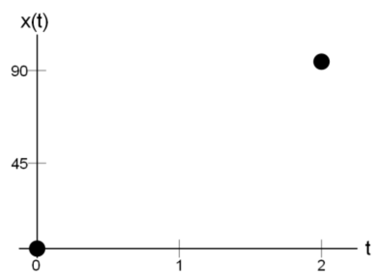
Limits- Motivation

The concept of a limit is at the heart of calculus. Before formally introducing limits, let's begin with a practical example that will illustrate why the study of limits is needed in the first place.

Rates of Change

A rate of change is a measure of how one quantity changes with respect to another. A practical example is *velocity*. When measuring velocity, we would like to know how the position, x, is changing with respect to time, t. Let's say you drove on a straight road for 2 hours for 90 miles.

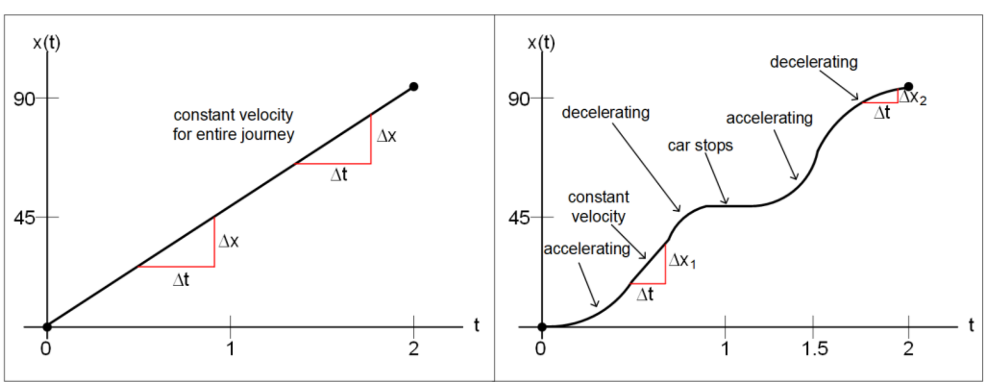
Intuitively we know that the velocity is equal to the change in position divided by the change in time, i.e. 60 miles/hour. For the problem above we can plot the two points we know on a position versus time graph as shown below.



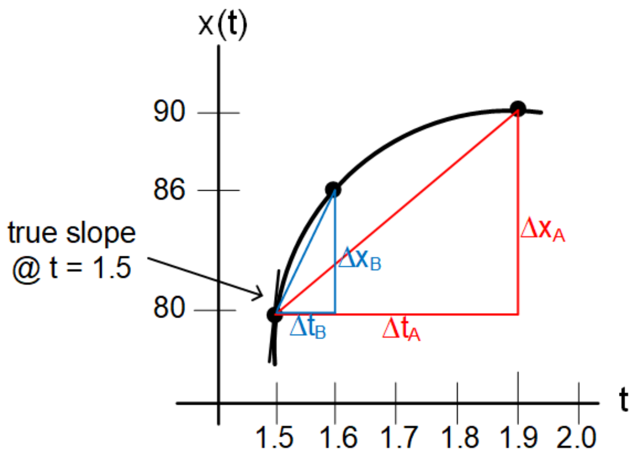
In this case it seems simple enough to compute the velocity as follows:

$$v = \frac{\Delta x}{\Delta t} = \frac{x(2) - x(0)}{2 - 0} = \frac{90 - 0}{2} = 45\, miles/hr$$

This leads us to believe that the car drove at constant velocity of 45 mph for the entire two-hour drive, which we know is not realistic. More than likely the car accelerated, decelerated, and probably came to complete stops at some instances of time. The two plots below highlight this distinction. The first plot shows the more unrealistic case where the car traveled at a constant velocity for the entire time, while the second plot shows a more realistic case.



The first plot highlights the fact that the velocity is equivalent to the slope of the position versus time curve and that no matter where we measure the slope, i.e. velocity, we get a constant value. Measuring the slope in the second plot, however, will results in different answers depending on where in time it is measured. For the second plot let's see how we might compute the velocity at $t = 1.5$. The figure below zooms in between $t = 1.5$ and $t = 2$.



As a first estimate of the velocity at $t = 1.5$ we may use the times 1.5 and 2 to find the slope.

$$v_A = \frac{\Delta x_A}{\Delta t_A} = \frac{x(t_2) - x(t_2)}{t_2 - t_1} = \frac{90 - 80}{2 - 1.5} = 20\ miles/hour$$

As you can see this slope, (shown by the red line), seems quite different from what we expect the slope to be, (shown by the black line) at $t = 1.5$. To make this estimate better we allow Δt_A to shrink to Δt_B, as shown. The velocity is now computed as follows:

$$v_B = \frac{\Delta x_B}{\Delta t_B} = \frac{86 - 80}{1.6 - 1.5} = 40\ miles/hour$$

This slope, (shown by the blue line), looks much closer to the "true" value, however as we see it's still not quite right.

What these values above are computing is referred to as the *average velocity*. It is the constant velocity value we would need to travel over the measured time interval to reach the desired destination. However, it is not the only way we could have arrived at the destination. Instead of the average velocity, we are attempting to measure the instantaneous velocity, which is the velocity at a particular instant in time. To find this value it seems we need to shrink the value of Δt as much as possible. Of course, if we make Δt strictly zero the result would be an undefined value since Δx would also be zero and we would end up with $v = 0/0$. So how exactly do we allow Δt to get as close to zero as possible and still end up with a defined velocity. The answer is we use the concept of a limit! The velocity at $t = 1.5$ is expressed in two different ways using the notion of a limit.

The velocity is equal to $\frac{\Delta x}{\Delta t}$ in the limit as Δt approaches 0	***The velocity is equal to $\frac{x(t_2)-x(t_1)}{t_2-t_1}$ in the limit as t_2 approaches t_1***
$v = \lim_{\Delta t \to 0} \left\{ \frac{\Delta x}{\Delta t} \right\}$	$v = \lim_{t_2 \to t_1} \left\{ \frac{x(t_2) - x(t_1)}{t_2 - t_1} \right\}$

Although we don't yet know how to compute these limits, with this simple example we hopefully understand the importance of limits in describing various real-world scenarios.

Final Summary for Limits- Motivation

Average -vs- Instantaneous Velocity
The average velocity over a time interval $[t_1, t_2]$: $$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{x(t_2) - x(t_1)}{t_2 - t_1}$$ The instantons velocity at a time, t_1 can be defined in terms of a limit and written in two different forms: $$v = \lim_{\Delta t \to 0} \left\{ \frac{\Delta x}{\Delta t} \right\} \qquad v = \lim_{t_2 \to t_1} \left\{ \frac{x(t_2) - x(t_1)}{t_2 - t_1} \right\}$$

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