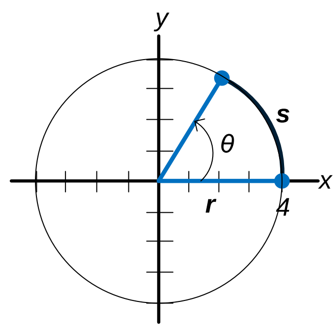
Trigonometry – Radian Measure and The Unit Circle

Measurements can be referenced to any number of so-called 'units'. For example, lengths can be measured in inches, feet, meters, etc. The same is true for angle measurements. So far, we have used degrees to measure angles. In this lesson we introduce an alternate unit called 'radians'. Although it is far more common to use degrees in common speech, radians is much more common in science and engineering work. We also formally introduce the concept of a unit circle as it relates to the trigonometric functions.

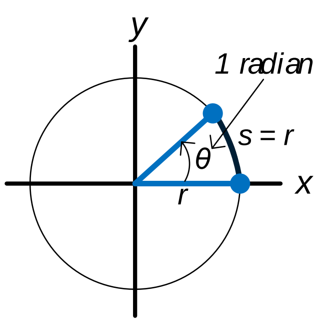
Radians

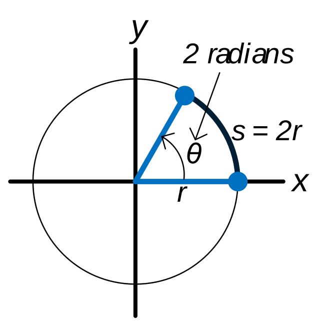
Using a rectangular coordinate system, we measure an angle in standard position using the origin as the vertex and the positive x-axis as the initial ray. The angle is formed by rotating a second ray counterclockwise. Choosing a fixed location along the x-axis, this action sweeps out an arc, s, on a circle with radius equal to the starting location on the x-axis, i.e. $x = 4$.



The radian is defined in terms of this radius, r, and the length of the arc, s, as follows:

Sweeping an angle of 1 radian results in an arc length that is equal to the length of the radius. Similarly, sweeping an angle of 2 radians results in an arc length that if equal to twice the length of the radius.





With this we can write a simple relationship between the arc length and the radius as follows.

$$s = \theta r$$

Next, we recall a familiar elementary formula for the circumference of a circle.

$$C = 2\pi r$$

The circumference represents the arc length after one full rotation, i.e. $s = C$ for $\theta = 2\pi$. Since one full rotation is also known to be equivalent to 360°, we can say $360^{\circ} = 2\pi$ or $180^{\circ} = \pi$. We can use the relationship to convert between degrees and radians as summarized below.

Degrees to Radians	***Radians to Degrees***
Multiply the angle in degree by $\frac{\pi}{180^{\circ}}$. Example: $30^{\circ} \cdot \left(\frac{\pi}{180^{\circ}}\right) = \frac{\pi}{6}\ radians$	Multiple the angle in radians by $\frac{180^{\circ}}{\pi}$. Example: $\frac{\pi}{3} radians \cdot \frac{180^{\circ}}{\pi} = 60^{\circ}$

The unit circle is one of the most important tools for understanding and working with trigonometric functions. Before moving on to this let's do some examples with converting radian and degree angle measures.

Example 1: Convert each degree measure to radians.

a. 45° *b.* -270° *c.* 249.8°

Solution:

a. $45^{\circ} = 45 \cdot \left(\frac{\pi}{180^{\circ}}\right) = \frac{\pi}{4}\ radians$

b. $-270^{\circ} = -270 \cdot \left(\frac{\pi}{180^{\circ}}\right) = -\frac{3\pi}{2}\ radians$

c. $249.8^{\circ} = 249.8 \cdot \left(\frac{\pi}{180^{\circ}}\right) \cong 4.36\ radians$

Example 2: Convert each radian measure to degree.

a. $\frac{9\pi}{4}$ *b.* $-\frac{5\pi}{6}$ *c.* 4.25

Solution:

a. $\frac{9\pi}{4} = \frac{9\pi}{4} \cdot \left(\frac{180^{\circ}}{\pi}\right) = 405^{\circ}$

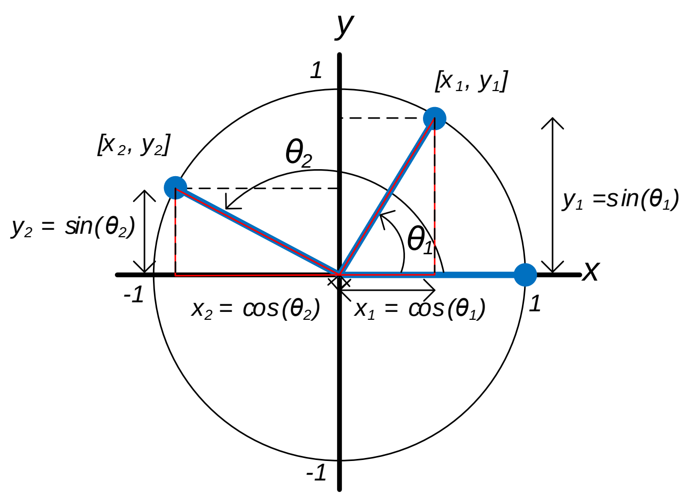
b. $-\frac{5\pi}{6} = -\frac{5\pi}{6} \cdot \left(\frac{180^{\circ}}{\pi}\right) = -150^{\circ}$

c. $4.25 = 4.25 \cdot \left(\frac{180^{\circ}}{\pi}\right) \cong 243.5^{\circ}$

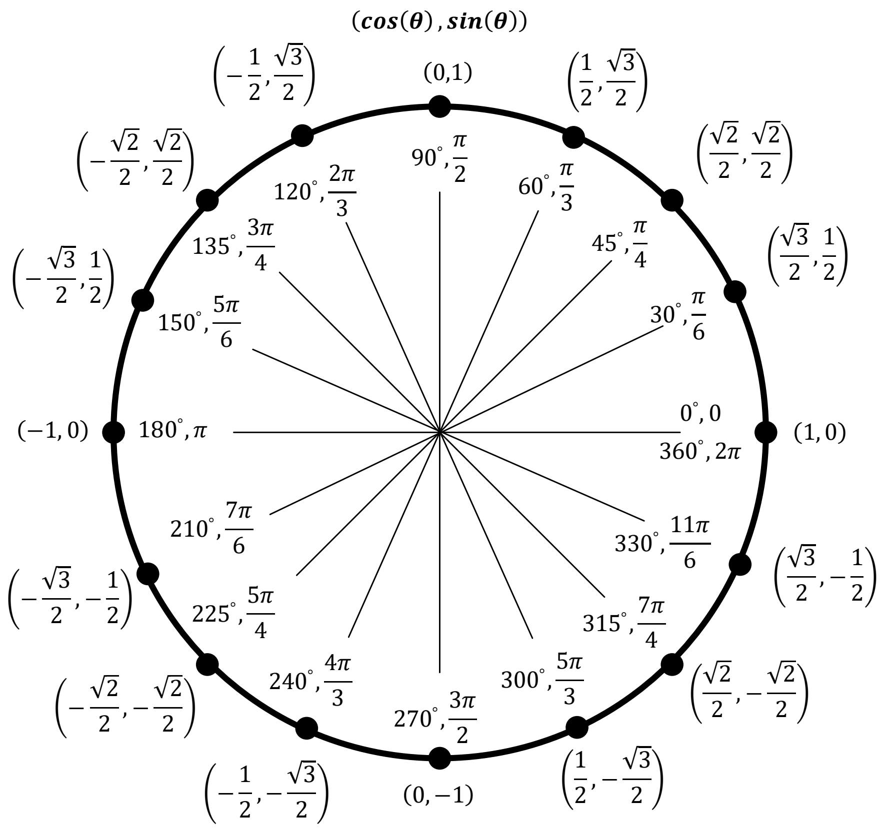
The Unit Circle

In the previous lesson we extended the trigonometric functions beyond 90°,or $\pi/2$, by drawing a right triangle with a hypotenuse length of 1 in each quadrant of the x-y plane. Tracing the endpoint of the hypotenuse for all such triangles as we move the rotating ray counterclockwise, we obtain a unit circle. Note that each angle, θ_i, corresponds to a point on the circle, $[x_i, y_i]$. With $h = 1$ these coordinates correspond directly to the cosine and sine functions as shown.

$$cos(\theta_i) = \frac{x_i}{h} = x_i \qquad sin(\theta_i) = \frac{y_i}{h} = y_i$$



Using the special angles, we can create a reference unit circle as shown below.



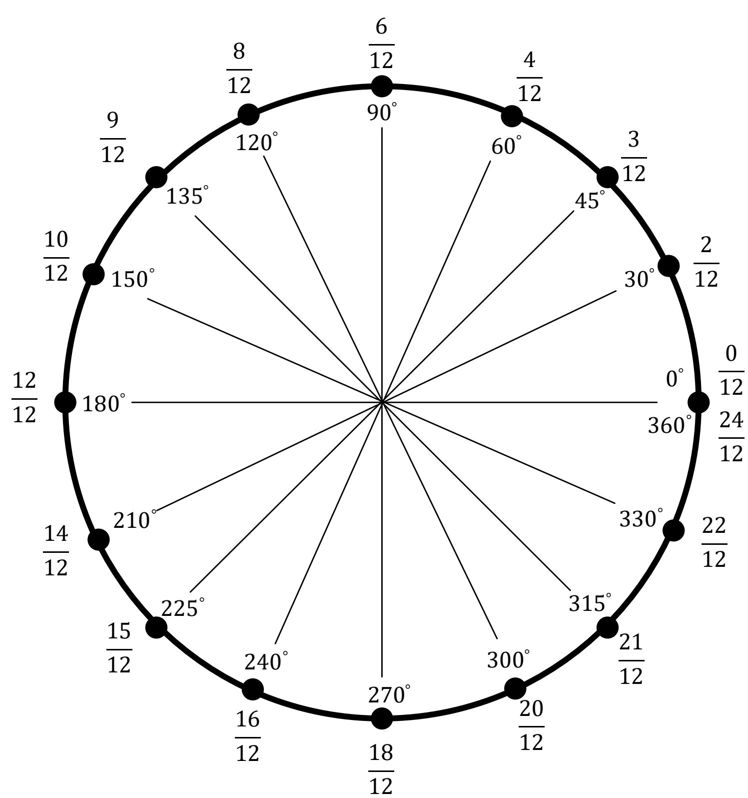
This unit circle should be used as a reference when evaluating trigonometric functions. However, rather than trying to fully commit the figure to memory, I suggest you understand how to create it from fundamental principles. It contains the same information that was in the table from the previous lesson, (except for the radian measure), but organized on the unit circle, which highlights the various symmetries as they apply to the cosine and sine values.

- ***Symmetric with respect to x-axis***
  Flipping the cosine and sine values about the x-axis we note that the cosine values, which refer to the x-axis, remain the same, and the sine values, which refer to the y-axis, flip in sign.
- ***Symmetric with respect to y-axis***
  Flipping the cosine and sine values about the y-axis we note that the sine values, which refer to the y-axis, remain the same, and the cosine values, which refer to the x-axis, flip in sign.

We can use these symmetries, along with the 'trick' for memorizing the special angles in the first quadrant we learned two lessons ago, to reproduce the unit circle.

When producing the unit circle its generally more difficult to remember the radian values compared to the degree values. For this we describe one method that can help.

Imagine dividing the circle into 12 pieces, i.e. 15° each. Next, we pull out a factor of π from the radian values giving us a measure that goes from 0 to 2 in steps of $1/12$. Knowing this we can easily reproduce the radian values with simple increments of $1/12$. The figure below shows the degree and radian values (less the π). Be sure to verify for yourself, by reducing the fractions and multiplying by π, that the values shown here correspond to the values from above. Note that we sometimes have increments of 30°, i.e. $2/12$, e.g. between 60° and 90°.



Example 3: Given the quadrant where θ exists, find its exact value for each of the following equations.

a. $Q2\colon\ sin(\theta) = \frac{1}{2}$ *b.* $Q2\colon\ cos(\theta) = -\frac{1}{2}$ *c.* $Q3\colon\ tan(\theta) = \sqrt{3}$

d. $Q3\colon\ sec(\theta) = -\frac{2\sqrt{3}}{3}$ *e.* $Q4\colon\ tan(\theta) = -1$ *f.* $Q4\colon\ cos(\theta) = \frac{\sqrt{3}}{2}$

Solution: For this example, as well as example 4 and 5, you should start by reproducing the unit circle without consulting the one given above. When complete verify it with the one from above and then solve the examples using your unit circle as your reference. To begin drawing the unit circle it usually helps to create a quick table for the first quadrant using our 'trick' from a previous lesson.

$\boldsymbol{\theta}$	$\mathbf{cos}(\boldsymbol{\theta})$	$\mathbf{sin}(\boldsymbol{\theta})$	$\mathbf{tan}(\boldsymbol{\theta}) = \frac{\mathbf{sin}(\boldsymbol{\theta})}{\mathbf{cos}(\boldsymbol{\theta})}$
$0^\circ, \frac{0}{12}\pi = 0$	$\frac{1}{2}\sqrt{4} = 1$	$\frac{1}{2}\sqrt{0} = 0$	0
$30^\circ, \frac{2}{12}\pi = \frac{\pi}{6}$	$\frac{1}{2}\sqrt{3} = \frac{\sqrt{3}}{2}$	$\frac{1}{2}\sqrt{1} = \frac{1}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
$45^\circ, \frac{3}{12}\pi = \frac{\pi}{4}$	$\frac{1}{2}\sqrt{2} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}\sqrt{2} = \frac{\sqrt{2}}{2}$	1
$60^\circ, \frac{4}{12}\pi = \frac{\pi}{3}$	$\frac{1}{2}\sqrt{1} = \frac{1}{2}$	$\frac{1}{2}\sqrt{3} = \frac{\sqrt{3}}{2}$	$\sqrt{3}$
$90^\circ, \frac{6}{12}\pi = \frac{\pi}{2}$	$\frac{1}{2}\sqrt{0} = 0$	$\frac{1}{2}\sqrt{4} = 1$	$\frac{1}{0} = UND$

a. In the first quadrant $sin(\theta) = \frac{1}{2}$ for $\theta = 30^\circ$. The sine value stays positive above the x-axis, i.e. 1st and 2nd quadrant. In the 2nd quadrant 150° corresponds to 30°. You can visualize this by imagining flipping the Q1 triangle over the y-axis. Therefore, in the 2nd quadrant $sin(\theta) = \frac{1}{2}$ for

$$\theta = 150^\circ or\ \frac{150}{180}\pi = \frac{5}{6}\pi\ radians$$

b. In the first quadrant $cos(\theta) = \frac{1}{2}$ for $\theta = 60^\circ$. The cosine value is negative on the left side of the y-axis, i.e. 2nd and 3rd quadrant. In the 2nd quadrant 120° corresponds to 60°. You can visualize this by imagining flipping the Q1 triangle over the y-axis. Therefore, in the 2nd quadrant $cos(\theta) = \frac{1}{2}$ for

$$\theta = 120^\circ or\ \frac{120}{180}\pi = \frac{2}{3}\pi\ radians$$

c. In the first quadrant $tan(\theta) = \sqrt{3}$ for $\theta = 60^\circ$. The tangent value is positive in the 1st and 3rd quadrant. In the 3rd quadrant 240° corresponds to 60°. Therefore, in the 3rd quadrant $tan(\theta) = \sqrt{3}$ for

$$\theta = 120^\circ or\ \frac{120}{180}\pi = \frac{2}{3}\pi\ radians$$

d. Since secant is the inverse of cosine, we can rewrite the question as follows.

$$\frac{1}{sec(\theta)} = -\frac{3}{2\sqrt{3}} = -\frac{\sqrt{3}}{2} = cos(\theta)$$

In the first quadrant $cos(\theta) = \frac{\sqrt{3}}{2}$ for $\theta = 30^\circ$. The cosine value is negative on the left side of the y-axis, i.e. 2nd and 3rd quadrant. In the 3rd quadrant 210° corresponds to 30°. Therefore, in the 3rd quadrant $cos(\theta) = -\frac{\sqrt{3}}{2}$ for

$$\theta = 210^\circ or\ \frac{210}{180}\pi = \frac{7}{6}\pi\ radians$$

e. In the first quadrant $tan(\theta) = 1$ for $\theta = 45^\circ$. The tangent value is negative in the 2nd and 4th quadrant. In the 4th quadrant 315° corresponds to 45°. You can visualize this by imagining flipping the Q1 triangle over the y-axis. Therefore, in the 4th quadrant $tan(\theta) = -1$ for

$$\theta = 315^\circ or\ \frac{315}{180}\pi = \frac{7}{4}\pi\ radians$$

f. In the first quadrant $cos(\theta) = \frac{\sqrt{3}}{2}$ for $\theta = 30^\circ$. The cosine value is positive in the 1st and 4th quadrant. In the 4th quadrant 330° corresponds to 30°. You can visualize this by imagining flipping the Q1 triangle over the y-axis. Therefore, in the 4th quadrant $cos(\theta) = \frac{\sqrt{3}}{2}$ for

$$\theta = 330^\circ or\ \frac{330}{180}\pi = \frac{11}{6}\pi\ radians$$

Example 4: Find the exact function values without using a calculator

a. $tan\left(\frac{\pi}{3}\right)$ *b.* $cos\left(\frac{2\pi}{3}\right)$ *c.* $sin\left(-\frac{5\pi}{6}\right)$

d. $tan\left(-\frac{7\pi}{3}\right)$ *e.* $csc\left(-\frac{11\pi}{6}\right)$ *f.* $cot(-13\pi)$

Solution:

a. Using the table from example 3 we find that

$$tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

b. In this case we can use our alternate version of the unit circle by removing the π term in the angle and changing the fraction to have a denominator of 12. This gives us an 'angle' of $8/12$ or $8 \cdot 15^{\circ} = 120^{\circ}$, which is in the 2nd quadrant where the cosine value is negative. In the 2nd quadrant 120° corresponds to 60°. The cosine of 60° is $1/2$. Therefore, we have

$$cos\left(\frac{2\pi}{3}\right) = cos(120^{\circ}) = -\frac{1}{2}$$

c. We can again start by converting the angle to degrees by removing the π term and changing the fraction to have a denominator of 12, i.e. $-10/12$ or $-10 \cdot 15^{\circ} = -150^{\circ}$. Next, we can change to a positive angle using coterminal angles by adding 360°.

$$-150^{\circ} + 360^{\circ} = 210^{\circ}$$

210° is in the 3rd quadrant where the sine is negative. Furthermore, in the 3rd quadrant 210° corresponds to 30°. The sine of 30° is $1/2$, therefore,

$$sin\left(-\frac{5\pi}{6}\right) = sin(-150^{\circ}) = sin(210^{\circ}) = -\frac{1}{2}$$

d. We start by changing the angle similar to the previous example, starting with $-28/12$. In degrees we have $-28 \cdot 15^{\circ} = -420^{\circ}$. Adding $2 \cdot 360^{\circ}$ we find the coterminal angle.

$$-420^{\circ} + 2 \cdot 360^{\circ} = 300^{\circ}$$

Which is in the 4th quadrant where the tangent is negative. Finally, since 300° corresponds to 60° for which the tangent is $\sqrt{3}$. Therefore,

$$tan\left(-\frac{7\pi}{3}\right) = tan(-420^{\circ}) = tan(300^{\circ}) = -\sqrt{3}$$

e. In this case, $-11/6 = -22/12$, and $-22 \cdot 15^{\circ} = -330^{\circ}$. Adding 360° we find a coterminal angle of 30°. Since 30° is in the 1st quadrant we can directly use the simple table from above to find $sin(30^{\circ}) = 1/2$. Finally, since $csc(\theta) = 1/sin(\theta)$ we have

$$csc\left(-\frac{11\pi}{6}\right) = csc(-330^{\circ}) = csc(30^{\circ}) = 2$$

f. Each time we rotate $\pi = 180^{\circ}$ we travel halfway around the circle. You will notice that when we rotate on odd integer number of times, we always end up at 180°. Therefore

$$cot(-13\pi) = cot(180^{\circ}) = \frac{cos(180^{\circ})}{sin(180^{\circ})} = \frac{-1}{0} = UND$$

Example 5: Find the exact values of θ over the given interval that satisfy the given conditions.

a. $[0,2\pi): sin(\theta) = -\frac{\sqrt{3}}{2}$ *b.* $[0,2\pi): cos(\theta) = -\frac{1}{2}$ *c.* $[0,2\pi): cos^2(\theta) = \frac{1}{2}$

d. $[0,2\pi): tan^2(\theta) = 3$ *e.* $[-2\pi,\pi): 3\,tan^2(\theta) = 1$ *f.* $[-2\pi,\pi): sin^2(\theta) = \frac{1}{2}$

Solution:

a. From our simple 1st quadrant table, we know that $sin(60^{\circ}) = \frac{\sqrt{3}}{2}$. We also know that the sine is negative in quadrants 3 and 4. In the 3rd quadrant 60° corresponds to 210° and in the 4th quadrant it corresponds to 300°. Therefore,

$$sin(\theta) = -\frac{\sqrt{3}}{2} \text{ for } \theta = [210^{\circ}, 300^{\circ}] = \left[\frac{7}{6}\pi, \frac{5}{3}\pi\right].$$

b. In this case, $cos(60^{\circ}) = \frac{1}{2}$. Furthermore, the cosine is negative in quadrants 2 and 3. In the 2nd quadrant 60° corresponds to 120° and in the 3rd quadrant it corresponds to 210°. Therefore, $cos(\theta) = -\frac{1}{2}$ for $\theta = [120^{\circ}, 210^{\circ}] = \left[\frac{2}{3}\pi, \frac{7}{6}\pi\right]$.

c. In this case, we start by taking the square root of both sides.

$$cos(\theta) = \pm\left(\frac{1}{\sqrt{2}}\right) = \pm\left(\frac{\sqrt{2}}{2}\right)$$

Cosine is positive in the 1st and 2nd quadrant and $cos(\theta) = \frac{\sqrt{2}}{2}$ for $\theta = 45^\circ$ and $\theta = 135^\circ$ respectively. Furthermore, the cosine is negative in the 3rd and 4th quadrant where $cos(\theta) = -\frac{\sqrt{2}}{2}$ for $\theta = 225^\circ$ and $\theta = 315^\circ$ respectively. Therefore,

$$cos^2(\theta) = \frac{1}{2} \text{ for } \theta = [45^\circ, 135^\circ, 225^\circ, 315^\circ]$$

d. Taking the square root of both sides again we have

$$tan(\theta) = \pm(\sqrt{3})$$

In the 1st quadrant the tangent is $\sqrt{3}$ for $\theta = 60^\circ$ and in the 3rd quadrant the tangent is $\sqrt{3}$ for $\theta = 240^\circ$. Since the equation is also valid for $-\sqrt{3}$ we solutions in the 2nd and 4th quadrant also, i.e. $\theta = 120^\circ, 300^\circ$. Finally, we can write the solution as follows:

$$tan^2(\theta) = 3 \text{ for } \theta = [60^\circ, 120^\circ, 240^\circ, 300^\circ]$$

e. We start by dividing by 3 and taking the square root of both sides.

$$tan(\theta) = \pm\left(\frac{1}{\sqrt{3}}\right) = \pm\left(\frac{\sqrt{3}}{3}\right)$$

We again have solution in all 4 quadrants. In the 1st quadrant the angle is 30°. The corresponding angles in the other 3 quadrants are $150^\circ, 210^\circ$ and 330°. However, in this case since the range for solutions is $[-2\pi, \pi) = [-360^\circ, 180^\circ)$, we find the corresponding coterminal angles for each solution.

$$\begin{aligned} 30^\circ - 360^\circ &= -330^\circ \\ 150^\circ - 360^\circ &= -210^\circ \\ 210^\circ - 360^\circ &= -150^\circ \\ 330^\circ - 360^\circ &= -30^\circ \end{aligned}$$

The final solution set is

$$[-330^\circ, -210^\circ, -150^\circ, -30^\circ, 30^\circ, 150^\circ]$$

f. Start again with the square root and find

$$sin(\theta) = \pm\left(\frac{\sqrt{2}}{2}\right)$$

Similar to problem *c*. we have

$$\theta = [45^\circ, 135^\circ, 225^\circ, 315^\circ]$$

Since the solution range is the same as in problem *e.,* we use the same technique.

$$\begin{aligned} &45^\circ - 360^\circ = -315^\circ \\ &135^\circ - 360^\circ = -225^\circ \\ &225^\circ - 360^\circ = -135^\circ \\ &315^\circ - 360^\circ = -45 \end{aligned}$$

The final solution set is then

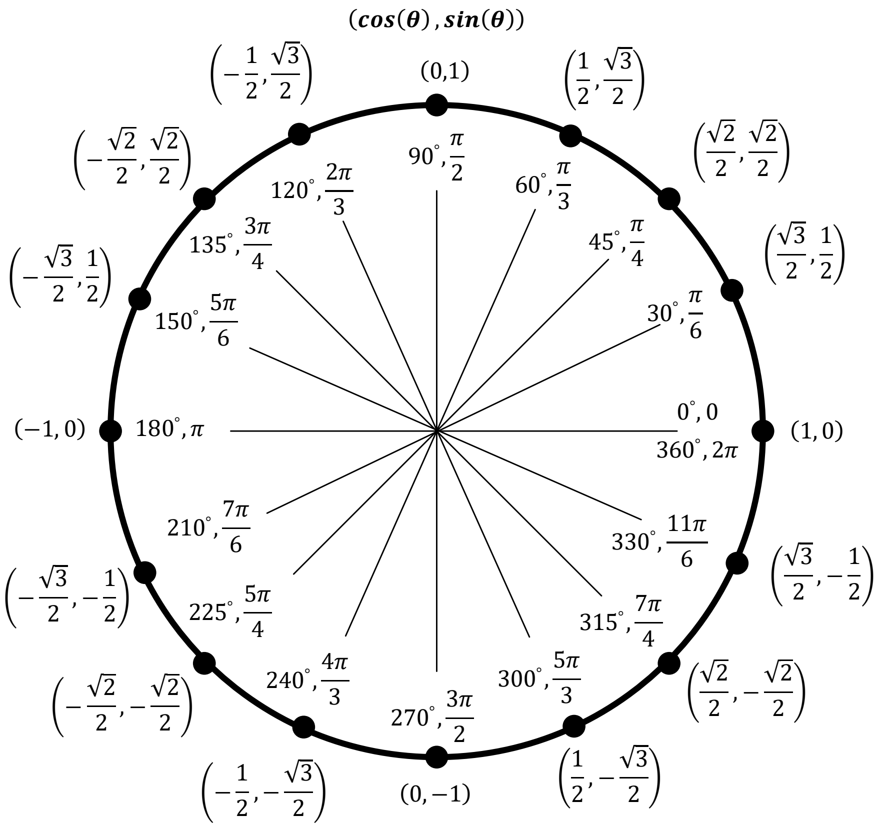
$$[-315^\circ, -225^\circ, -135^\circ, -45^\circ, 45^\circ, 135^\circ]$$

__Final Summary for Trigonometry – Radian Measure and The Unit Circle__

Degrees and Radians
One complete revolution around a circle is 360° or 2π radians. Knowing this we can convert between these two angle measures as shown below.

Degrees to Radians	***Radians to Degrees***
Multiply the angle in degree by $\frac{\boldsymbol{\pi}}{\mathbf{180^\circ}}$. Example: $30^\circ \cdot \left(\frac{\pi}{180^\circ}\right) = \frac{\pi}{6}\, radians$	Multiple the angle in radians by $\frac{\mathbf{180^\circ}}{\boldsymbol{\pi}}$. Example: $\frac{\pi}{3} radians \cdot \frac{180^\circ}{\pi} = 60^\circ$

The Unit Circle
The unit circle can be used to solve basic trigonometric equations which are related to the special angles, i.e. $0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$.



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