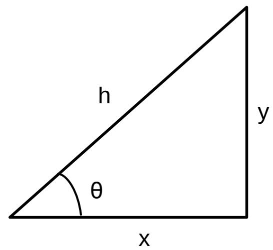
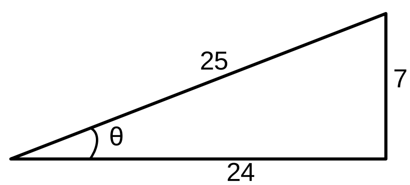
Trigonometry – Right Triangle Trigonometry

We mentioned in the previous lesson that trigonometric functions are special functions that are used to express relationships between the side lengths and angles of *right* triangles. More specifically, there are six trigonometric functions that are defined as ratios of the lengths of the sides of a right triangle as shown below.



$\sin(\theta) = \dfrac{opposite}{hypotenuse} = \dfrac{y}{h}$	$\csc(\theta) = \dfrac{1}{\sin(\theta)} = \dfrac{hypotenuse}{opposite} = \dfrac{h}{y}$
$\cos(\theta) = \dfrac{adjacent}{hypotenuse} = \dfrac{x}{h}$	$\sec(\theta) = \dfrac{1}{\cos(\theta)} = \dfrac{hypotenuse}{adjacent} = \dfrac{h}{x}$
$\tan(\theta) = \dfrac{opposite}{adjacent} = \dfrac{y}{x}$	$\cot(\theta) = \dfrac{1}{\tan(\theta)} = \dfrac{adjacent}{opposite} = \dfrac{x}{y}$

Let's look at the following example:



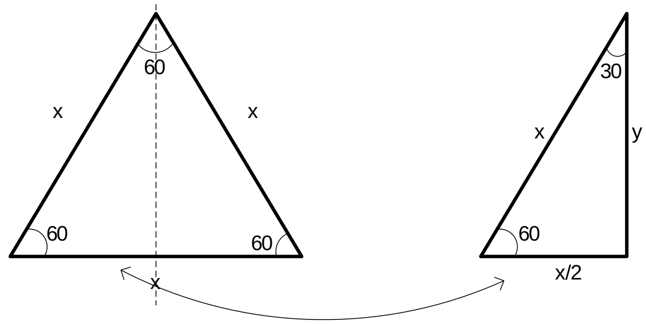
Using the side lengths given, the six trigonometric functions are evaluated as follows:

$sin(\theta) = \dfrac{opposite}{hypotenuse} = \dfrac{7}{25}$	$\csc(\theta) = \dfrac{1}{\sin(\theta)} = \dfrac{25}{7}$
$cos(\theta) = \dfrac{adjacent}{hypotenuse} = \dfrac{24}{25}$	$\sec(\theta) = \dfrac{1}{\cos(\theta)} = \dfrac{25}{24}$
$tan(\theta) = \dfrac{opposite}{adjacent} = \dfrac{7}{24}$	$\cot(\theta) = \dfrac{1}{\tan(\theta)} = \dfrac{24}{7}$

Evaluating trigonometric functions for most angles results in irrational numbers and are not easy to compute manually. Generally, the easiest way to evaluate any of the trigonometric functions for an arbitrary angle is to use a standard scientific calculator. For example, the sine function evaluated for an angle of 38.6° is given below.

$$sin(38.6^{\circ}) = 0.6238795967$$

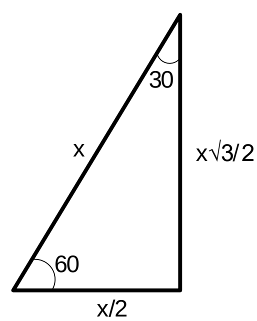
There are, however, some frequently used angles, e.g. 30°, 45°, and 60°, that should be committed to memory. The relationships can be derived using an equilateral and right triangle. We start by deriving the values for 30° and 60° using an equilateral triangle as shown below.



By cutting the triangle in half we have created a right triangle with 30° and 60° angles. Next, we write y as a function of x so that all side lengths are functions of x only. Using the Pythagorean Theorem, we have

$$y = \sqrt{x^2 - \left(\frac{x}{2}\right)^2} = \sqrt{x^2 - \frac{x^2}{4}} = \sqrt{x^2\left(1 - \frac{1}{4}\right)} = x\sqrt{\left(\frac{3}{4}\right)} = x\frac{\sqrt{3}}{2}$$

The right triangle can now be drawn with all sides being a function of x.



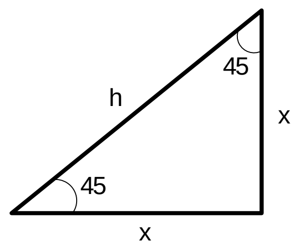
The six trigonometric functions can then be derived for 30° and 60° respectively. We illustrate for the sine and cosine below.

$$sin(30) = \frac{opp}{hyp} = \frac{\frac{\cancel{x}}{2}}{\cancel{x}} = \frac{1}{2} \qquad cos(30) = \frac{adj}{hyp} = \frac{\frac{\cancel{x}\sqrt{3}}{2}}{\cancel{x}} = \frac{\sqrt{3}}{2}$$

$$sin(60) = \frac{opp}{hyp} = \frac{\frac{\cancel{x}\sqrt{3}}{2}}{\cancel{x}} = \frac{\sqrt{3}}{2} \qquad cos(60) = \frac{adj}{hyp}\frac{\frac{\cancel{x}}{2}}{\cancel{x}} = \frac{1}{2}$$

The remaining trigonometric functions can be similarly computed.

Next, we'll derive values for 45° using an isosceles right triangle as shown below.



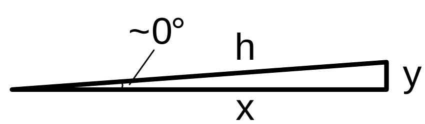
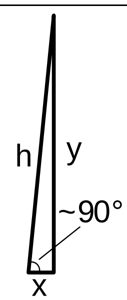
We start by writing all sides as functions of x.

$$h = \sqrt{x^2 + x^2} = \sqrt{2x^2} = \sqrt{2}x$$

Once again, we illustrate the values for the sine and cosine of 45° below.

$$sin(45) = \frac{opp}{hyp} = \frac{\cancel{x}}{\sqrt{2}\cancel{x}} = \frac{1}{\sqrt{2}} \qquad cos(45) = \frac{adj}{hyp} = \frac{\cancel{x}}{\sqrt{2}\cancel{x}} = \frac{1}{\sqrt{2}}$$

Lastly, we note two other "*special*" angles, 0° and 90°. The figures below show what these triangles look like when the angles approach 0° and 90°, i.e. $\theta \to 0^{\circ}$ and $\theta \to 90^{\circ}$. Without being rigorous we could easily imagine that when $\theta = 0^{\circ}$ we will have $h = x$ and $y = 0$, and when $\theta = 90^{\circ}$ we will have $h = y$, and $x = 0$.

	$\theta \to 0^{\circ}$	$x = x$ $h = x$ $y = 0$
	$\theta \to 90^{\circ}$	$y = y$ $h = y$ $x = 0$

With this the sine and cosine for 0° and 90° are given.

$$sin(0) = \frac{opp}{hyp} = \frac{0}{x} = 0 \qquad cos(0) = \frac{adj}{hyp} = \frac{x}{x} = 1$$

$$sin(90) = \frac{opp}{hyp} = \frac{y}{y} = 1 \qquad cos(90) = \frac{adj}{hyp} = \frac{0}{y} = 0$$

Computing the remaining functions for all special angles we create the table shown below.

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$	$\csc(\theta)$	$\sec(\theta)$	$\cot(\theta)$
0°	0	1	0	∞	1	∞
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{2}$
90°	1	0	∞	1	∞	0

Committing this table to memory will prove helpful. Fortunately, is not as difficult as it may first seem. Firstly, notice that the sine and cosine functions are the most important trigonometric functions in the sense that all others function values can be derived from their values. In other words, once we know that values of $\sin(\theta)$ and $\cos(\theta)$ the others can be determined as follows:

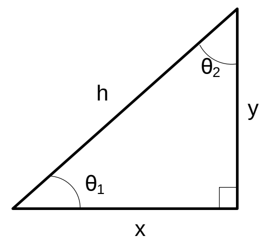
$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)} \quad \sec(\theta) = \frac{1}{\cos(\theta)} \quad \cot(\theta) = \frac{1}{\tan(\theta)}$$

Therefore, by memorizing only two columns we can reproduce the entire table. Secondly, notice that the cosine values are just a reverse reading of the sine values. Therefore, we need only really memorize *one* column from this entire table! Finally, if we rewrite the values for $\sin(\theta)$ in a different format we notice a very simple pattern. Examining the table below we see that each entry has a factor of $1/2$ multiplied by the square root of the numbers 0 through 4. Verify for yourself that the values shown below match the table from above.

θ	$\sin(\theta)$	$\cos(\theta)$
0°	$\left(\frac{1}{2}\right)\sqrt{0}$	$\left(\frac{1}{2}\right)\sqrt{4}$
30°	$\left(\frac{1}{2}\right)\sqrt{1}$	$\left(\frac{1}{2}\right)\sqrt{3}$
45°	$\left(\frac{1}{2}\right)\sqrt{2}$	$\left(\frac{1}{2}\right)\sqrt{2}$
60°	$\left(\frac{1}{2}\right)\sqrt{3}$	$\left(\frac{1}{2}\right)\sqrt{1}$
90°	$\left(\frac{1}{2}\right)\sqrt{4}$	$\left(\frac{1}{2}\right)\sqrt{0}$

Let's finish this lesson with examples to practice what we learned.

Example 1: Find the unknown side lengths for the right triangle shown below using the given measurements.



a. $x = 5, y = 12$ *b.* $\theta_1 = 24^\circ, x = 8$ *c.* $\theta_2 = 65^\circ, h = 12$

Solution:

a. In this case we can use the Pythagorean theorem to find the unknown side length, i.e. the hypotenuse.

$$h = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

b. We can start by using the cosine function to find the length of the hypotenuse.

$$cos(24) = \frac{8}{h}$$

Cross multiplying, we have

$$h = \frac{8}{cos(24)} \cong 8.76$$

With two sides known we can again use the Pythagorean theorem. However, for illustration purposes we'll use the tangent function to find the height, y.

$$tan(24) = \frac{y}{8} \rightarrow y = 8\,tan(24) \cong 3.56$$

c. In this case, we use both the cosine and sine functions to find y and x, respectively. Note we are using the upper angle.

$$cos(65) = \frac{y}{12} \rightarrow y = 12\,cos(65) \cong 5.07$$

$$sin(65) = \frac{x}{12} \rightarrow x = 12\,sin(65) \cong 10.88$$

Note, we could also use the lower angle, $\theta_1 = (90^\circ - \theta_2) = 25^\circ$, as shown.

$$cos(25) = \frac{x}{12} \rightarrow x = 12\,cos(25) \cong 10.88$$

$$sin(25) = \frac{y}{12} \rightarrow y = 12\,sin(25) \cong 5.07$$

Example 2: For each expression find the exact value.

a. $tan(30)$ *b.* $csc(45)$ *c.* $cos(60)$

Solution: Instead of using the larger table from above let's create the smaller table for sine and cosine only.

$\boldsymbol{\theta}$	$\mathbf{sin}(\boldsymbol{\theta})$	$\mathbf{cos}(\boldsymbol{\theta})$
$\mathbf{0}^{\circ}$	$\left(\frac{1}{2}\right)\sqrt{0} = 0$	$\left(\frac{1}{2}\right)\sqrt{4} = 1$
$\mathbf{30}^{\circ}$	$\left(\frac{1}{2}\right)\sqrt{1} = \frac{1}{2}$	$\left(\frac{1}{2}\right)\sqrt{3} = \frac{\sqrt{3}}{2}$
$\mathbf{45}^{\circ}$	$\left(\frac{1}{2}\right)\sqrt{2} = \frac{\sqrt{2}}{2}$	$\left(\frac{1}{2}\right)\sqrt{2} = \frac{\sqrt{2}}{2}$
$\mathbf{60}^{\circ}$	$\left(\frac{1}{2}\right)\sqrt{3} = \frac{\sqrt{3}}{2}$	$\left(\frac{1}{2}\right)\sqrt{1} = \frac{1}{2}$
$\mathbf{90}^{\circ}$	$\left(\frac{1}{2}\right)\sqrt{4} = 1$	$\left(\frac{1}{2}\right)\sqrt{0} = 0$

a. Since, $tan(\theta) = \frac{sin(\theta)}{cos(\theta)}$, we have

$$tan(30) = \frac{sin(30)}{cos(30)} = \frac{\left(\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

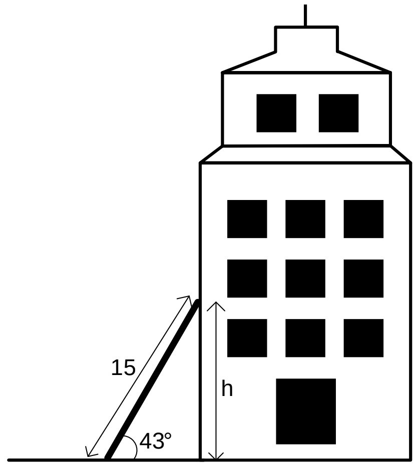
b. Since, $csc(\theta) = \frac{1}{sin(\theta)}$, we have

$$csc(45) = \frac{1}{sin(45)} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

c. In this case, we can read directly from our table.

$$cos(60) = \frac{1}{2}$$

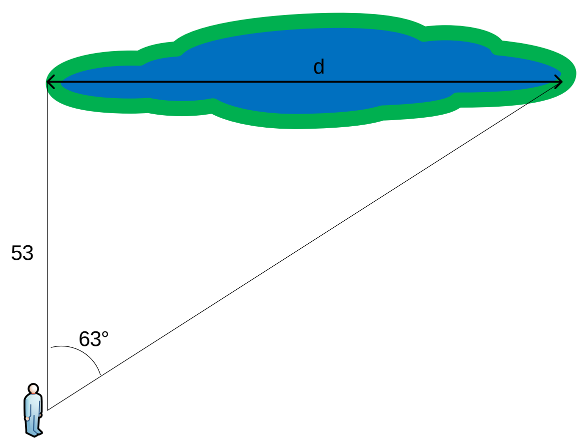
Example 3: How high up the side of the building can a 15 m ladder reach if it makes a 43° angle with the ground?



Solution: The ladder and the building form a right triangle. Therefore, we can use the sine function to find the height the ladder reaches up the side of the building.

$$sin(43) = \frac{h}{15} \rightarrow h = 15\, sin(43) \cong 10.23\ m$$

Example 4: You want to estimate the distance across long but narrow lake. You start at one end of the lake and measure out to a distance of 53 m. You then look through a surveyors tool and find the other end of the lake to be at an angle of 63°. Show how you can use this information to estimate the distance across the lake.



Solution: Assuming we walked perpendicular to the length of the lake a right triangle is formed and we can use the tangent function to find the distance, d, as follows.

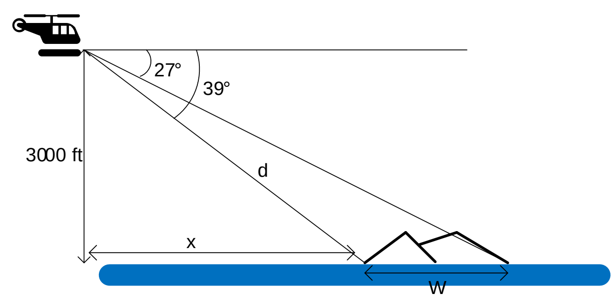
$$tan(63) = \frac{d}{53} \rightarrow d = 53\, tan(63) \cong 104\ m$$

Example 5: A ski slope at a mountain has an angle of elevation of 25°. The vertical height of the mountain is 1808 feet. How long is the ski slope?

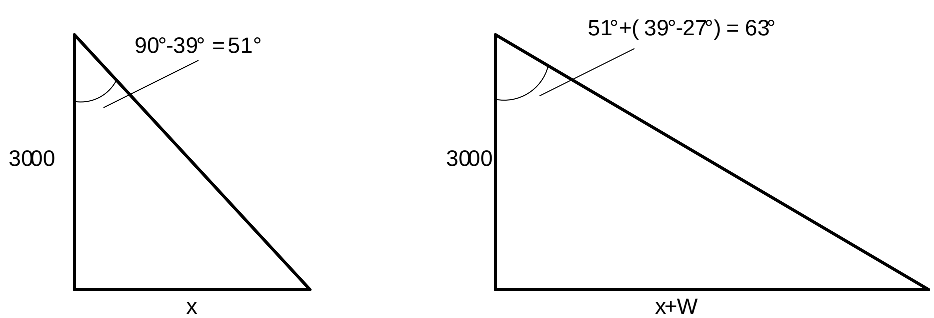
Solution: The distance of the slope can be determined using the sine function.

$$sin(25) = \frac{1808}{d} \rightarrow d = \frac{1808}{sin(25)} \cong 4278\ feet$$

Example 6: A surveyor in a helicopter is trying to determine the width of an island. The helicopter is flying at an altitude of 3000 ft. The surveyor determines the angles of inclination to the front and back of the island as shown. Find the width, W, of the island. Then find the shortest distance, d, for the helicopter to travel to land on the island.



Solution: To find the width of the island we notice that we can draw two right triangles as shown below.



Next, we use the tangent function to find expressions for x and W.

$$tan(51) = \frac{x}{3000}$$
$$x = 3000\, tan(51)$$

$$tan(63) = \frac{x+W}{3000}$$
$$x + W = 3000\, tan(63)$$
$$W = 3000\, tan(63) - x$$

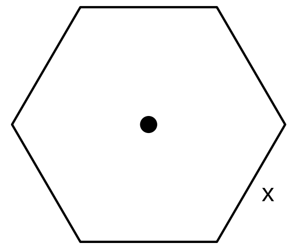
We can then solve for W by substituting the first equation into the second.

$$\begin{aligned} W &= 3000\, tan(63) - 3000\, tan(51) \\ &= 3000(tan(63) - tan(51)) \\ W &\cong 2183\ ft \end{aligned}$$

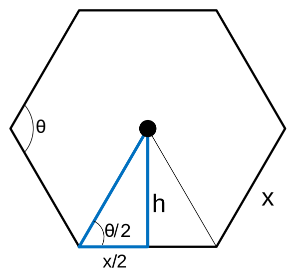
We can find the distance, d, using any of the trigonometric functions, or even the Pythagorean theorem. We compute below using the cosine function.

$$cos(51) = \frac{3000}{d} \rightarrow\ d = \frac{3000}{cos(51)} \cong 4767\ ft$$

Example 7: Use trigonometry to derive a formula for the area of a regular hexagon with side length x.



Solution: To start we draw a right triangle by dropping a line from the center of the hexagon as shown below. This triangle has a base length of $x/2$, and a height of h. The height can be determined using trigonometry, but first we need to find the angle, θ.



Recall that the sum of the interior angles of a polygon with n sides is given by

$$\theta_T = (n-2)180^{\circ}$$

Furthermore, since the figure is a *regular* polygon all interior angles are equal. Therefore,

$$\theta = \frac{\theta_T}{6} = \frac{(6-2)180^{\circ}}{6} = \frac{2}{3}180^{\circ} = 120^{\circ}$$

With the angle being known we can use the tangent function to find the height of the triangle as a function of x.

$$tan(\theta/2) = \frac{h}{x/2} \rightarrow h = \frac{x\,tan(60)}{2} \rightarrow h = \frac{\sqrt{3}}{2}x$$

The area of the right triangle is one half of the base times the height.

$$A_T = \frac{1}{2}(b)(h) = \left(\frac{1}{2}\right)\left(\frac{x}{2}\right)\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{8}x^2$$

The hexagon consists of 12 of these triangles. Therefore, the area of a regular hexagon with side length, x, is given as

$$A_H = 12A_T$$

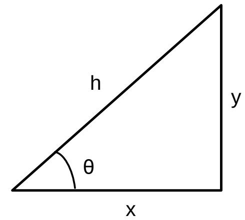
$$A_H = 12\frac{\sqrt{3}}{8}x^2$$

$$A_H = \frac{3\sqrt{3}}{2}x^2$$

Final Summary for Trigonometry – Right Triangle Trigonometry

Basic Right Angle Trigonometric Functions

Six trigonometric functions are defined as ratios of the lengths of the sides of a right triangle.



$\sin(\theta) = \frac{opposite}{hypotenuse} = \frac{y}{h}$	$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{hypotenuse}{opposite} = \frac{h}{y}$
$\cos(\theta) = \frac{adjacent}{hypotenuse} = \frac{x}{h}$	$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{hypotenuse}{adjacent} = \frac{h}{x}$
$\tan(\theta) = \frac{opposite}{adjacent} = \frac{y}{x}$	$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{adjacent}{opposite} = \frac{x}{y}$

Trigonometric Function Values for Special Angles

$\boldsymbol{\theta}$	$\mathbf{sin}(\boldsymbol{\theta})$	$\mathbf{cos}(\boldsymbol{\theta})$	$\mathbf{tan}(\boldsymbol{\theta})$	$\mathbf{csc}(\boldsymbol{\theta})$	$\mathbf{sec}(\boldsymbol{\theta})$	$\mathbf{cot}(\boldsymbol{\theta})$
$\mathbf{0}^\circ$	0	1	0	∞	1	∞
$\mathbf{30}^\circ$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\mathbf{45}^\circ$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\mathbf{60}^\circ$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{2}$
$\mathbf{90}^\circ$	1	0	∞	1	∞	0

The above table can be easily recalled using the following relationships.

$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$	$\csc(\theta) = \frac{1}{\sin(\theta)}$	$\sec(\theta) = \frac{1}{\cos(\theta)}$	$\cot(\theta) = \frac{1}{\tan(\theta)}$

And by memorizing the much simpler table with since and cosine only.

$\boldsymbol{\theta}$	$\mathbf{sin}(\boldsymbol{\theta})$	$\mathbf{cos}(\boldsymbol{\theta})$
$\mathbf{0}^\circ$	$\left(\frac{1}{2}\right)\sqrt{0}$	$\left(\frac{1}{2}\right)\sqrt{4}$
$\mathbf{30}^\circ$	$\left(\frac{1}{2}\right)\sqrt{1}$	$\left(\frac{1}{2}\right)\sqrt{3}$
$\mathbf{45}^\circ$	$\left(\frac{1}{2}\right)\sqrt{2}$	$\left(\frac{1}{2}\right)\sqrt{2}$
$\mathbf{60}^\circ$	$\left(\frac{1}{2}\right)\sqrt{3}$	$\left(\frac{1}{2}\right)\sqrt{1}$
$\mathbf{90}^\circ$	$\left(\frac{1}{2}\right)\sqrt{4}$	$\left(\frac{1}{2}\right)\sqrt{0}$

By: ferrantetutoring