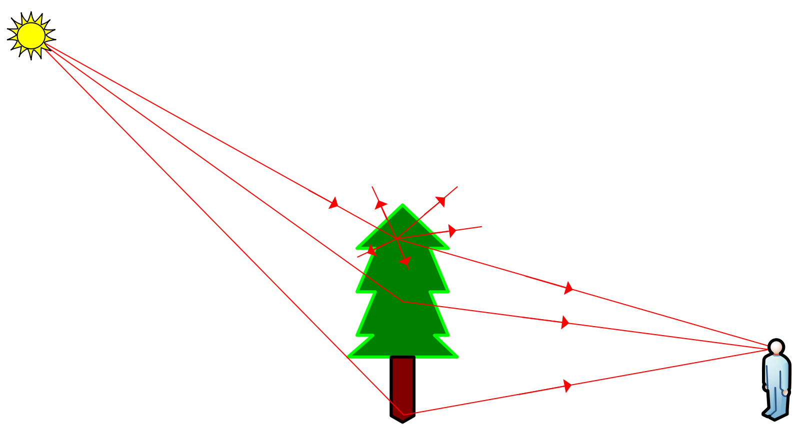
Geometric Optics – Mirrors

Light is a form of electromagnetic radiation, which according to Maxwell's equations, can be described as a wave. Modeling light as a wave enables us to better understand certain aspects of light, such as interference and diffraction. However, other aspects of light, such as reflection, refraction, and the formation of images from mirrors and lenses, can be much more easily understood using a so-called ray model of light. A ray is an idealization that is meant to represents an extremely narrow beam of light. The ray model of light assumes light travels in straight line paths called rays. As you will see, this model aligns very well with our intuitive understanding of light. According to the ray model, an object that does not emit light is 'seen' because of light rays emitted from a source, e.g. the sun. These light rays strike each and every point on the object before reaching our eyes. We then subconsciously process these rays to recreate an image of the object.

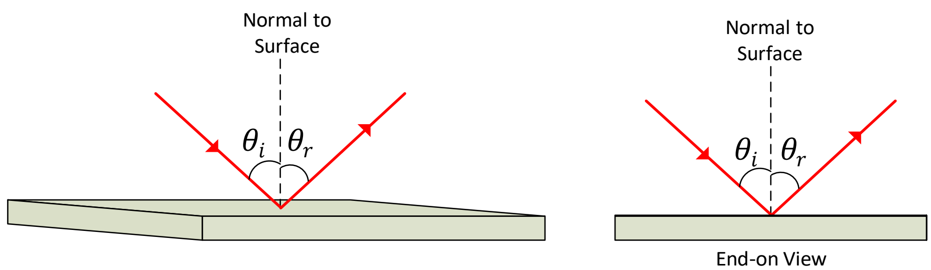


As shown in the figure, although light rays leave each point in many directions, we can assume a single ray reaches our eyes from each point on the object. The study of sight and the behavior of light using a ray model is referred to as *geometric optics*. We begin our study of geometric optics looking at images formed by mirrors. A mirror is a flat surface that reflects most of the light rays that are incident on that surface. Any object that reflects a sizable portion of incident light rays can be considered a mirror, e.g. water. However, more commonly we refer to a mirror as a specifically manufactured glass that is coated with a thin layer of metal such that it reflects nearly 100% of the incident light rays. Before diving into studying exactly how images are formed using mirrors, we need to understand how light rays are reflected from surfaces.

Reflection:

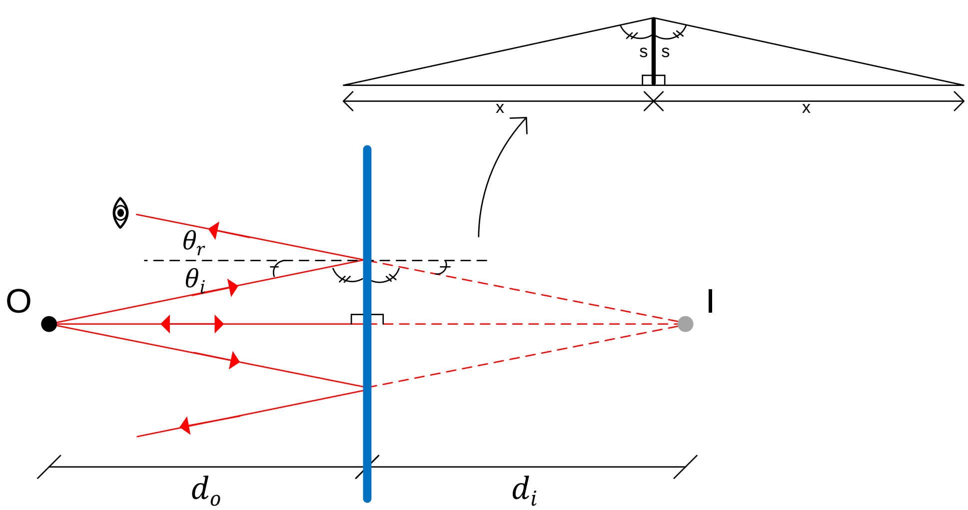
When a narrow beam of light strikes a highly reflective flat surface the beam will reflect from that surface at an angle, called the angle of reflection, that is directly related to the angle for which the beam struck the surface, called the incident angle. Both angles are measured with respect to the normal to the surface. The angles are related through the so called *Law of Reflection*.

Law of Reflection
When a narrow beam of light, e.g. a light ray, strikes a flat and reflective surface the following applies. **The angle of incidence, θ_i, equals the angle of reflection, θ_r.**



Plane Mirrors:

For our purposes we define a mirror as a perfectly smooth surface that reflects all incident rays according to the law of reflection. Furthermore, a plane mirror is one that is perfectly flat. As mentioned, an object that does not emit light is 'seen' because of light rays emitted from a source, e.g. the sun, striking each and every point on the object before reaching our eyes. In the case where we are viewing the object through a mirror, the rays are first reflected from the surface of the mirror before reaching our eyes. Our eyes, along with our brains, then attempt to recreate an image of the object. To see how this process works let's imagine a point object, O, placed a distance, d_o, from a plane mirror, as shown below.



The figure shows three specific light rays that are bounced from the object and incident on the mirror. The first ray is incident at 90° and therefore reflects straight back to the object. The second ray reaches the mirror at a point above the first ray and is incident at an angle, θ_i. This ray is shown to reflect from the mirror with a reflection angle, $\theta_r = \theta_i$. The reflected ray enters the eye of an observer. The observer, assuming that rays travel in straight lines, traces the ray back along the dotted line as shown. All three rays are shown traced back in a similar manner. The point at which these rays converge is where the observer will form an image of the object as shown. Using some simple geometry, we can prove the triangles are similar so that the lengths are the same. Using the convention that distances on the side of the mirror where the object is placed are positive and are negative on the opposite, we have

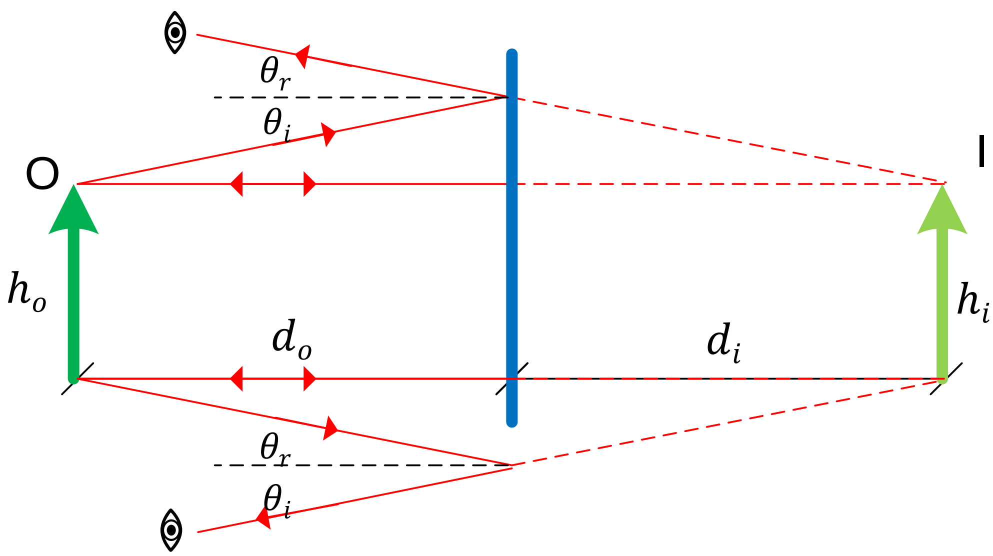
$$d_i = -d_o$$

Where d_i is referred to as the image distance.

Finally, since the rays do not actually pass through the image, we refer to this as a *virtual image*. When the rays do pass through the image, which we will see is sometimes the case for curved mirrors, we call these *real images*. Real images would appear on paper or film if placed at the image location, whereas virtual images would not.

Extended Objects:

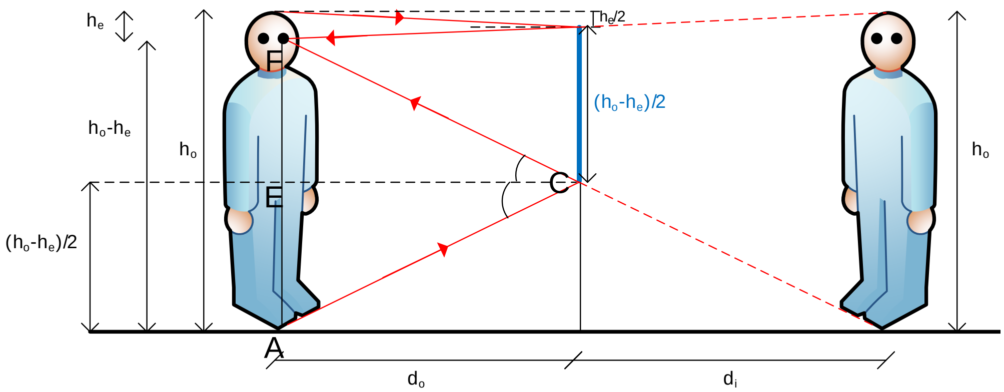
An extended object is represented by an upright arrow as shown in the figure below.



Each small portion of the object that faces the mirror acts like a point source. Tracing each of these points, as we did with the single point source above, we can perceive the entire image. Doing so for the top and bottom portion only we see that the image formed is the same height as the object, $h_i = h_o$.

Example 1: What mirror length is required for a person to see their entire body?

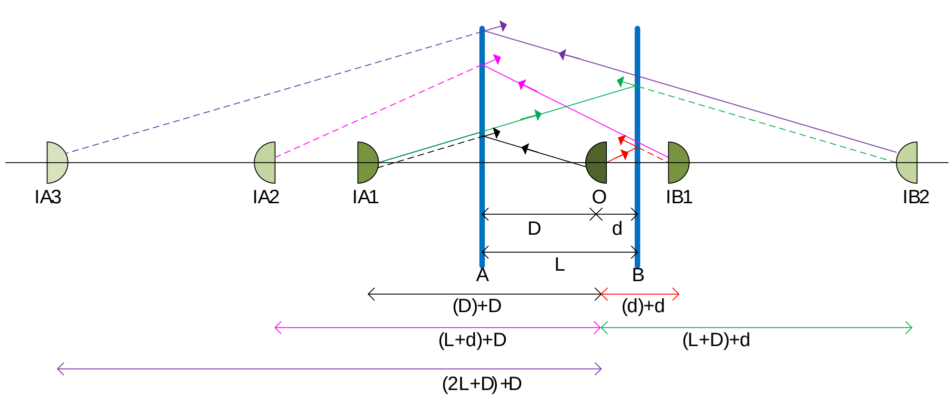
Solution: The figure below shows a person of height, h_o, with eyes that are a distance h_e from the top of their head.



The first ray we draw leaves the foot of the person, reflects off the mirror and impinges on the eye of the observer. Since the angle of incidence is equal to the angle of reflection the triangle AEC is congruent to triangle FEC. Therefore, the ray reaches the mirror at a height of $h_o - h_e/2$. Next, we draw a ray that leaves the top of the persons head, reflects from the mirror, and impinges on the eye of the observer. Using the same argument with congruent triangles we see the ray hits the mirror $h_e/2$ below the persons head. With this we see that the length of mirror required is $h_o - h_e/2$. Neglecting the small distance, h_e, we can say in general that a full length mirror need only be half the size of the person for them to see their entire image. Note that this result doesn't depend on the distance the person is from the mirror.

Example 2: In a mirror maze each wall is covered with mirrors. The result is a confusing montage of reflections that make it fun, (and sometimes frustrating), to walk through. In a mirror maze the image formed from one mirror is used as the object of the next, and so on. To illustrate this, take the case of two mirrors facing each other at $L = 2.0\ m$ apart. You stand at $D = 1.5\ m$ away from one of these mirrors and look into it. How far away do the first three images of yourself appear in the mirror in front of you? Which way are these images oriented?

Solution: The figure below illustrates the scenario.



The first set of images are directly created from the person, labeled O.

Image $IA1$ created from O:

- The distance from O to mirror A is D. Therefore, the image is located $D + D$ meters in front of O and is oriented to the right.

Image $IB1$ created from O:

- The distance from O to mirror B is d. Therefore, the image is located $d + d$ meters behind O and is oriented to the right.

The second set of images are created from the first set of images.

Image $IA2$ created from image $IB1$:

- The distance from $IB1$ to mirror A is $L + d$. Therefore, $IA2$ is located $(L + d) + D$ meters in front of O and is oriented to the left.

Image $IB2$ created from image $IA1$:

- The distance from $IA1$ to mirror B is $L + D$. Therefore, $IB2$ is located $(L + D) + d$ meters in behind O oriented to the left.

The third image seen in mirror A is created from the second image in mirror B.

Image $IA3$ created from image $IB2$:

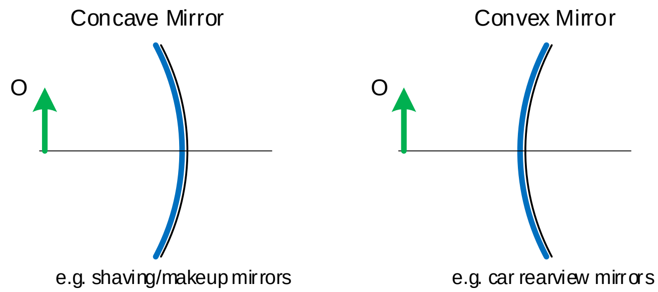
- The distance from $IB2$ to mirror A is $2L + D$. Therefore, $IA3$ is located $(2L + D) + D$ meters in front of O and is oriented to the right.

Using the values from D, d, and L from above the first three images in mirror A are at the following distances and orientation from the person, O.

Image in Mirror $\boldsymbol{A}$	**Distance from $\boldsymbol{O}$**	**Orientation**
$IA1$	$D + D = 2D = 2 \cdot 1.5 = \mathbf{3\ m}$	Right
$IA2$	$(L + d) + D = (2 + 0.5) + 1.5 = \mathbf{4\ m}$	Left
$IA3$	$(2L + D) + D = 2L + 2D = 2 \cdot 2 + 2 \cdot 1.5 = \mathbf{7\ m}$	Right

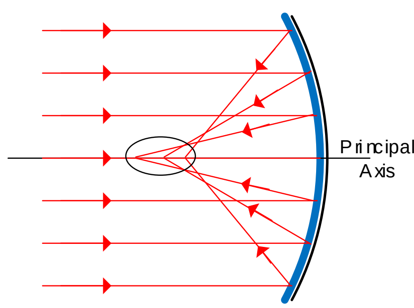
Curved Mirrors:

The most common curved mirrors are spherical, which means they from a section of a sphere. If the reflection takes place on surface considered the inside of the sphere the mirror is called concave. If the reflection takes place on the outside surface the mirror is called convex.

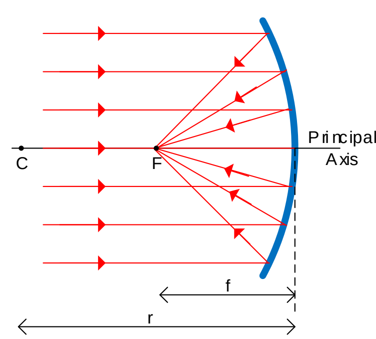


Concave Mirrors:

The rays that arrive at a concave mirror from an object that is very far away are very nearly parallel. Using a concave mirror and the law of reflection for each incident ray we can argue that each ray will reflect in nearly the same spot on what we refer to as the principal axis as shown in the figure below.



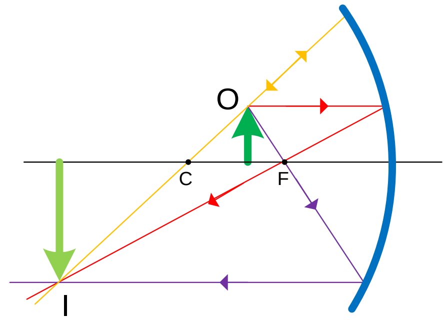
In order to create a sharp image all parallel rays should come to a single point. If the mirror is small compared to its radius of curvature it can be approximate that all parallel rays fall on a single point, called the focus as shown in the figure below.



r is the radius of the sphere if complete, i.e. curvature.
f is the focal length and is equal to $r/2$.
F is the focal point. It is the image point for an object that is infinitely far away along the principal axis.

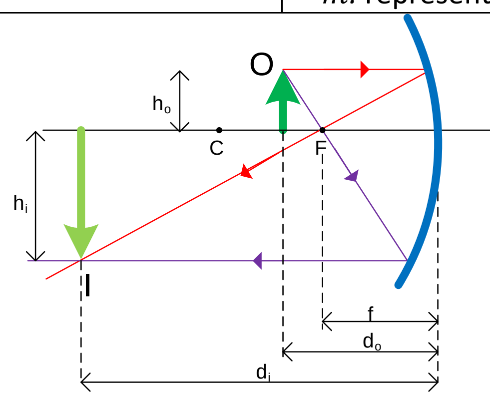
The figure above tells us that for an object that is infinitely far away the image will appear, (actually it will have no height so not seen), at the focal point. To illustrate with objects that are not infinitely far away we place an object between C and F. To find the image location we can draw several rays from the object and use the law of reflection to see where they intersect. However, to simplify the process, we can use the following three rays:

- Ray 1: Is drawn parallel to the axis and therefore passes through F upon reflection.
- Ray 2: Is drawn passing through F and therefore is parallel to the axis upon reflection.
- Ray 3: Is drawn passing through C and therefore reflects back along the same path.



Note: Only two of these rays are actually required but the third can serve as a check.

The image formed is inverted, larger, and a real image. Objects can be placed at different locations along the principle axis and we can draw the rays to roughly determine the size and location of the resulting image. Unfortunately, high accuracy is difficult to achieve because of the small angles required. Fortunately, geometry can be used to derive equations that give the distance, orientation, and size, of the image. They are provided below without proof.

<table>
<tr><th colspan="2">Mirror Equations</th></tr>
<tr><td>$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$$</td><td>$$m = \frac{h_i}{h_0} = -\frac{d_i}{d_0}$$
m: represents the lateral magnification</td></tr>
<tr><td colspan="2">
</td></tr>
<tr><th colspan="2">Conventions:</th></tr>
<tr><td colspan="2">

- Object height, h_o, and image height, h_i, are positive if upright and negative if inverted.
- Object distance, d_o, the image distance, d_i, and the focal length, f, are positive if they are located on the side of where the light rays originate and negative otherwise.
- The magnification, m, is positive for an upright image and negative for an inverted image.
- The image is considered a *real image* if light rays pass through it and a *virtual image* is the light rays do not pass through it.

</td></tr>
</table>

Example 3: A 1.0 cm high object is placed 10 cm in front of a concave mirror whose radius of curvature is 30 cm.

a.) Determine the position and magnification of the image analytically.
b.) Draw a ray diagram to locate the approximate location and size of the image.

Solution: For part a. we directly use the mirror equations. The focal length is given by

$$f = \frac{r}{2} = \frac{30}{2} = 15\ cm$$

The image distance can then be found as follows:

$$\frac{1}{d_i} = \frac{1}{f} - \frac{1}{d_o}$$

$$d_i = \left(\frac{1}{15} - \frac{1}{10}\right)^{-1} = -30\ cm$$

Which indicates the image is on the non-reflecting side of the mirror and is virtual.

Next, the height of the image can be found using the magnification equation.

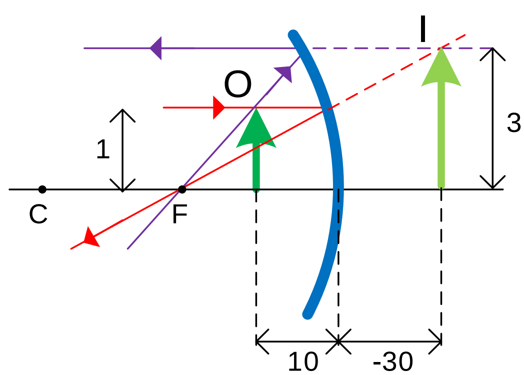
$$m = \frac{h_i}{h_0} = -\frac{d_i}{d_0}$$

$$h_i = -\frac{d_i}{d_0} h_0$$

$$h_i = -\frac{-30}{10} 1 = 3\ cm$$

Which indicates that the image is upright.

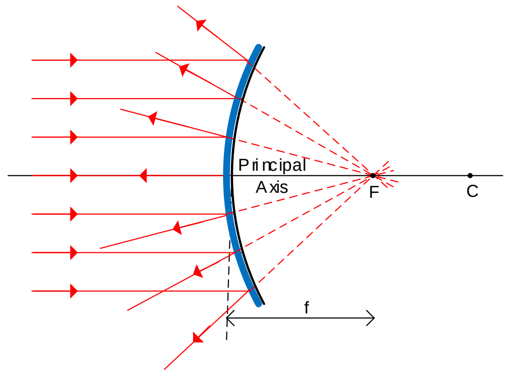
For part b. we sketch a ray diagram to illustrate this scenario.



Note, since the rays do not pass through the object the image is virtual. Furthermore, as you can see the diagram is not 100% accurate but does give the relative location and magnification of the image.

Convex Mirrors:

The ray tracing technique, as well as the mirror equations that were used for concave mirrors, can also be applied to convex mirrors. The main difference for convex mirrors is the focal point is behind the reflecting surface. Therefore, the focal length must be considered negative.



Example 4: A convex rearview mirror has a radius of convergence of 16 m. Determine the location and magnification of an object placed 10 m from the mirror.

a.) Determine the position and magnification of the image analytically.
b.) Draw a ray diagram to locate the approximate location and size of the image.

Solution: For part a. we directly use the mirror equations, taking care to treat the focal length as negative.

$$f = \frac{r}{2} = \frac{-16}{2} = -8\,m$$

The image distance can then be found as follows:

$$\frac{1}{d_i} = \frac{1}{f} - \frac{1}{d_o}$$

$$d_i = \left(\frac{1}{-8} - \frac{1}{10}\right)^{-1} = -\frac{40}{9}$$

$$d_i \cong -4.4\,m$$

Which indicates the image is on the non-reflecting side of the mirror and is virtual.

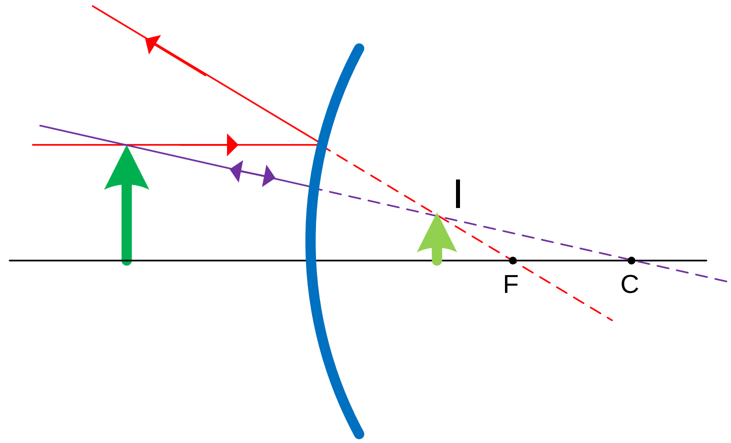
Next, the magnification is given as follows:

$$m = -\frac{\left(-\frac{40}{9}\right)}{10}$$

$$m = \frac{4}{9} \cong 0.44$$

Which indicates that the image is upright and reduced in size.

For part b. we sketch a ray diagram to illustrate this scenario.



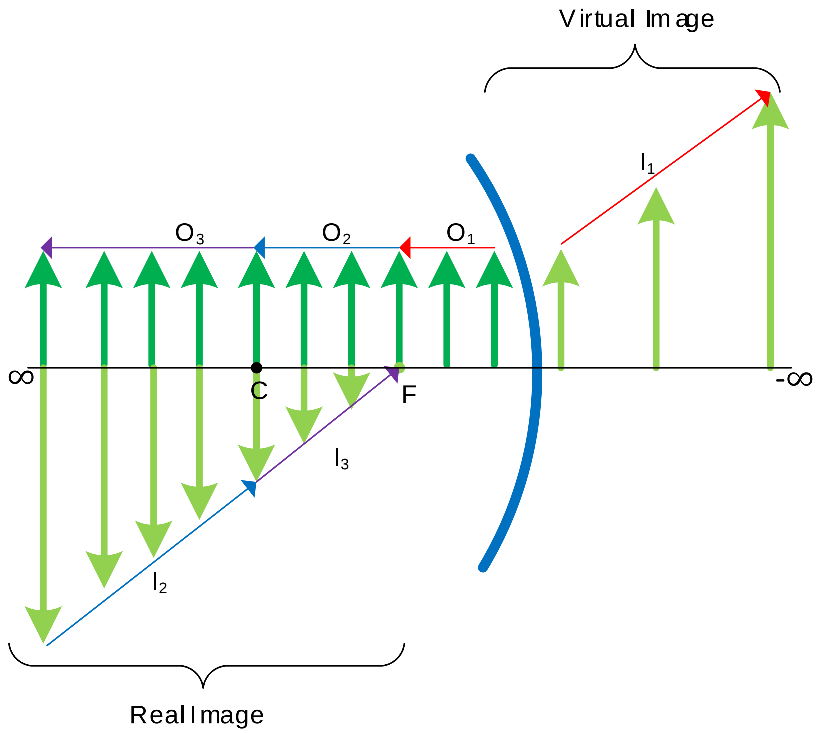
Note, since the rays do not pass through the object the image is virtual.

We see that as the location of the object varies the image location and size changes. Furthermore, as the image location switches mirror sides, assuming it does, it changes between a real and virtual image. The pattern can be investigated by drawing ray diagrams for different locations of the object or by rearranging the mirror equations to give new equations for d_i and m as functions of d_o, which can then be plotted for a given f. As an example, for the concave mirror we can write the following relationships.

$$d_i(d_o) = \frac{f d_o}{d_o - f} \qquad\qquad m(d_o) = -\frac{f}{d_o - f}$$

You can investigate these equations to verify they match with the figures we show below.

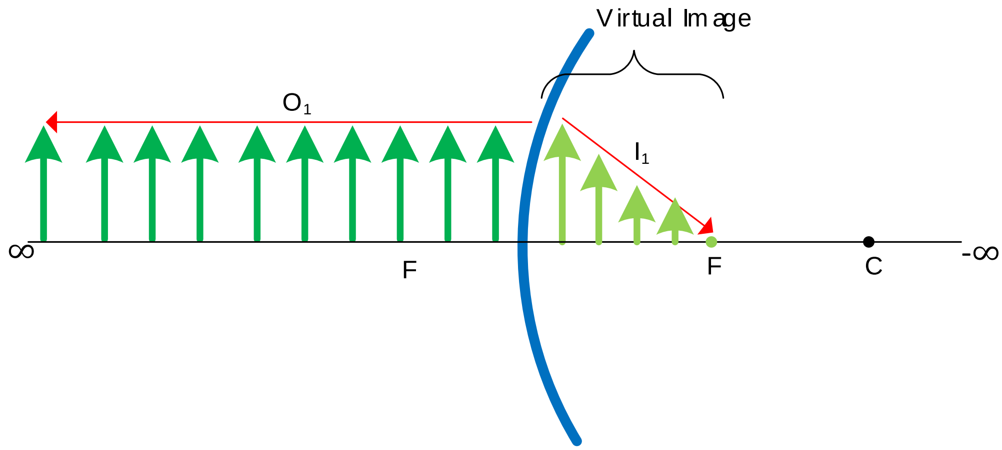
The first figure shows how the image varies as a function of the object distance for a concave mirror.



The illustration shows the object and image location moving over three regions which are explained below.

- $O_1, I_1, (\, 0 < d_o < f)$:
  - As the object moves away from the mirror towards the focal point, a *virtual* image is formed that moves further away from the mirror and gets larger.
- $(d_o = f)$:
  - As the object continues to move away from the mirror and approaches the focal point the virtual image approaches a distance of negative infinity with height approaching infinity. Note if the object were approaching the focal point from the other side an inverted *real* image would be approaching a distance of positive infinity with a height approaching infinity.
- $O_2, I_2, (\, f < d_o \leq c)$:
  - As the object moves from the focal point to the center point an inverted *real* image begins to move in from a distance of positive infinity and begins to decrease in size. When $d_o = c$ the inverted real image has a distance, $d_i = d_o$, and a height of $h_i = -h_o$.
- $O_3, I_3, (\, c < d_o < \infty)$:
  - As the object moves from the center point to a distance of positive infinity an inverted real image moves closer to the focal point and its size continues to diminish. Note as the object approaches positive infinity the image approaches zero in height at the focal point.

The next figure shows how the image varies as a function of the object distance for a convex mirror.



The illustration shows the object and image location moving over a single region which is explained below.

- $O_1, I_1, (\, 0 < d_o < \infty)$:
  - As the object moves away from the mirror towards positive infinity, a *virtual* image is formed that moves further away from the mirror and gets smaller. Note that the image starts with the same height as the object and approaches zero in height at the focal point when the object is at positive infinity.

Refraction:

In the above analysis we have assumed that all light rays incident on a surface are reflected from that surface, e.g. the surface of the mirror. In reality when light is incident on an object some of the light is reflected while some may pass through the object. Focusing instead on the portion of the light that is not reflected we can say that when a light ray passes from one medium, e.g. air, into another medium, e.g. water, the ray is bent. This bending of the light ray is called *refraction*. The amount of bending is dependent on what we refer to as the *index of refraction* of the medium. The index of refraction, n, in turn is related to the relative speed of light in the given medium.

$$n = \frac{c}{v}$$

Where c is the speed of light in a vacuum, $c \cong 3E8\ m/s$.

Since the speed, v, can never exceed c, the index of refraction will be a number greater than or equal to one, $n \geq 1$.

The relationship between the incident angle and the refracted angle is given by Snell's Law.

<table>
<tr><th colspan="2">Snell's Law of Refraction</th></tr>
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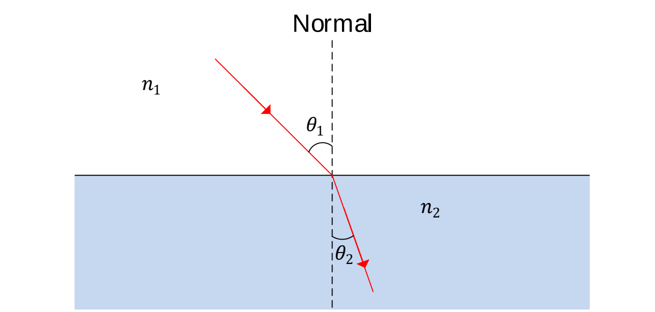
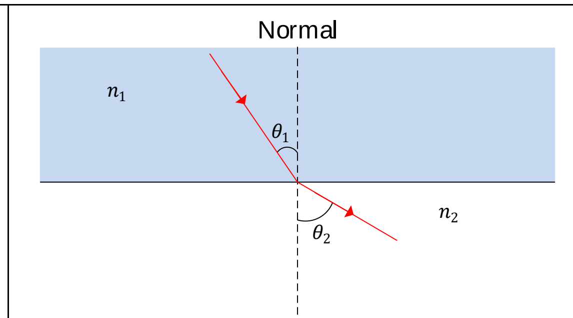
$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

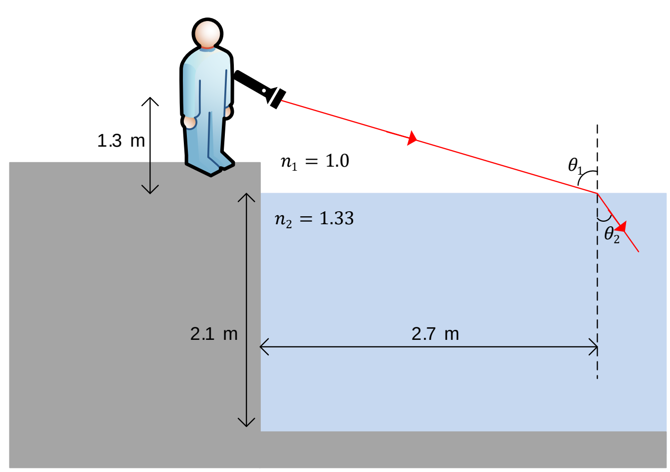
Where θ_1 is the angle if incidence and θ_2 is the angle of refraction; n_1 and n_2 are the respective indices of refraction of the materials.</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>$n_2 > n_1$ therefore, $\theta_1 > \theta_2$</td><td>$n_1 > n_2$ therefore, $\theta_2 > \theta_1$</td></tr>
</table>

The rules governing the direction of the refracted ray, (towards or away from the normal), shown above can be more easily seen by rewriting Snell's law as follows.

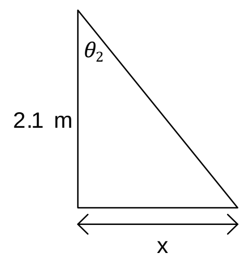
$$\frac{n_1}{n_2} = \frac{sin(\theta_2)}{sin(\theta_1)}$$

Since the sine function monotonically increases for $0 < \theta < 90^{\circ}$, when $n_1 > n_2$, θ_2 must be grater than θ_1. Similarly, when $n_2 > n_1$ we must have $\theta_2 > \theta_1$.

Example 5: A night watchman shines a narrow beam of light into a pool with a water depth of 2.1 m. The flashlight stands at 1.3 m above the water and the light beam hits the surface of the water at a point 2.7 m from the where he stands. Where does the spot of light hit the bottom of the pool, measured from the wall beneath his foot?



Solution: The horizontal distance the light travels is indicated by the right triangle formed from the point at which the beam strikes the surface of the water.

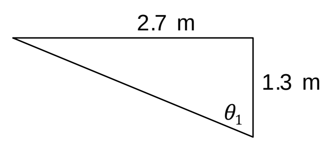


$$x = 2.1\, tan(\theta_2)$$

As we know, the refracting angle, θ_2, is related to the angle of incidence, θ_1, through Snell's law.

$$n_1\, sin(\theta_1) = n_2\, sin(\theta_2)$$
$$\theta_2 = sin^{-1}\left(\frac{n_1\, sin(\theta_1)}{n_2}\right)$$

The angle of incidence can be found using the right triangle formed similar to the triangle above.



In this case, we have

$$\theta_1 = tan^{-1}\left(\frac{2.7}{1.3}\right) \cong 64.3^{\circ}$$

Using this result and the two equations from above we can solve for x as follows.

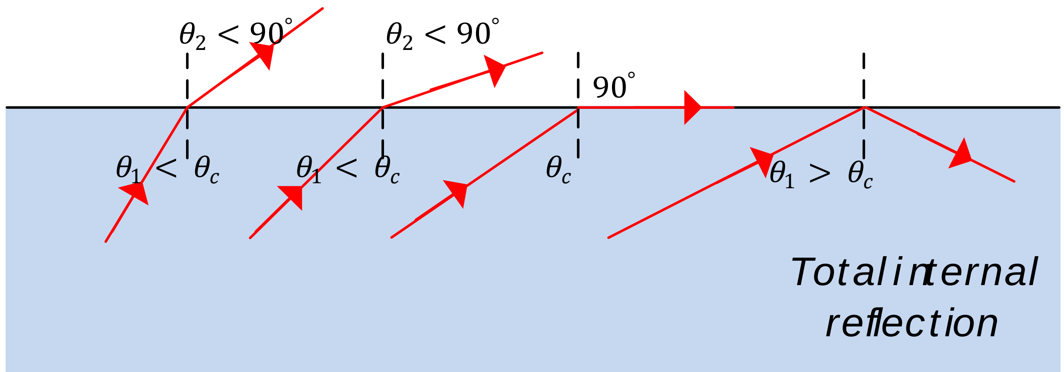
$$x = 2.1\, tan\left(sin^{-1}\left(\frac{n_1\, sin(\theta_1)}{n_2}\right)\right)$$
$$= 2.1\, tan\left(sin^{-1}\left(\frac{1\, sin(64.3^{\circ})}{1.33}\right)\right) \cong 1.9\, m$$

Finally, the distance from the pool wall to the spot where the light strikes the ground, D, is found as follows.

$$D = 2.7 + x$$
$$\cong 2.7 + 1.9 = 4.6\, m$$

Total Internal Reflection:

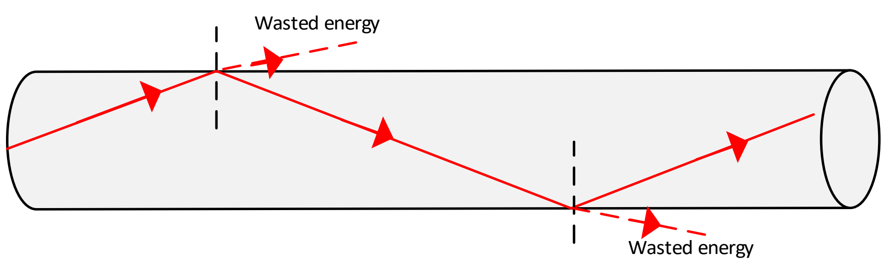
Recall that when light passes from a material with a higher index of refraction than the material for which the light enters, the light ray bends away from the normal. An example is a light ray going from water to air as shown below.



Note from the figure, as the angle of incidence gets larger the light beam bends more towards the surface of the water. Even more interesting is when the angle of incidence reaches the so called critical angle, θ_c, the light beam moves along the surface of the water and an observer above the water will not see the light. The reflecting angle at this point is 90°, which we can use to find an expression for the critical incident angle as follows.

$$n_1\, sin(\theta_1) = n_2\, sin(\theta_2)$$
$$sin(\theta_c) = \frac{n_2}{n_1} sin(90^\circ)$$
$$sin(\theta_c) = \frac{n_2}{n_1}$$

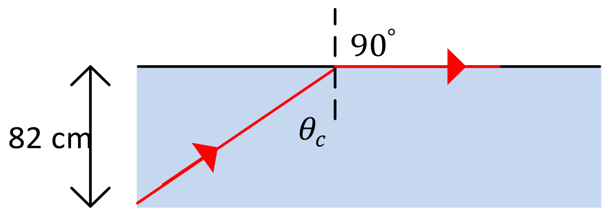
Incident angles larger than θ_c will reflect back into the water. This is referred to as total internal reflection and is one of the main principles used to send light signals through fiber optic cables. Any light that escapes the cable causes a loss in signal getting to the intended receiver.



Example 6: A beam of light is emitted in a pool of water from a depth of 82 cm. Where must it strike the air-water interface, relative to the spot directly above it, in order that the light does *not* exit the water? The index of refraction of water is 1.33.

Solution: As shown the light beam must strike the interface at the critical angle, θ_c.

$$sin(\theta_c) = \frac{n_2}{n_1}$$
$$\theta_c = sin^{-1}\left(\frac{1}{1.33}\right) \cong 48.8^{\circ}$$

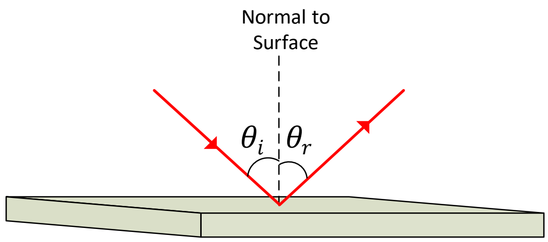
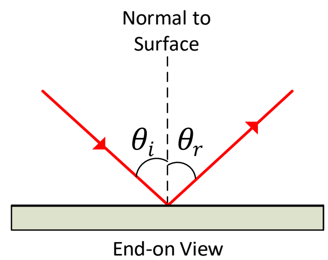
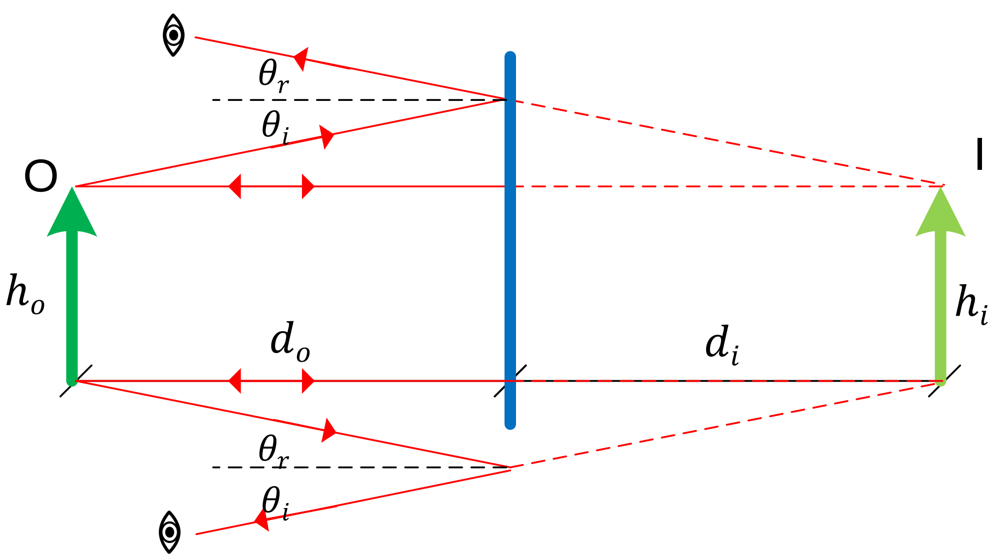


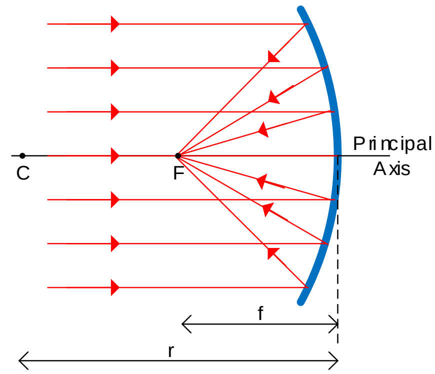
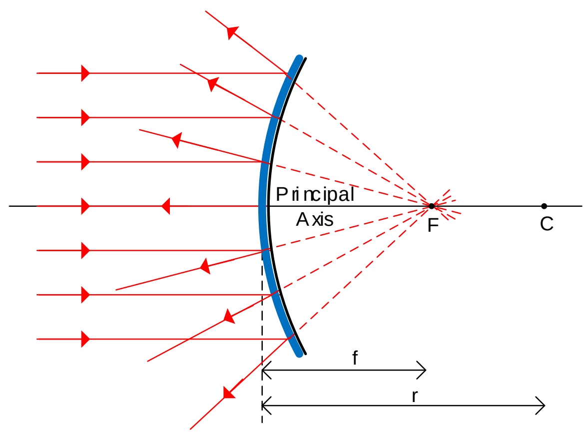
Next, we use the right angle formed to find the horizontal distance.

$$x = 82\,tan(\theta_c)$$
$$= 82\,tan(48.8^{\circ}) \cong 93.5\ cm$$

Note incident angles larger than the critical angle will also experience total internal refection. Therefore, the light ray can strike the surface at distances greater than the value computed above also.

Final Summary for Geometric Optics – Mirrors

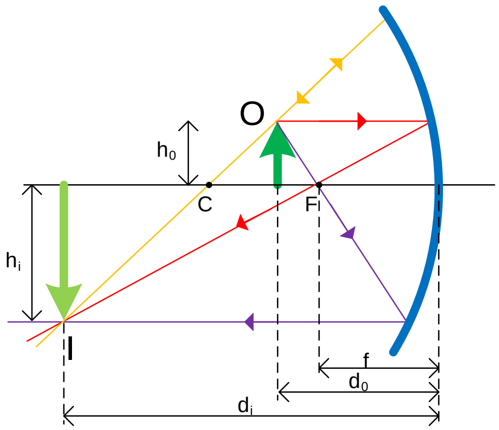
Law of Reflection
When a narrow beam of light, e.g. a light ray, strikes a flat and reflective surface the following applies. **The angle of incidence, θ_i, equals the angle of reflection, θ_r** 
Plane Mirrors
For plane mirrors the image is always virtual, upright, same height, and same distance behind the mirror as the object is in front of the mirror. 

Spherically Curved Mirrors
Spherical mirrors can be concave, where the reflecting surface is on the inside of the sphere, or convex, where the reflecting surface is on the outside of the sphere. Concave mirrors focus parallel lights rays to a point in front of the mirror called the focal point, F.    Convex mirrors focus parallel lights rays to a point behind the mirror, also called the focal point, F.    The distance of the point F to the mirror is called the focal length, f, and $f = \frac{r}{2}$ Where r is the radius of curvature of the mirror, where C marks the center of the sphere.

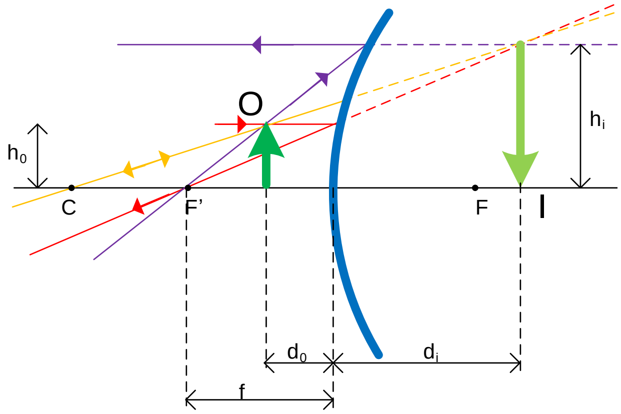
Mirror Equations

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$$	$$m = \frac{h_i}{h_0} = -\frac{d_i}{d_0}$$ m: represents the lateral magnification

Concave Mirror



Convex Mirror



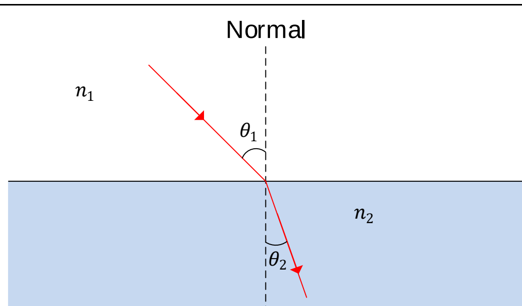
Conventions:

- Object height, h_o, and image height, h_i, are positive if upright and negative if inverted.
- Object distance, d_o, the image distance, d_i, and the focal length, f, are positive if they are located on the side of where the light rays originate and negative otherwise.
- The magnification, m, is positive for an upright image and negative for an inverted image.
- The image is considered a *real image* if light rays pass through it and a *virtual image* is the light rays do not pass through it.

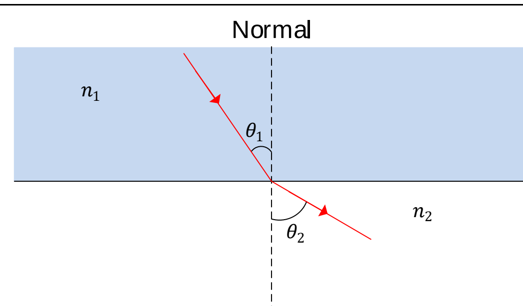
Snell's Law of Refraction

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

Where θ_1 is the angle if incidence and θ_2 is the angle of refraction; n_1 and n_2 are the respective indices of refraction of the materials.



$n_2 > n_1$ therefore, $\theta_1 > \theta_2$



$n_1 > n_2$ therefore, $\theta_2 > \theta_1$

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