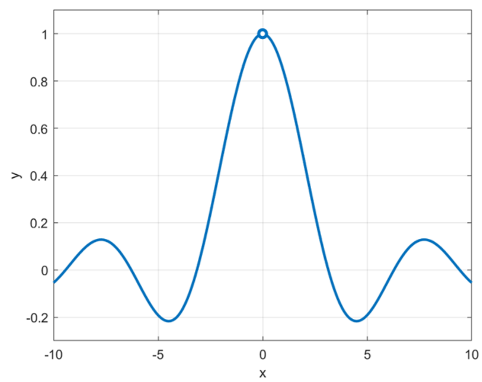
Limits – Numerically and Graphical Evaluation

In this section we formally introduce limits and study both a numerical and graphical method for evaluating limits. In a later section we show how some limits can also be solved using algebraic techniques, however the numerical and graphical techniques provide much more useful insight when first studying limits.

We start our investigation of a limit with the following function.

$$f(x) = \frac{\sin(x)}{x}$$



Note this function is undefined at $x = 0$ since $f(x) = \frac{\sin(0)}{0} = \frac{0}{0}$. However, using the concept of a limit, we can define $f(x)$ for values of x arbitrarily close to 0. The table below lists values of $f(x)$ as x approaches zero from both the left and the right.

x	$f(x)$	x	$f(x)$
1	0.8414709848	-1	0.8414709848
0.5	0.**9**588510772	-0.5	0.**9**588510772
0.1	0.**99**83341665	-0.1	0.**99**83341665
0.01	0.**9999**833334	-0.01	0.**9999**833334
0.001	0.**999999**8333	-0.001	0.**999999**8333

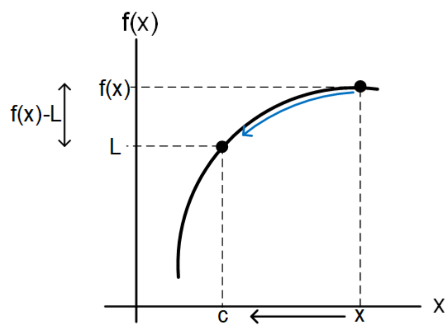
The table shows that as x approaches 0 from either side, $f(x)$ approaches the value of 1.0. More formally we say: “The limit of $f(x)$ as x approaches 0 is equal to 1.0”, and we write.

$$\lim_{x\to 0}\left\{\frac{\sin(x)}{x}\right\} = 1$$

This is an example of how to solve a limit problem using a numerical method. Before doing more examples let’s give a proper definition of a limit.

Definition of a Limit:
If $f(x)$ is defined for all x in an open interval containing c, but not necessarily defined at c, we can say that: The limit of $f(x)$ as x approaches c is equal to the number L, if $\lvert f(x) - L \rvert$ can be made arbitrarily small by taking x sufficiently close (but not equal) to c. In this case we can write: $$\lim_{x \to c}\{f(x)\} = L$$

Note that the absolute value used in the above definition ensures that the same value of L holds as x approaches c from either the left or right side. The figure below illustrates the above definition when x approaches c from the right side. A similar picture holds when x approaches c from the left side.



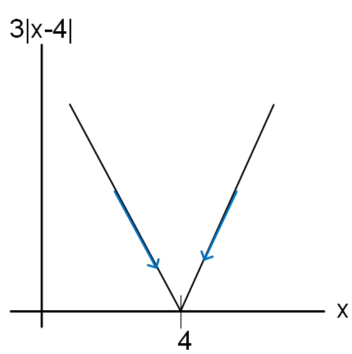
Before returning to additional numerical examples, let's use the definition from above to prove the following limit.

$$\underset{x \to 4}{lim}\{f(x)\} = 13, \qquad where\ f(x) = 3x + 1$$

According to the definition above we should evaluate $\lvert f(x) - 13 \rvert$ and try to determine if we can make this quantity arbitrarily small by taking x as close as possible to 4.

$$\lvert f(x) - 13 \rvert = \lvert 3x + 1 - 13 \rvert = \lvert 3x - 12 \rvert = 3\lvert x - 4 \rvert$$

The figure below is a graph of $3\lvert x - 4 \rvert$, which shows that the closer x gets to 4 from both the left or right, the closer the function value gets to zero. Therefore, $\lvert f(x) - 13 \rvert$ does indeed become arbitrarily small by taking x sufficiently close to 4, and the limit is thus proven.



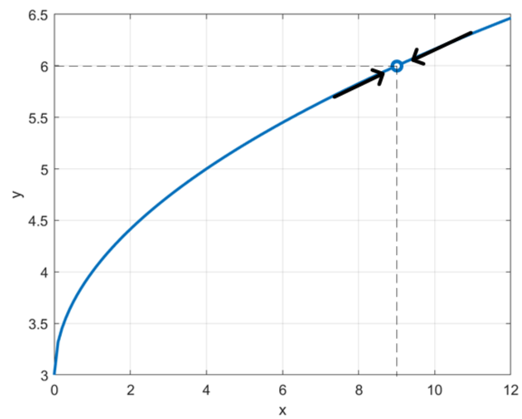
Let's now return to the numerical technique and attempt to find the following two limits.

a.) $\lim_{x\to 9}\left\{\frac{x-9}{\sqrt{x}-3}\right\}$ b.) $\lim_{x\to 4}\{x^2\}$

a.) If we try to evaluate the function at $x = 9$, the result is the undefined expression $0/0$, which is formally called an "indeterminant form". Therefore, we attempt to evaluate the function numerically to determine the limit. As you can see from both the table and the plot below, the function value approaches 6 as x approaches 9 from both the left and right. In this case, we write

$$\lim_{x\to 9}\left\{\frac{x-9}{\sqrt{x}-3}\right\} = 6$$

x from left	$f(x)$	x from right	$f(x)$
8.9	**5.9**8329	9.1	**6.0**1662
8.99	**5.99**833	9.01	**6.00**1666
8.999	**5.999**83	9.001	**6.000**167
8.9999	**5.9999**833	9.0001	**6.0000**167



b.) In this case evaluating the function at $x = 4$ results in a defined value, $4^2 = 16$. We can also evaluate this function numerically to make sure that it approaches 16 from both the left and the right. However, we should recognize that this function is continuous and therefore it's obvious that this requirement holds. The conclusion we can draw is that for continuous functions, which we will formally define later, the limit is equal to the function value.

$$\lim_{x\to 4}\{x^2\} = 4^2 = 16$$

One-Sided Limits

The definition we gave for the limit above requires that the function approaches L when x approaches c from both the left side and the right side. In some cases, however, $f(x)$ may approach L from one side but not the other. For these cases we define the following one-sided limits.

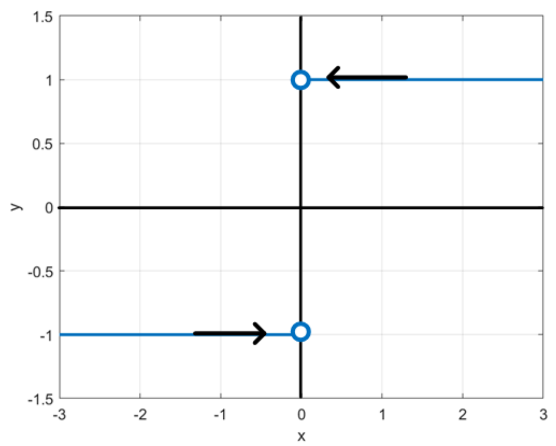
Left-sided limit **Right-sided limit**

$\lim_{x\to c^-}\{f(x)\}$ $\lim_{x\to c^+}\{f(x)\}$

And the limit itself exists when both one-sided limits exists and are equal.

As an example, let's evaluate the one-sided limit of $f(x) = \frac{x}{|x|}$ as x approaches 0.

We can numerically evaluate this limit, but by graphing the function we can more conveniently see exactly how the function behaves near $x = 0$. The figure and resulting analysis is shown below.



Right-Sided Limit

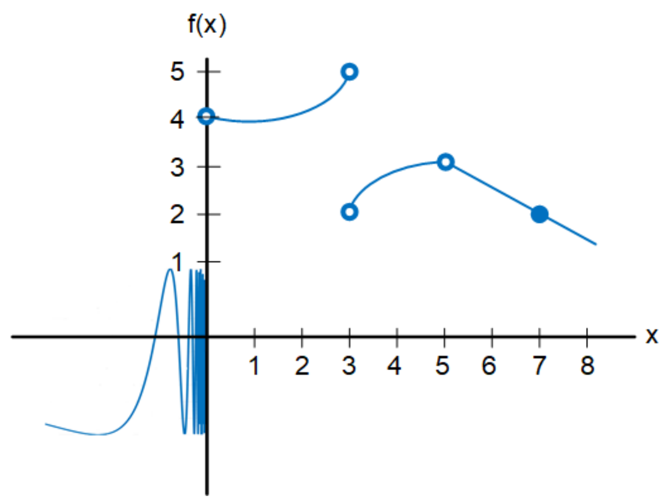
$$\lim_{x\to 0^+} \left\{\frac{x}{|x|}\right\} = \lim_{x\to 0^+} \left\{\frac{x}{x}\right\} = \lim_{x\to 0^+} \{1\} = +1$$

Left-Sided Limit

$$\lim_{x\to 0^-} \left\{\frac{x}{|x|}\right\} = \lim_{x\to 0^-} \left\{\frac{-x}{x}\right\} = \lim_{x\to 0^-} \{-1\}$$
$$= -1$$

And since these limits are not equal, the $\lim_{x\to 0} \left\{\frac{x}{|x|}\right\}$ does not exist, i.e. $\lim_{x\to 0} \left\{\frac{x}{|x|}\right\} = DNE$.

Let's take another example using the piecewise function below and evaluate the limit at the points $c = 0, 3, 5, 7$.



- $c = 0$: The left-sided limit does not exist since the function seems to oscillate to the left of $x = 0$, however the right-sided limit is equal to 4. Finally, since the left-sided limit does not exist the two-sided limit does not exist either.

$\lim_{x\to 0^-} \{f(x)\} = DNE$	$\lim_{x\to 0^+} \{f(x)\} = 4$	$\lim_{x\to 0} \{f(x)\} = DNE$

- $c = 3$: The left-sided limit is equal to 5 and the right-sided limit is equal to 2. In this case, even though both one-sided limits exist they are not equal, and so the two-sided limit does not exist.

$\lim_{x \to 3^-} \{f(x)\} = 5$	$\lim_{x \to 3^+} \{f(x)\} = 2$	$\lim_{x \to 3} \{f(x)\} = DNE$

- $c = 5$: The left-sided limit and the right-sided limit are both equal to 3. In this case, since both one-sided limits exist, *and* they are equal, the two-sided limit exists and is equal to the one-sided limit. However, note that the function value at $x = 5$ does not exist, $f(5) = DNE$.

$\lim_{x \to 5^-} \{f(x)\} = 3$	$\lim_{x \to 5^+} \{f(x)\} = 3$	$\lim_{x \to 5} \{f(x)\} = 3$

- $c = 7$: In this case the function is continuous at $x = 7$, and therefore the left-sided limit, right-sided limit, and the two-sided limit are all equal to $f(7) = 2$.

$\lim_{x \to 7^-} \{f(x)\} = f(7) = 2$	$\lim_{x \to 7^+} \{f(x)\} = f(7) = 2$	$\lim_{x \to 7} \{f(x)\} = f(7) = 2$

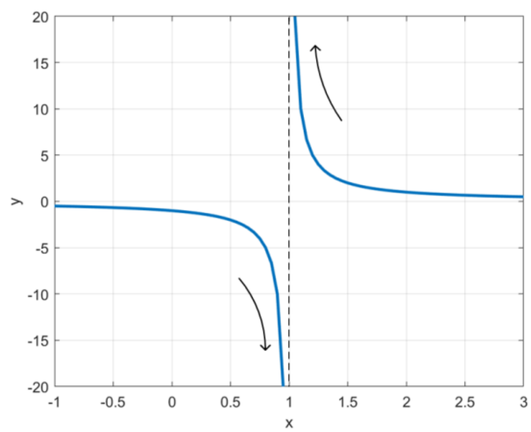
Infinite Limits

Some functions tend to ∞ or $-\infty$ as x approaches a value, c. We can also say that these limits do not exist, however, to distinguish them from the cases above we can say that these functions have an infinite limit. For example, we can write the following.

- $\lim_{x \to c} \{f(x)\} = \infty$ if $f(x)$ increases without bound as x approaches c.
- $\lim_{x \to c} \{f(x)\} = -\infty$ if $f(x)$ decreases without bound as x approaches c.

The following 3 examples illustrate infinite limits.

$$f(x) = \frac{1}{x-1}$$



Left-Sided Limit

$$\lim_{x \to 1^-} \left\{ \frac{1}{x-1} \right\} = -\infty$$

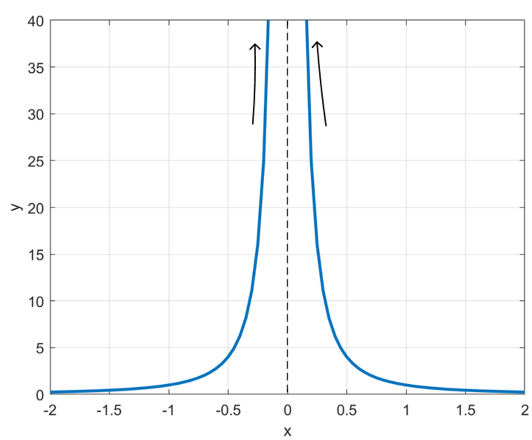
Right-Sided Limit

$$\lim_{x \to 1^+} \left\{ \frac{1}{x-1} \right\} = \infty$$

Two-Sided Limit

$$\lim_{x \to 1} \left\{ \frac{1}{x-1} \right\} = DNE$$

$$f(x) = \frac{1}{x^2}$$



Left-Sided Limit

$$\lim_{x \to 0^-} \left\{ \frac{1}{x^2} \right\} = \infty$$

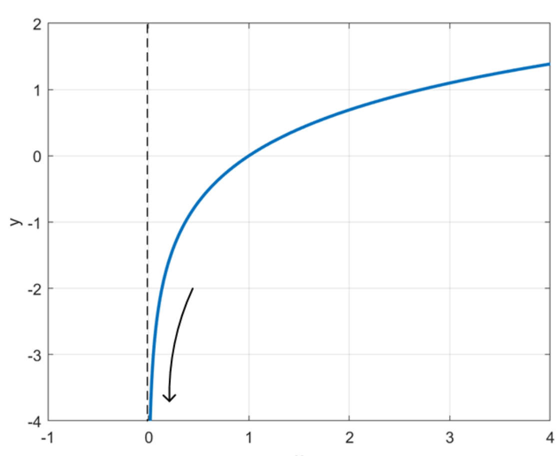
Right-Sided Limit

$$\lim_{x \to 0^+} \left\{ \frac{1}{x^2} \right\} = \infty$$

Two-Sided Limit

$$\lim_{x \to 0} \left\{ \frac{1}{x^2} \right\} = \infty$$

$$f(x) = \ln(x)$$



Left-Sided Limit

$$\lim_{x \to 0^-} \{\ln(x)\} = DNE$$

Right-Sided Limit

$$\lim_{x \to 0^+} \{\ln(x)\} = -\infty$$

Two-Sided Limit

$$\lim_{x \to 0} \{\ln(x)\} = DNE$$

Final Summary for Limits – Numerical and Graphically Evaluation

Definition of a Limit

If $f(x)$ is defined for all x in an open interval containing c, but not necessarily defined at c, we can say that:

The limit of $f(x)$ as x approaches c is equal to the number L if

$|f(x) - L|$ can be made arbitrarily small by taking x sufficiently close (but not equal) to c. In this case we can write:

$$\lim_{x \to c}\{f(x)\} = L$$

$x \to c$

And we can say: "The limit of $f(x)$ as x approaches c is equal to L"

One-Sided Limits

Left-Sided Limit:

The limit of $f(x)$ as x approaches c from values less than c is equal to L.

$$\lim_{x \to c^-}\{f(x)\} = L$$

Right-Sided Limit:

The limit of $f(x)$ as x approaches c from values greater than c is equal to L.

$$\lim_{x \to c^+}\{f(x)\} = L$$

- The limit itself exists only when both one-sided limits exists and are equal.

$$\lim_{x \to c}\{f(x)\} = L \text{ if and only if } \lim_{x \to c^-}\{f(x)\} = \lim_{x \to c^+}\{f(x)\} = L$$

Infinite Limits

- $\lim_{x \to c}\{f(x)\} = \infty$ if $f(x)$ increases without bound as x approaches c.
- $\lim_{x \to c}\{f(x)\} = -\infty$ if $f(x)$ decreases without bound as x approaches c.

Note: The same one-sided limit rules apply to infinite limits.

By: ferrantetutoring