Trigonometry – Trigonometric Functions of Non-Acute Angles

In the previous section we introduced the trigonometric functions for right triangles, i.e. $0^{\circ} \leq \theta \leq 90^{\circ}$. In this section we extend the trigonometric functions to all angles. We do this with the rectangular coordinate system, using the x and y coordinates of a point as the side lengths of a right triangle. With this the applicability of trigonometric functions is greatly extended.

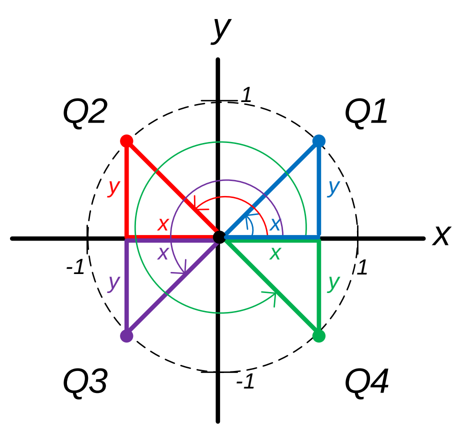
Angles greater than 90°

First note that changing the length of the hypotenuse of a right triangle along a straight line does not change the interior angles. Because of this we can use an arbitrary hypotenuse length for defining trigonometric angles. For convenience we will use a hypotenuse length of 1.

To extend the trigonometric functions beyond 90° we'll start by drawing a unit circle centered at the origin of the x-y plane. A right triangle can be constructed using a line drawn from the origin to any point on the circle as the hypotenuse. There are four different cases for each of the four quadrants in terms of the sign, i.e. positive or negative, of the side lengths.

- Quadrant 1:
  - x and y side lengths are both positive.
- Quadrant 2:
  - x is negative and y is positive.
- Quadrant 3:
  - x and y side lengths are both negative.
- Quadrant 4:
  - x is positive and y is negative.

In all cases the hypotenuse, $h = \sqrt{x^2 + y^2}$, is positive.



We use the triangles shown above to extend the trigonometric functions for $0^{\circ} \leq \theta < 360^{\circ}$. This is accomplished by defining a so-called reference angle, β, which represents the interior angle of the triangles and is between 0° and 90°. To find the value of a trigonometric function where $0^{\circ} \leq \theta < 360^{\circ}$ we proceed as follows.

1. Find the reference angle, β, which is a function of θ and the quadrant where θ exists.

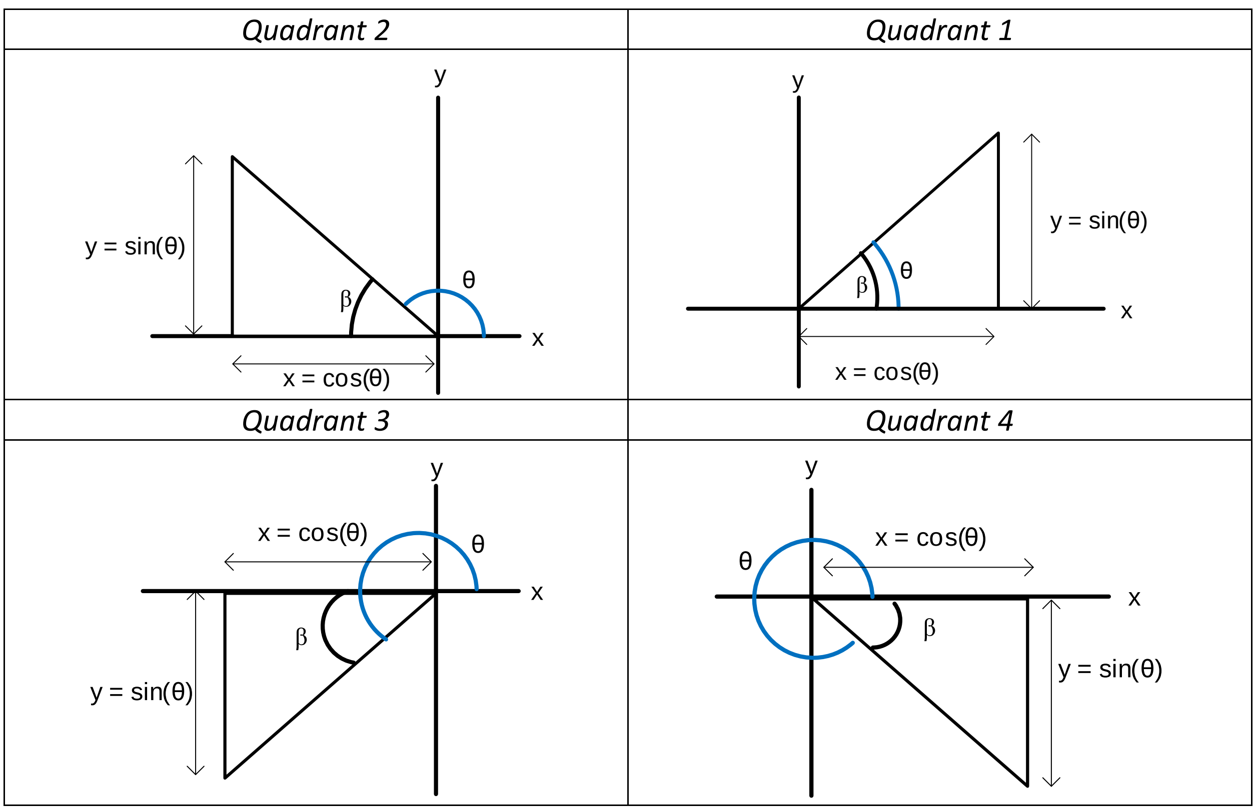
Quadrant 1	Quadrant 2	Quadrant 3	Quadrant 4
$\beta = \theta$	$\beta = 180^{\circ} - \theta$	$\beta = \theta - 180^{\circ}$	$\beta = 360^{\circ} - \theta$

2. Evaluate the required trigonometric function using this angle, e.g. $sin(\beta)$, to find the magnitude of the value.

3. Determine the sign of the value based on the quadrant of θ.
   - The sign of the various trigonometric functions are determined by the sign of the x and y coordinates as summarized below. Recall, on the unit circle $h = 1$.

<table>
<tr>
<td>$sin(\theta) = \frac{\boldsymbol{y}}{h} = y, \qquad csc(\theta) = \frac{1}{sin(\theta)}$</td>
<td>Follows the sign of the y coordinate, i.e.<table><tr><td>$Q1$ and $Q2$</td><td>Positive</td></tr><tr><td>$Q3$ and $Q4$</td><td>Negative</td></tr></table></td>
</tr>
<tr>
<td>$cos(\theta) = \frac{\boldsymbol{x}}{h} = x, \qquad sec(\theta) = \frac{1}{cos(\theta)}$</td>
<td>Follows the sign of the x coordinate, i.e.<table><tr><td>$Q1$ and $Q4$</td><td>Positive</td></tr><tr><td>$Q2$and $Q3$</td><td>Negative</td></tr></table></td>
</tr>
<tr>
<td>$tan(\theta) = \frac{\boldsymbol{y}}{\boldsymbol{x}}, \qquad cot(\theta) = \frac{1}{tan(\theta)}$</td>
<td>Positive when x and y are same signs and negative when they are opposite, i.e.<table><tr><td>$Q1$ and $Q3$</td><td>Positive</td></tr><tr><td>$Q2$and $Q4$</td><td>Negative</td></tr></table></td>
</tr>
</table>

The figures below illustrates the above procedure. You should verify each explanation from above by examining each of the figures.



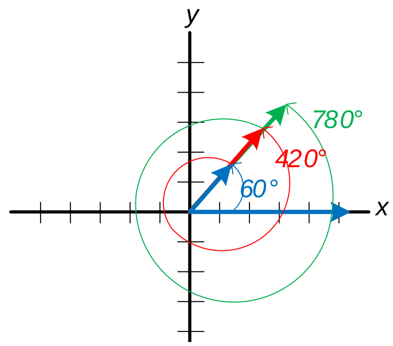
Using the procedure above we construct the table below giving values of sine, cosine, and tangent for special angles between 0° and 360°.

Quadrant	$\boldsymbol{\theta}$	$\boldsymbol{\beta}$	$\mathbf{sin}(\boldsymbol{\theta})$	$\mathbf{cos}(\boldsymbol{\theta})$	$\mathbf{tan}(\boldsymbol{\theta})$
First Quadrant (x and y positive)	0°	0°	0	1	0
	30°	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
	45°	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
	60°	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
	90°	90°	1	0	∞
Second Quadrant (x negative, y positive)	120°	60°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
	135°	45°	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
	150°	30°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
	180°	0°	0	-1	
Third Quadrant (x and y negative)	210°	30°	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
	225°	45°	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
	240°	60°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
	270°	90°	-1	0	$-\infty$
Fourth Quadrant (x positive, y negative)	300°	60°	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
	315°	45°	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
	330°	30°	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
	360°	0°	0	1	0

Finally, to extend beyond $0^{\circ} \leq \theta \leq 360^{\circ}$ we use the notion of coterminal angles. Recall the definition of coterminal angles.

Coterminal Angles: Angles that have the same terminal side but have gone through a different number of rotations to arrive there.

$$\theta_{ct(n\pm)} = \theta \pm n \cdot 360^{\circ}$$



Using the sine as an example trigonometric function, we can write the following.

$$sin(\theta \pm n \cdot 360^{\circ}) = sin(\theta), \qquad for\ n = 0, 1, 2, \ldots.$$

Let's do some examples to practice the concepts introduced in the lesson.

Example 1: Find the exact values of the six trigonometric functions for each given angle.

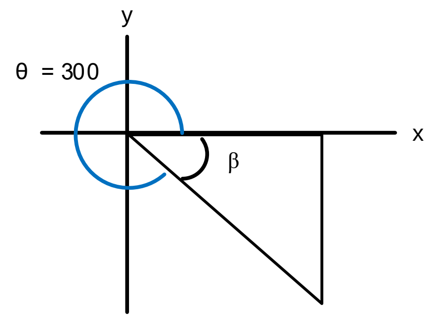
a. 300° *b.* 135° *c.* 210°

d. -225° *e.* 1305° *f.* -390°

Solution: We'll solve the problems assuming that we do not have access to the larger table above, but we have used the memorization technique from the previous lesson to recall the following small table. Furthermore, we will detail each step

$\boldsymbol{\theta}$	$\mathbf{sin(\boldsymbol{\theta})}$	$\mathbf{cos(\boldsymbol{\theta})}$
$\mathbf{0^{\circ}}$	$\left(\frac{1}{2}\right)\sqrt{0} = 0$	$\left(\frac{1}{2}\right)\sqrt{4} = 1$
$\mathbf{30^{\circ}}$	$\left(\frac{1}{2}\right)\sqrt{1} = \frac{1}{2}$	$\left(\frac{1}{2}\right)\sqrt{3} = \frac{\sqrt{3}}{2}$
$\mathbf{45^{\circ}}$	$\left(\frac{1}{2}\right)\sqrt{2} = \frac{\sqrt{2}}{2}$	$\left(\frac{1}{2}\right)\sqrt{2} = \frac{\sqrt{2}}{2}$
$\mathbf{60^{\circ}}$	$\left(\frac{1}{2}\right)\sqrt{3} = \frac{\sqrt{3}}{2}$	$\left(\frac{1}{2}\right)\sqrt{1} = \frac{1}{2}$
$\mathbf{90^{\circ}}$	$\left(\frac{1}{2}\right)\sqrt{4} = 1$	$\left(\frac{1}{2}\right)\sqrt{0} = 0$

a. The given angle, 300°, corresponds to a right triangle in the fourth quadrant as shown below.



Once we draw this picture it is straightforward to see that the reference angle, β, is 60°, which we can also find by using the formula from above.

$$\beta = 360^{\circ} - \theta = 360^{\circ} - 300^{\circ} = 60^{\circ}$$

However, rather than trying to memorize the various formulas its always better to draw the angle and determine visually.

Next, we find the magnitude of the sine and cosine using the small table from above.

$$sin(60) = \frac{\sqrt{3}}{2} \qquad cos(60) = \frac{1}{2}$$

In the fourth quadrant sine is negative and cosine is positive, therefore,

$$sin(300) = -\frac{\sqrt{3}}{2} \qquad cos(300) = \frac{1}{2}$$

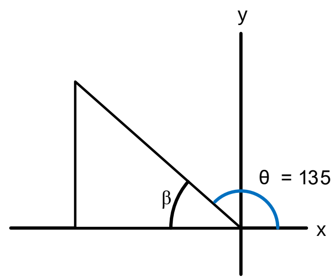
Finally, all other trigonometric functions can be derived from these two values as shown.

$$sin(300) = -\frac{\sqrt{3}}{2} \qquad csc(300) = \frac{1}{sin(300)} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$cos(300) = \frac{1}{2} \qquad sec(300) = \frac{1}{cos(300)} = 2$$

$$tan(300) = \frac{sin(300)}{cos(300)} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3} \qquad cot(300) = \frac{1}{tan(300)} = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

b. In this case we have a triangle in the second quadrant.



Therefore, $\beta = 180^{\circ} - \theta = 180^{\circ} - 135^{\circ} = 45^{\circ}$. The magnitudes for sine and cosine are then

$$sin(45) = \frac{\sqrt{2}}{2} \qquad cos(45) = \frac{\sqrt{2}}{2}$$

And since x is negative and y is positive, we find

$$sin(135) = \frac{\sqrt{2}}{2} \qquad cos(135) = -\frac{\sqrt{2}}{2}$$

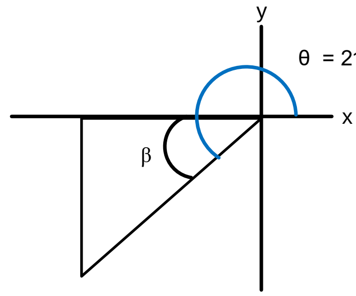
Finally, the six trigonometric functions are as shown below.

$$sin(135) = \frac{\sqrt{2}}{2} \qquad csc(135) = \frac{1}{sin(135)} = \sqrt{2}$$

$$cos(135) = -\frac{\sqrt{2}}{2} \qquad sec(135) = \frac{1}{cos(135)} = -\sqrt{2}$$

$$tan(135) = \frac{sin(135)}{cos(135)} = -1 \qquad cot(135) = \frac{1}{tan(135)} = -1$$

c. In this case we have a triangle in the third quadrant.



Therefore, $\beta = \theta - 180^{\circ} = 210^{\circ} - 180^{\circ} = 30^{\circ}$. The magnitudes for sine and cosine are then

$$sin(30) = \frac{1}{2} \qquad cos(30) = \frac{\sqrt{3}}{2}$$

And since x and y are both negative we find

$$sin(210) = -\frac{1}{2} \qquad cos(210) = -\frac{\sqrt{3}}{2}$$

Finally, the six trigonometric functions are as shown below.

$$sin(210) = -\frac{1}{2} \qquad csc(210) = \frac{1}{sin(210)} = -2$$

$$cos(210) = -\frac{\sqrt{3}}{2} \qquad sec(210) = \frac{1}{cos(210)} = -\frac{2\sqrt{3}}{3}$$

$$tan(210) = \frac{sin(210)}{cos(210)} = \frac{\sqrt{3}}{3} \qquad cot(210) = \frac{1}{tan(210)} = \sqrt{3}$$

d. In this case, we start by finding the least positive coterminal angle.

$$-225^{\circ} + 360^{\circ} = 135^{\circ}$$

Which is the same angle from problem b. Therefore, we have

$$sin(135) = \frac{\sqrt{2}}{2} \qquad csc(135) = \frac{1}{sin(135)} = \sqrt{2}$$

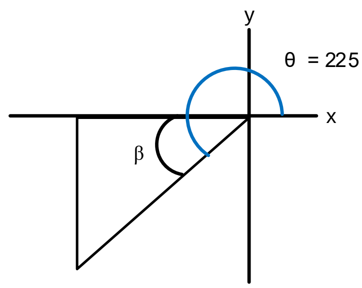
$$cos(135) = -\frac{\sqrt{2}}{2} \qquad sec(135) = \frac{1}{cos(135)} = -\sqrt{2}$$

$$tan(135) = \frac{sin(135)}{cos(135)} = -1 \qquad cot(135) = \frac{1}{tan(135)} = -1$$

e. We again start by finding the least positive coterminal angle.

$$1305^\circ - (3 \cdot 360^\circ) = 225^\circ$$

Which is in the third quadrant.



Therefore, $\beta = \theta - 180^\circ = 225^\circ - 180^\circ = 45^\circ$. The magnitudes for sine and cosine are then

$$sin(45) = \frac{\sqrt{2}}{2} \qquad cos(45) = \frac{\sqrt{2}}{2}$$

And since x and y are both negative we find

$$sin(225) = -\frac{\sqrt{2}}{2} \qquad cos(225) = -\frac{\sqrt{2}}{2}$$

Finally, the six trigonometric functions are as shown below.

$$sin(225) = -\frac{\sqrt{2}}{2} \qquad csc(225) = \frac{1}{sin(225)} = -\sqrt{2}$$

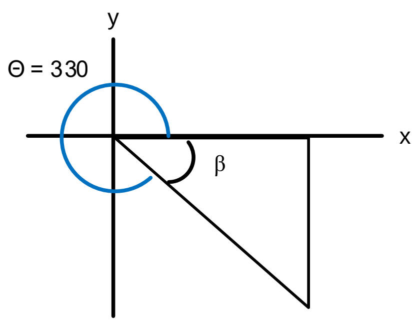
$$cos(225) = -\frac{\sqrt{2}}{2} \qquad sec(225) = \frac{1}{cos(225)} = -\sqrt{2}$$

$$tan(225) = \frac{sin(225)}{cos(225)} = 1 \qquad cot(225) = \frac{1}{tan(225)} = 1$$

f. We again start by finding the least positive coterminal angle.

$$-390^\circ + 360^\circ = -30^\circ + 360^\circ = 330^\circ$$

Which is in the fourth quadrant.



Therefore, $\beta = 360^\circ - \theta = 360^\circ - 330^\circ = 30^\circ$. The magnitudes for sine and cosine are then

$$sin(30) = \frac{1}{2} \qquad cos(30) = \frac{\sqrt{3}}{2}$$

And since x is positive and y is negative we find

$$sin(330) = -\frac{1}{2} \qquad cos(330) = \frac{\sqrt{3}}{2}$$

Finally, the six trigonometric functions are as shown below.

$$sin(330) = -\frac{1}{2} \qquad csc(330) = \frac{1}{sin(330)} = -2$$

$$cos(330) = \frac{\sqrt{3}}{2} \qquad sec(330) = \frac{1}{cos(330)} = \frac{2}{3\sqrt{3}}$$

$$tan(330) = \frac{sin(330)}{cos(330)} = -\frac{\sqrt{3}}{3} \qquad cot(330) = \frac{1}{tan(330)} = -\sqrt{3}$$

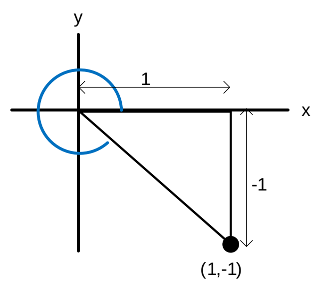
Example 2: Let the given point in the x-y plane represent the endpoint of the hypotenuse of a right triangle. Use this point to evaluate the six trigonometric functions of θ.

a. $(-1,1)$ *b.* $(-12,5)$ *c.* $(6,-9)$

d. $(1,-\sqrt{3})$ *e.* $(-3,-4)$ *f.* $(-9,14)$

Solution:

a. In this case we can draw a right triangle based on the x-y coordinate given.



There is no need to determine the angle since we know the two side lengths. Furthermore, the hypotenuse is given by the Pythagorean Theorem.

$$h = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

The six trigonometric functions can be found using the side lengths as follows.

$$sin(\theta) = \frac{y}{h} = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \qquad csc(\theta) = \frac{h}{y} = -\sqrt{2}$$

$$cos(\theta) = \frac{x}{h} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad sec(\theta) = \frac{h}{x} = \sqrt{2}$$

$$tan(\theta) = \frac{y}{x} = \frac{-1}{1} = -1 \qquad cot(\theta) = \frac{x}{y} = -1$$

b. We can find the six trigonometric functions in a similar way since we again know the side lengths. First, we find the hypotenuse.

$$h = \sqrt{(-12)^2 + (5)^2} = \sqrt{169} = 13$$

$$sin(\theta) = \frac{y}{h} = \frac{5}{13} \qquad csc(\theta) = \frac{h}{y} = \frac{13}{5}$$

$$cos(\theta) = \frac{x}{h} = -\frac{12}{13} \qquad sec(\theta) = \frac{h}{x} = -\frac{13}{12}$$

$$tan(\theta) = \frac{y}{x} = -\frac{5}{12} \qquad cot(\theta) = \frac{x}{y} = -\frac{12}{5}$$

c. With $x = 6$ and $y = -9$, the hypotenuse is

$$h = \sqrt{(6)^2 + (-9)^2} = \sqrt{117} = \sqrt{9 \cdot 13} = 3\sqrt{13}$$

$$sin(\theta) = \frac{y}{h} = -\frac{9}{3\sqrt{13}} = -\frac{3\sqrt{13}}{13}$$

$$cos(\theta) = \frac{x}{h} = \frac{6}{3\sqrt{13}} = \frac{2\sqrt{13}}{13}$$

$$tan(\theta) = \frac{y}{x} = -\frac{9}{6} = -\frac{3}{2}$$

$$csc(\theta) = \frac{h}{y} = -\frac{\sqrt{13}}{3}$$

$$sec(\theta) = \frac{h}{x} = \frac{\sqrt{13}}{2}$$

$$cot(\theta) = \frac{x}{y} = -\frac{2}{3}$$

d. With $x = 1$ and $y = -\sqrt{3}$, the hypotenuse is

$$h = \sqrt{(1)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{4} = 2$$

$$sin(\theta) = \frac{y}{h} = -\frac{\sqrt{3}}{2}$$

$$cos(\theta) = \frac{x}{h} = \frac{1}{2}$$

$$tan(\theta) = \frac{y}{x} = -\frac{\sqrt{3}}{1} = -\sqrt{3}$$

$$csc(\theta) = \frac{h}{y} = -\frac{2\sqrt{3}}{3}$$

$$sec(\theta) = \frac{h}{x} = 2$$

$$cot(\theta) = \frac{x}{y} = -\frac{\sqrt{3}}{3}$$

e. With $x = -3$ and $y = -4$, the hypotenuse is

$$h = \sqrt{(-3)^2 + (-4)^2} = \sqrt{25} = 5$$

$$sin(\theta) = \frac{y}{h} = -\frac{4}{5}$$

$$cos(\theta) = \frac{x}{h} = -\frac{3}{5}$$

$$tan(\theta) = \frac{y}{x} = \frac{4}{3}$$

$$csc(\theta) = \frac{h}{y} = -\frac{5}{4}$$

$$sec(\theta) = \frac{h}{x} = -\frac{5}{3}$$

$$cot(\theta) = \frac{x}{y} = -\frac{3}{4}$$

f. With $x = -9$ and $y = 14$, the hypotenuse is

$$h = \sqrt{(-9)^2 + (14)^2} = \sqrt{227}$$

$$sin(\theta) = \frac{y}{h} = \frac{14}{\sqrt{227}} = \frac{14\sqrt{227}}{227}$$

$$cos(\theta) = \frac{x}{h} = -\frac{9}{\sqrt{227}} = -\frac{9\sqrt{227}}{227}$$

$$tan(\theta) = \frac{y}{x} = -\frac{14}{9}$$

$$csc(\theta) = \frac{h}{y} = \frac{\sqrt{227}}{14}$$

$$sec(\theta) = \frac{h}{x} = -\frac{\sqrt{227}}{9}$$

$$cot(\theta) = \frac{x}{y} = -\frac{9}{14}$$

Example 3: Evaluate the trigonometric expression given.

a. $sin^2(120^\circ) + cos^2(120^\circ)$

b. $cos^2(60^\circ) + sec^2(150^\circ) - csc^2(210^\circ)$

c. $2\,tan^2(120^\circ) + 3\,sin^2(150^\circ) - cos^2(180^\circ)$

Solution: We evaluate trigonometric expressions just like any other expression. In this case we use the larger table as a reference.

a.

$$sin^2(120^\circ) + cos^2(120^\circ) = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1$$

b.

$$cos^2(60^\circ) + sec^2(150^\circ) - csc^2(210^\circ) = \left(\frac{1}{2}\right)^2 + \left(-\frac{2}{\sqrt{3}}\right)^2 - (-2)^2$$

$$= \frac{1}{4} + \frac{4}{3} - 4$$

$$= -\frac{29}{12}$$

c.

$$2\,tan^2(120^\circ) + 3\,sin^2(150^\circ) - cos^2(180^\circ) = 2\left(-\sqrt{3}\right)^2 + 3\left(\frac{1}{2}\right)^2 - (-1)^2$$

$$= 6 + \frac{3}{4} - 1$$

$$= \frac{23}{4}$$

Final Summary for Trigonometry – Trigonometric Functions of Non-Acute Angles

Trigonometric Functions Values Procedure

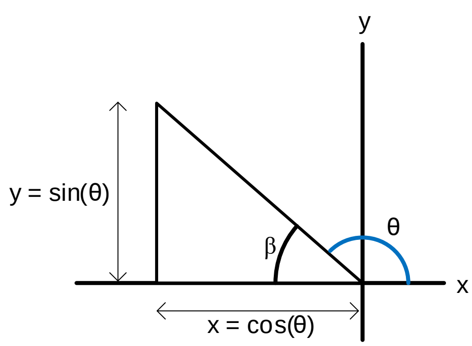
Procedure:

1. Find the reference angle, β, based on the quadrant of θ.
2. Evaluate the function based on the reference angle to find the magnitude, i.e. $\sin(\beta)$
3. Determine the sign of the value based on the quadrant of the original angle, θ.

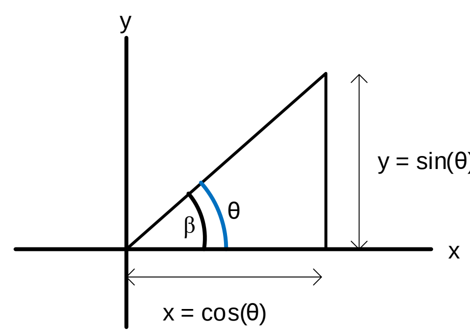
Quadrant Rules:

- Quadrant 1: $\beta = \theta$
  - Sine and cosine function are both positive.
- Quadrant 2: $\beta = 180^{\circ} - \theta$
  - Sine function is positive, cosine function is negative.
- Quadrant 3: $\beta = \theta - 180^{\circ}$
  - Sine function is negative, cosine function is negative.
- Quadrant 4: $\beta = 360^{\circ} - \theta$
  - Sine function is negative, cosine function is positive.

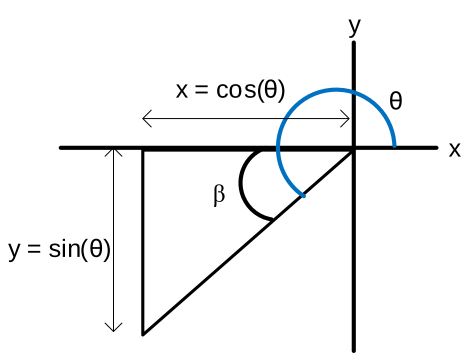
Quadrant 2



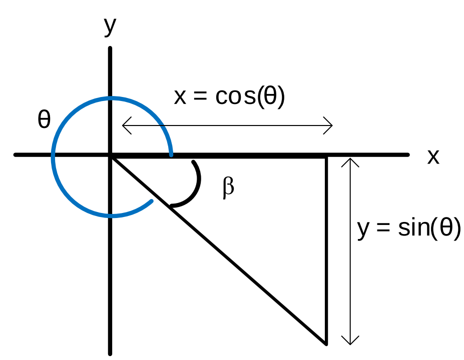
Quadrant 1



Quadrant 3



Quadrant 4



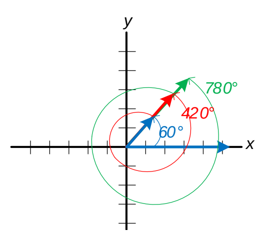
Trigonometric Functions Table for Special Angles $0^{\circ} \le \theta \le 360^{\circ}$

Quadrant	$\boldsymbol{\theta}$	$\boldsymbol{\beta}$	$\mathbf{sin}(\boldsymbol{\theta})$	$\mathbf{cos}(\boldsymbol{\theta})$	$\mathbf{tan}(\boldsymbol{\theta})$
First Quadrant (x and y positive)	0°	0°	0	1	0
	30°	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
	45°	45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
	60°	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
	90°	90°	1	0	∞
Second Quadrant (x negative, y positive)	120°	60°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
	135°	45°	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
	150°	30°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
	180°	0°	0	-1	
Third Quadrant (x and y negative)	210°	30°	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
	225°	45°	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
	240°	60°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
	270°	90°	-1	0	$-\infty$
Fourth Quadrant (x positive, y negative)	300°	60°	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
	315°	45°	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
	330°	30°	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
	360°	0°	0	1	0

Angles beyond $0^{\circ} \le \theta \le 360^{\circ}$

Coterminal Angles: Angles that have the same terminal side but have gone through a different number of rotations to arrive there.

$$\theta_{ct(n\pm)} = \theta \pm n \cdot 360^{\circ}$$



Using the sine as an example trigonometric function, we can write the following.

$$sin(\theta \pm n \cdot 360^{\circ}) = sin(\theta)\,, \qquad for\ n = 0, 1, 2, \ldots.$$

By: ferrantetutoring