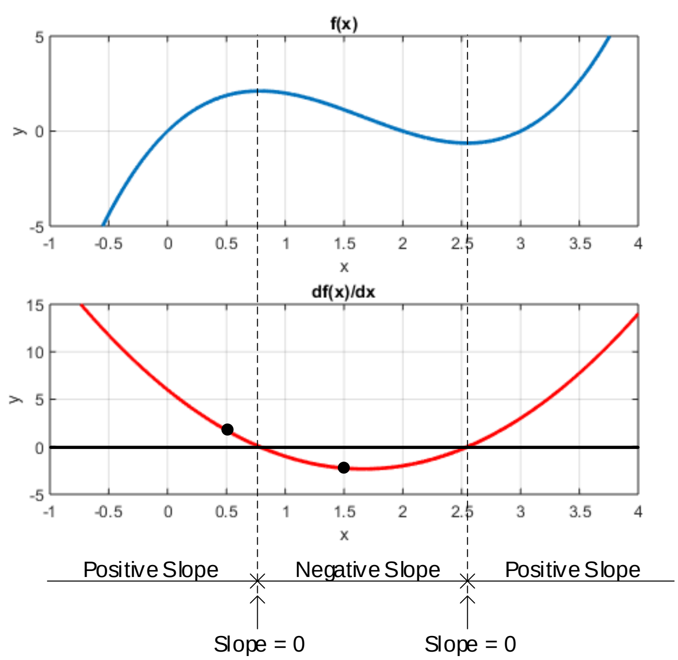
$$f'(x) = 3x^2 - 10x \; + 6$$

We can now compute the derivative for $x = 0.5, 1.5$.

$$\left.\frac{df(x)}{dx}\right|_{x=0.5} = 3(0.5)^2 - 10\cdot 0.5 \; + 6 \; = \; 1.75$$

$$\left.\frac{df(x)}{dx}\right|_{x=1.5} = 3(1.5)^2 - 10\cdot 1.5 \; + 6 \; = \; -2.25$$

Finally, the figures below show $f(x)$ and $f'(x)$.



Final Summary for Differentiation – Introduction to The Derivative

The Definitional of the Derivative

The derivative of a function, $f(x)$, with respect to x at a point $x = a$, which we refer to as $f'(a)$, is the limit of the difference quotient (if it exists):

$$f'(a) = \lim_{h \to 0} \left\{ \frac{f(a+h) - f(a)}{h} \right\}$$

If the limit exists, then we say the function is differentiable at the point $x = a$.

Note: The derivative at a point $x = a$ can also be interpreted as the slope of the tangent line to the graph of $f(x)$ at the point, $P = \big(a, f(a)\big)$. Using the point-slope formula, the tangent line can be expressed as:

$$y = f'(a)(x - a) + f(a)$$

The Derivative as a Function

The derivative of a function, $f(x)$, with respect to x is another function, $f'(x)$, defined as:

$$f'(x) = \lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} \right\}$$

The domain of $f'(x)$ consists of all values of x in the domain of $f(x)$ for which the limit above exists. We say that $f(x)$ is differentiable for wherever $f'(x)$ exists.

Various Notations for the Derivative

Assuming $y = f(x)$, notations for the derivative may be indicated as follows:

Lagrange Notation

$f'(x)$, $\quad f'$, $\quad y'(x)$, $\quad f'$

Leibniz Notation

$\dfrac{df(x)}{dx}$, $\quad \dfrac{df}{dx}$, $\quad \dfrac{dy(x)}{dx}$, $\quad \dfrac{dy}{dx}$

And to specify the value of the derivative for a fixed value of x, e.g. a:

Lagrange Notation

$f'(a)$, $\quad y'(a)$

Leibniz Notation

$\left.\dfrac{df(x)}{dx}\right|_{x=a}$ $\quad \left.\dfrac{dy(x)}{dx}\right|_{x=a}$

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