Trigonometry – Angle and Triangle Basics

Trigonometry is a field of mathematics that studies the relationships between the side lengths and the measures of the angles of triangles. Applications of trigonometry appear in a wide variety of fields. In addition to more sophisticated science and engineering application trigonometry also plays a vital role in architecture, construction, navigation, and many others. Special functions, called trigonometric functions, are defined specifically as relationships between side lengths and angles for right triangles. Before formally introducing these functions this first lesson reviews some basic angle and triangle relationships.

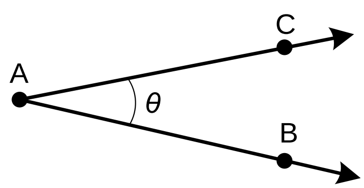
Angles

A *line*, denoted as , $\overleftrightarrow{AB}$, is a one-dimensional figure that extends forever in both directions.

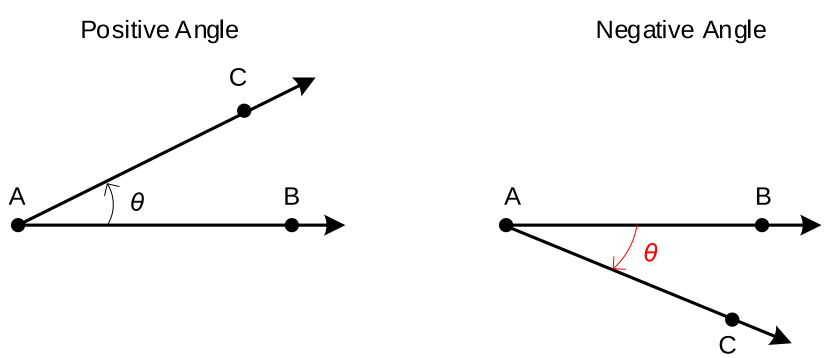
A *line segment*, denoted as , $\overline{AB}$, is a finite portion of a line.

A *ray*, denoted as , $\overrightarrow{AB}$, starts at a point A and continues forever in one direction.

Two rays can be placed so that they share a common endpoint, called the vertex.



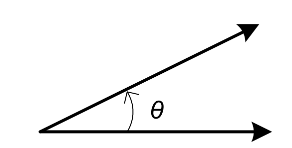
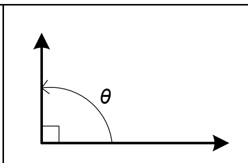
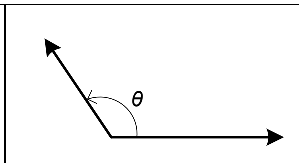
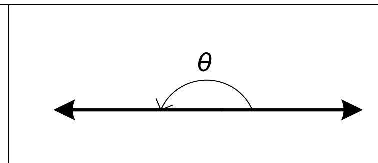
The measure of the 'distance' between the two rays is called the *angle,* θ. By leaving one ray fixed and rotating the other ray we change the measure of the angle. In trigonometry a counterclockwise rotation generates a positive measure, and a clockwise rotation generates a negative measure.



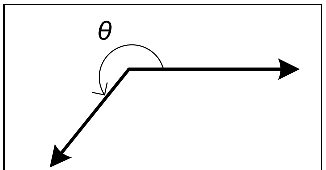
The most common unit to measure the angle is the *degree*. One full rotation of a ray is defined as 360 degrees, 360°. Therefore,

$$1^{\circ} = \frac{1}{360}\ of\ a\ single\ complete\ rotation$$

Other common angles and how we refer to them are shown below.

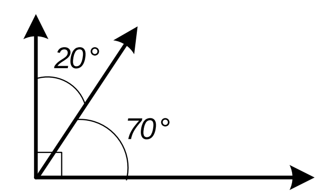
			
Acute Angle $0^{\circ} < \theta < 90^{\circ}$	Right Angle $\theta = 90^{\circ}$	Obtuse Angle $90^{\circ} < \theta < 180^{\circ}$	Straight Angle $\theta = 180^{\circ}$

Although we can always use the smaller angle as a measure, we sometimes need to refer to angles greater than 180°. We call this type of angle a reflex angle.

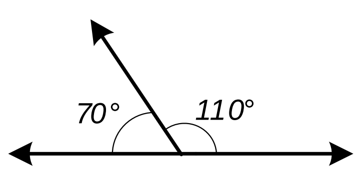

Reflex Angle $180^{\circ} < \theta < 360^{\circ}$

We also label certain relationships between angles, two of which are listed below.

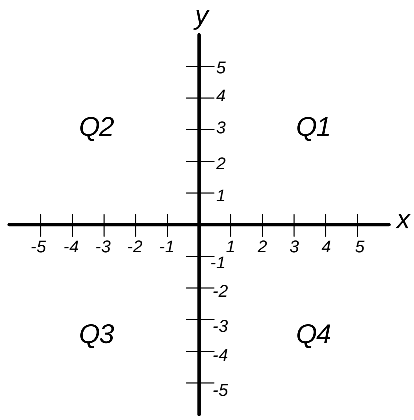
Complementary Angles: When two angles sum to 90°.



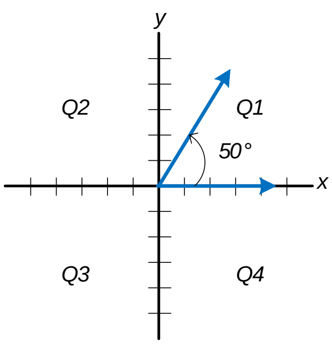
Supplementary Angles: When two angles sum to 180°.

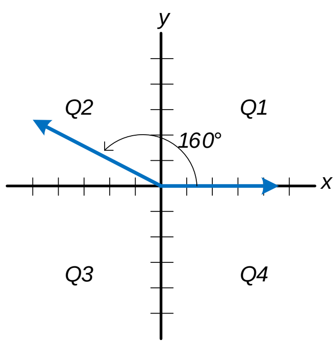


To keep track of points defined in a plane we reference them to a rectangular coordinate system. The standard is to define the point $(0,0)$ as the origin with increasing numbers going to the right for the x-axis and upward for the y-axis. In addition, the coordinate system is generally split into 4 quadrants as shown.



We can also use the rectangular coordinate system to define angles. In this case, the standard is to define the origin as the vertex and the positive x-axis as the initial ray. Angles defined in this manner are in ***standard position***. Two examples are shown below.





Note the relationship between quadrants and angles.

Quadrant I	$0^{\circ} < \theta < 90^{\circ}$
Quadrant II	$90^{\circ} < \theta < 180^{\circ}$
Quadrant III	$180^{\circ} < \theta < 270^{\circ}$
Quadrant IV	$270^{\circ} < \theta < 360^{\circ}$

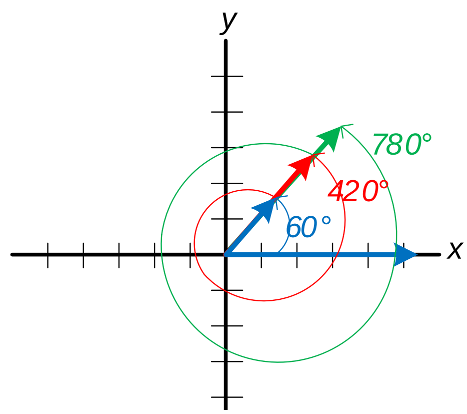
You may notice that as we rotate the ray beyond 360° the measure of the angle appears to repeat itself. Angles that have the same terminal side but have gone through a different number of rotations to arrive there are called *coterminal angles*. Each angle has an infinite number of coterminal angles. A relationship can be defined as

$$\theta_{ct(n\pm)} = \theta \pm n \cdot 360^{\circ}$$

For example, we can define two coterminal angles associated with a 60° angle as

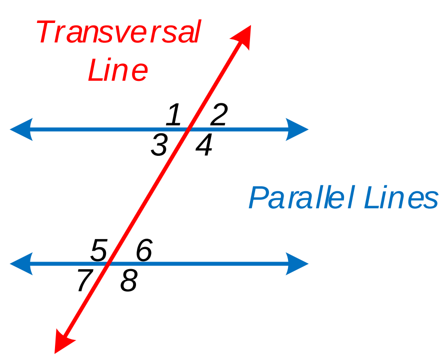
$$\theta_{ct(1)} = 60^{\circ} + 1 \cdot 360^{\circ} = 420^{\circ}$$

$$\theta_{ct(2)} = 60^{\circ} + 2 \cdot 360^{\circ} = 780^{\circ}$$

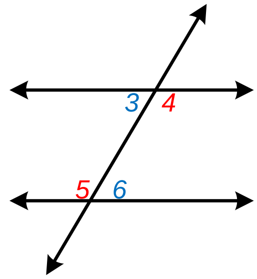
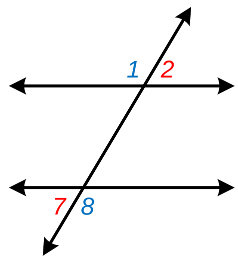
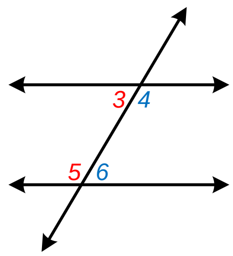
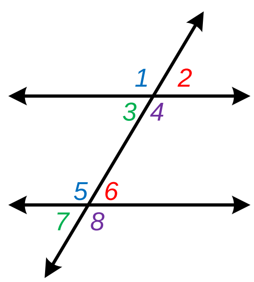


Next, we'll review some other angle relationships that you have probably seen before in your study of basic geometry.

Parallel lines are lines that lie in the same plane and do not intersect. When a line intersects two parallel lines it is called a *transversal*. This configuration of lines forms 8 angles that are all interrelated.



The table below defines the various relationships.

Alternate Interior Angles		*Angle measures are equal*
Alternate Exterior Angles		*Angle measures are equal*
Interior Angles on same side of transversal		*Angle measures add to* 180°
Corresponding Angles		*Angle measures are equal*

Next, we'll look at triangles. But before that let's do some examples to practice the above concepts.

Example 1: Find the complementary and supplementary angle that corresponds to each of the angles listed below.

a. 30° *b.* 54° *c.* 89°

Solution: Recall complementary angles sum to 90° and supplementary angles sum to 180°.

a.

$$\theta_{comp} = 90^\circ - 30^\circ = 60^\circ \qquad \theta_{supp} = 180^\circ - 30^\circ = 150^\circ$$

b.

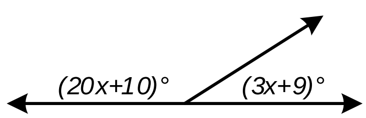
$$\theta_{comp} = 90^\circ - 54^\circ = 36^\circ \qquad \theta_{supp} = 180^\circ - 54^\circ = 126^\circ$$

c.

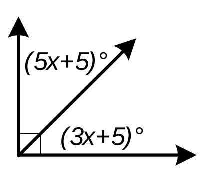
$$\theta_{comp} = 90^\circ - 89^\circ = 1^\circ \qquad \theta_{supp} = 180^\circ - 89^\circ = 91^\circ$$

Example 2: Find the measure of each unknown angle in the figures below.

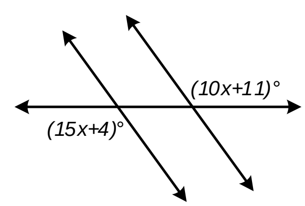
a.



b.



c.



Solution:

a. The two angles are supplementary; therefore, they should add to 180°.

$$(20x + 10) + (3x + 9) = 180$$
$$23x + 19 = 180$$
$$23x = 180 - 19$$
$$x = \frac{180 - 19}{23}$$
$$x = 7$$

The two angles can then be found by substitution

$$\begin{aligned} \theta &= 20x + 10 \\ &= 20 \cdot 7 + 10 \\ &= 150^\circ \end{aligned}$$

$$\begin{aligned} \theta &= 3x + 9 \\ &= 3 \cdot 7 + 9 \\ &= 30^\circ \end{aligned}$$

b. The two angles are complementary; therefore, they should add to 90°.

$$(5x + 5) + (3x + 5) = 90$$

$$8x + 10 = 90$$

$$8x = 90 - 10$$

$$x = \frac{90 - 10}{8}$$

$$x = 10$$

The two angles can then be found by substitution

$$\begin{aligned} \theta &= 5x + 5 \\ &= 5 \cdot 10 + 5 \\ &= 55^\circ \end{aligned}$$

$$\begin{aligned} \theta &= 3x + 5 \\ &= 3 \cdot 10 + 5 \\ &= 35^\circ \end{aligned}$$

c. In this case the angles shown are alternate exterior angles, and therefore are equal.

$$(15x - 54) = (10x + 11)$$

$$15x - 10x = 11 + 54$$

$$5x = 65$$

$$x = 13^\circ$$

The two angles can then be found by substitution. Both should evaluate to the same value.

$$\begin{aligned} \theta &= 15x - 54 \\ &= 15 \cdot 13 - 54 \\ &= 141^\circ \end{aligned}$$

$$\begin{aligned} \theta &= 10x + 11 \\ &= 10 \cdot 13 + 11 \\ &= 141^\circ \end{aligned}$$

Example 3: Find the angle of least positive measure that is coterminal with each angle given below.

a. 539° *b.* -541° *c.* 1000°

Solution: Recall coterminal angles are related by the following equation.

$$\theta_{ct(n\pm)} = \theta \pm n \cdot 360^\circ$$

Therefore, in all cases we subtract or add integer multiples of 360° until we find a positive angle that is less than 360°.

a. Subtracting the first 360° gives us the following

$$\theta = 539^\circ - 360^\circ = \mathbf{179}^\circ$$

Subtracting by 360° again would give results in a negative angle, therefore 179° is the least positive coterminal angle.

b. In this case, we start by adding 360°.

$$\theta = -541^\circ + 360^\circ = -181^\circ$$

Since this angle is still negative, we add another 360°.

$$\theta = -181^\circ + 360^\circ = \mathbf{179}^\circ$$

Which is again least positive coterminal angle.

c. In this case, we start out by subtracting $2 \cdot 360^\circ$.

$$\theta = 1000^\circ - 720^\circ = \mathbf{280}^\circ$$

Similar to *a*. subtracting another 360° would result in a negative angle, therefore 280° is the least positive coterminal angle.

Triangles

Triangles are at the heart of trigonometry. In this section we review some basic concepts that you may have seen in the past. The first formula you may remember relates to the sum of the interior angles of any polygon.

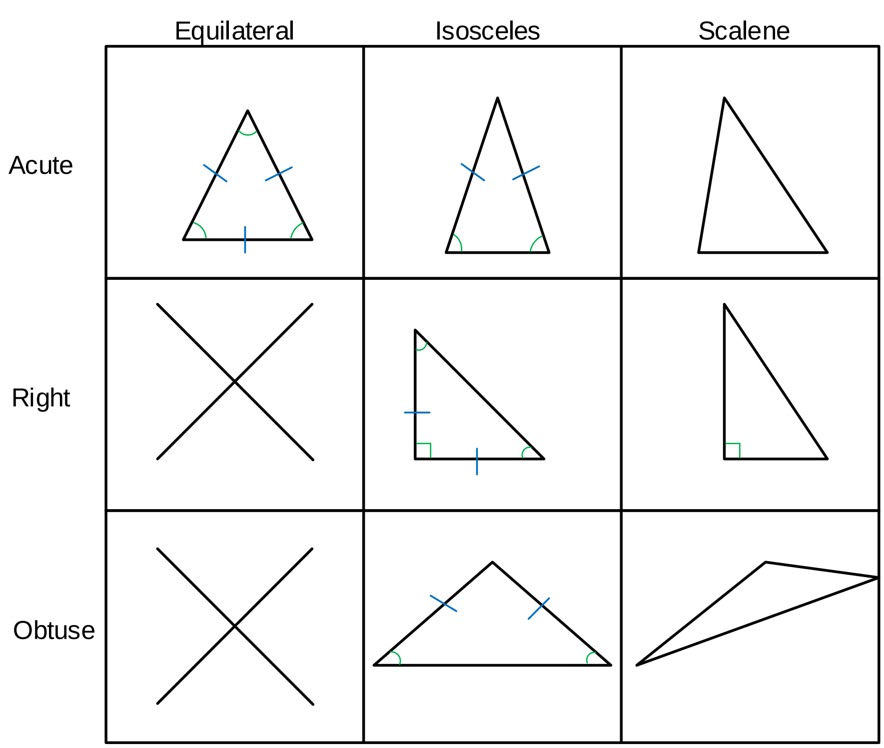
$$S = (n-2) \cdot 180^{\circ}$$

Where, n is the number of sides. In particular for triangles, where $n = 3$, we have

$$S_{tri} = (3-2) \cdot 180^{\circ} = 180^{\circ}$$

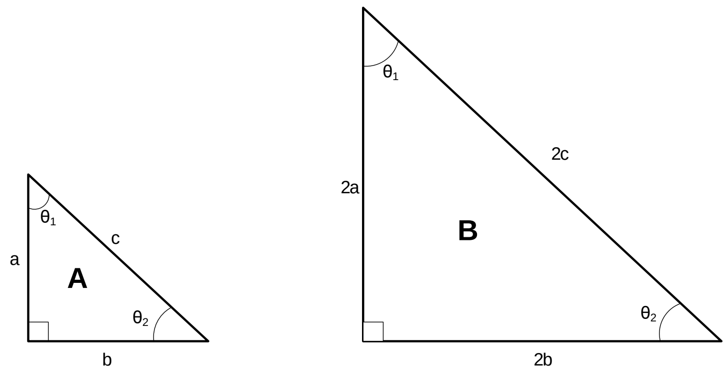
Triangle are classified according to the side lengths and interior angles as shown below.

- Side Lengths
  - ***Equilateral***
    - All three sides have equal length. This implies all 3 interior angles are 60°.
  - ***Isosceles***
    - Two sides have equal length. This implies 2 interior angles are the same.
  - ***Scalene***
    - No sides have the same length. This implies all 3 interior angles are different.
- Interior Angles
  - ***Acute***
    - All interior angles are acute.
  - ***Right***
    - One interior angle is a right angle.
  - ***Obtuse***
    - One interior angle is obtuse.



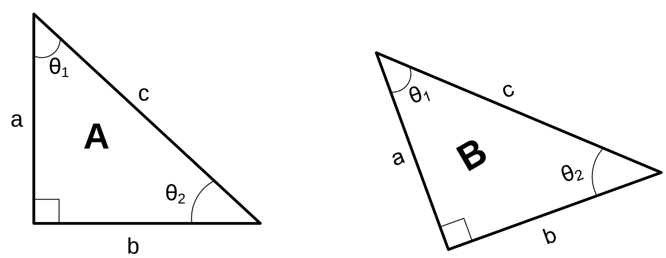
Lastly, there are two additional ways we use to classify how alike different triangles are.

Similar Triangles: Triangles that have the same exact shape, i.e. same interior angles and sides that are scaled versions of each other, but not the same size.



The example above shows triangle ***B*** is twice as big as triangle ***A***.

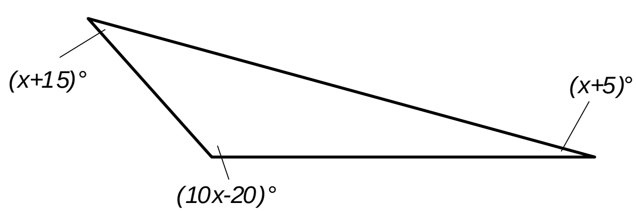
Congruent Triangles: Triangles that are the same size and the same shape.



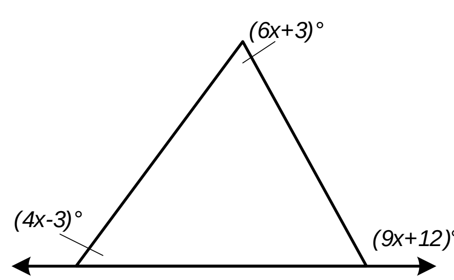
Note, in the example above triangle ***B*** is rotated with respect to triangle ***A***, however, the triangles are the same exact shape and size.

Example 4: Find the measure of each unknown angle in the figures below.

a.



b.



Solution:

a. The interior angles of a triangle sum to 180°. Therefore,

$$(x+15)+(10x-20)+(x+5)=180$$
$$x+10x+x=180-15+20-5$$
$$12x=180$$
$$x=\frac{180}{12}$$
$$x=15$$

The three interior angles are then.

$$\theta_1 = x+15 = 15+15 = 30^{\circ}$$
$$\theta_2 = 10x-20 = 10\cdot 15-20 = 130^{\circ}$$
$$\theta_3 = x+5 = 15+5 = 20^{\circ}$$

b. In this case the angle on the bottom right is an exterior angle. However, this angle and the corresponding interior angles are supplementary. Therefore, the interior angle, θ_3 is

$$\theta_3 = 180-(9x+12)=168-9x$$

Now, we can sum the interior angles to 180°.

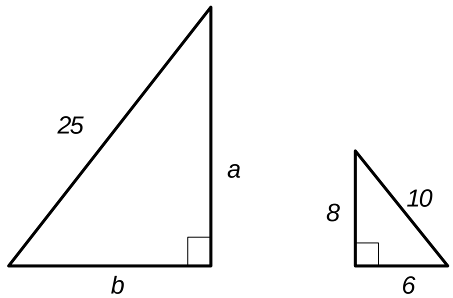
$$(4x-3)+(6x+3)+(168-9x)=180$$
$$4x+6x-9x=180+3-3-168$$
$$x=12$$

The three interior angles are then.

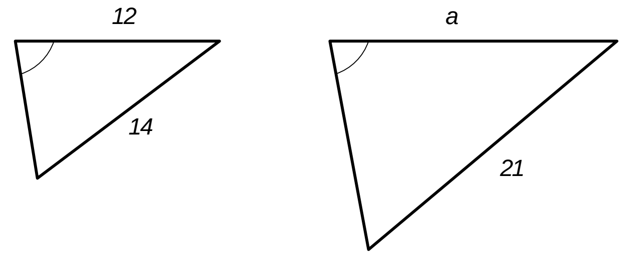
$$\theta_1 = 4x+3 = 4\cdot 12+3 = 45^{\circ}$$
$$\theta_2 = 6x+3 = 6\cdot 12+3 = 75^{\circ}$$
$$\theta_3 = 168-9x = 168-9\cdot 12 = 60^{\circ}$$

Example 5: Find the unknown side lengths for the pair of similar triangles shown below.

a.



b.



Solution:

a. Triangles that are similar are scaled versions of each other. We can determine the scale factor, x, based on the length of the hypotenuses.

$$10x = 25$$
$$x = \frac{25}{10}$$
$$x = 2.5$$

Therefore, each side is scaled by 2.5.

$$a = 2.5 \cdot 8 = 20 \qquad\qquad b = 2.5 \cdot 6 = 15$$

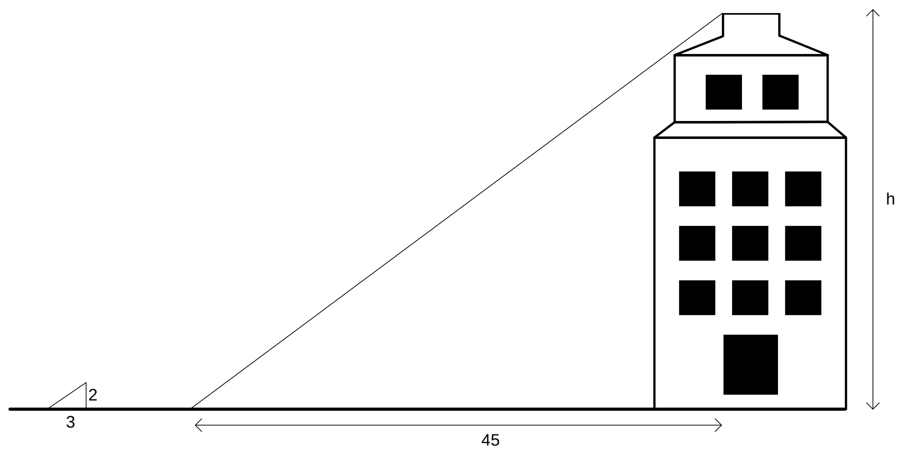
Note we can also create a proportion to find each side. We show it below for side a.

$$\frac{25}{10} = \frac{a}{8}$$
$$10 \cdot a = 25 \cdot 8$$
$$a = \frac{25 \cdot 8}{10}$$
$$a = 20$$

b. In this case, we can again set up a simple proportion.

$$\frac{21}{14} = \frac{a}{12}$$
$$14 \cdot a = 21 \cdot 12$$
$$a = \frac{21 \cdot 12}{14}$$
$$a = 18$$

Example 6: You would like to estimate the height of a building. You notice that it casts a shadow of 45m long. At the same time, you take a 2m stick and notice that it casts a shadow of 3m. What is a good estimate of the height if the building?



Solution: The triangle formed from the 2m stick and the building are approximately similar since they are both relative to the sun, which is far away. Using this fact we can use a simple proportion to estimate the height of the building.

$$\frac{3}{45} = \frac{2}{h}$$

$$3 \cdot h = 45 \cdot 2$$

$$h = \frac{45 \cdot 2}{3}$$

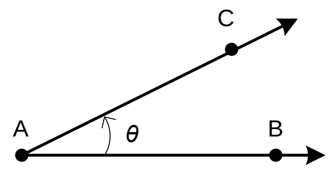
$$h = 30$$

Final Summary for Trigonometry – Angles and Triangle Basics

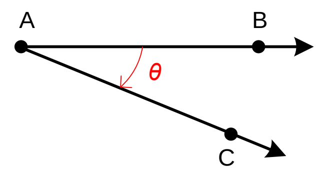
Angle Measurement

The measure of the ‘distance’ between the two rays is called the *angle,* θ. By leaving one ray fixed and rotating the other ray we change the measure of the angle. In trigonometry a counterclockwise rotation generates a positive measure, and a clockwise rotation generates a negative measure.

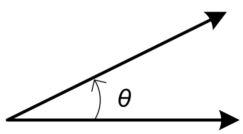
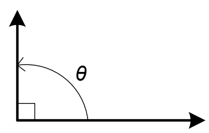
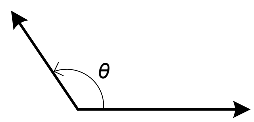
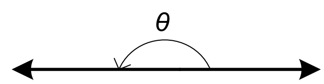
Positive Angle



Negative Angle



Angle Types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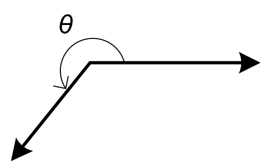
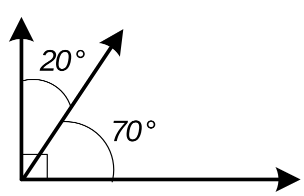
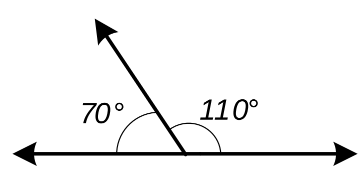
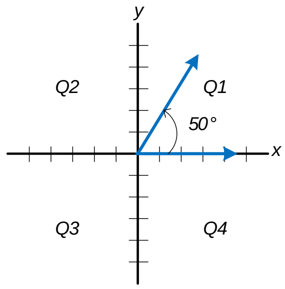
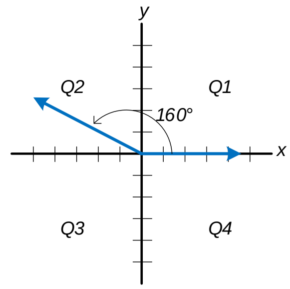

Reflex Angle $180^\circ < \theta < 360^\circ$

Complementary Angles: When two angles sum to 90°.	***Supplementary Angles***: When two angles sum to 180°.
	

Angles in Rectangular Coordinate System

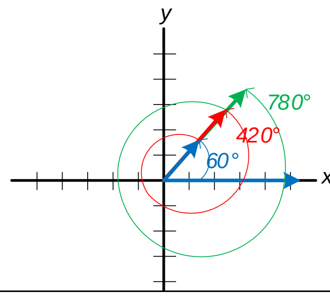
We can also use the rectangular coordinate system to define angles. In this case, the standard is to define the origin as the vertex and the positive x-axis as the initial ray. Angles defined in this manner are in ***standard position***. Two examples are shown below.



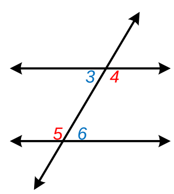
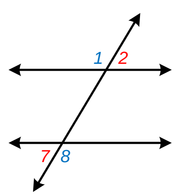
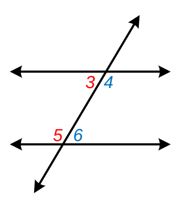
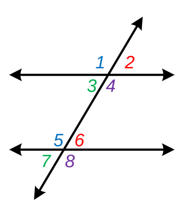


Coterminal Angles: Angles that have the same terminal side but have gone through a different number of rotations to arrive there.

$$\theta_{ct(n\pm)} = \theta \pm n \cdot 360^{\circ}$$



Angle Relationships

Alternate Interior Angles		*Angle measures are equal*
Alternate Exterior Angles		*Angle measures are equal*
Interior Angles on same side of transversal		*Angle measures add to* 180°
Corresponding Angles		*Angle measures are equal*

Triangle Classifications
Triangle are classified according to the following two criteria:

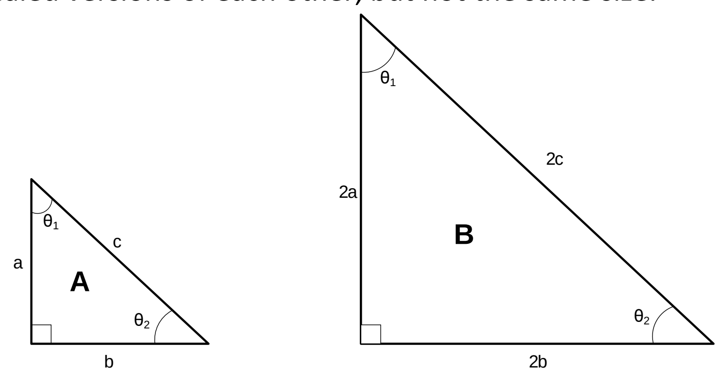
- Side Lengths
  - ***Equilateral***
    - All three sides have equal length. This implies all 3 interior angles are 60°.
  - ***Isosceles***
    - Two sides have equal length. This implies 2 interior angles are the same.
  - ***Scalene***
    - No sides have the same length. This implies all 3 interior angles are different.
- Interior Angles
  - ***Acute***
    - All interior angles are acute.
  - ***Right***
    - One interior angle is a right angle.
  - ***Obtuse***
    - One interior angle is obtuse.

Triangle Properties

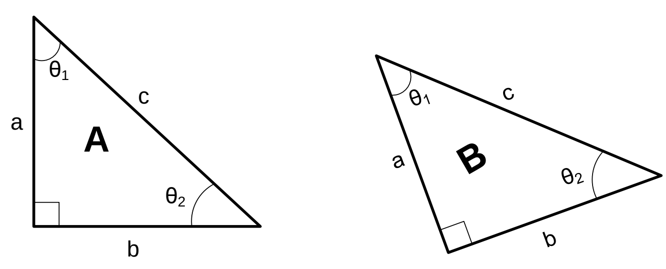
- Sum of Interior Angles of a Triangle is equal to 180°:

$$S_{tri} = 180^{\circ}$$

- ***Similar Triangles****:* Triangles that have the same exact shape, i.e. same interior angles and sides that are scaled versions of each other, but not the same size.



- ***Congruent Triangles:*** Triangles that are the same size and the same shape



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