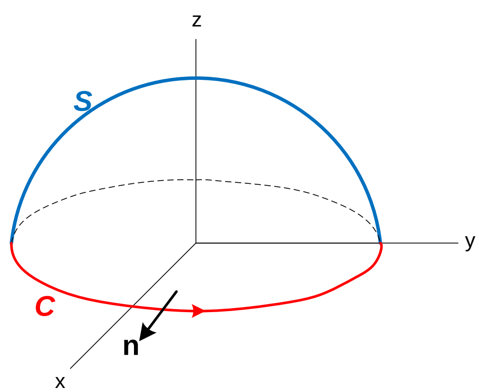
Fundamental Theorems of Vector Calculus – Stokes' Theorem

Stokes' Theorem is a direct extension of Green's Theorem to three dimensions. Recall that Green's Theorem relates a line integral over a closed curve that lives in two dimensions, i.e. a *planar curve*, to a double integral over the *planar region* bounded by the curve. Stokes' Theorem is able to deal with curves that are not restricted to a single plane. The theorem relates the line integral over a closed *curve* that lives in three dimensions to a double integral over the *surface* bounded by the curve, i.e. a surface integral. Before formally stating the theorem, we need to understand exactly how we define a surface boundary.

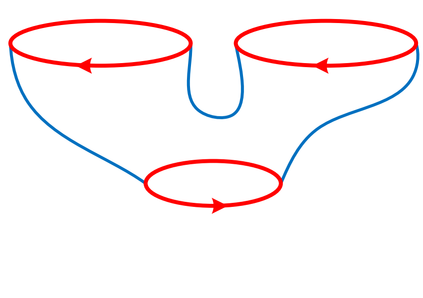
Surfaces and Surface Boundaries

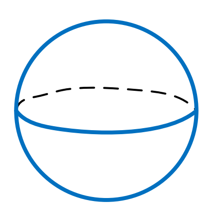
Different surfaces may have different types of boundaries. For example, the surface below has a single simple closed curve as its boundary. We define the orientation of the curve as follows:

- When you walk around the curve with your body pointing out in the direction of the normal vector, you should be walking in such a way that the surface is to your left side.



Other surfaces may have more complex boundaries. For example, the surface on the left below has three simple closed curves as it boundary. You will notice that each curve is oriented in a way that satisfies the condition above. Lastly, we can have a ***closed surface***, which does *not* have a boundary. An example is a sphere, which is shown on the right below.





Stokes' Theorem Introduction

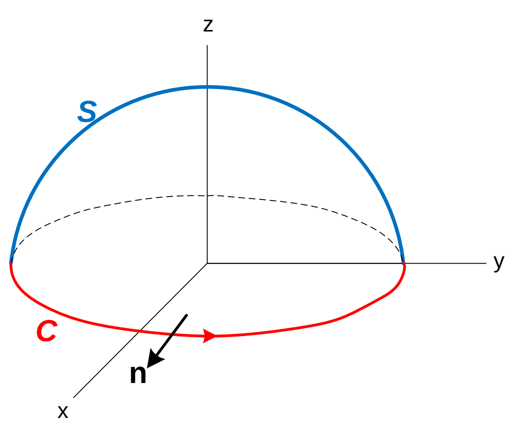
As mentioned, Stokes' Theorem is a direct extension of Green's Theorem. The difference being that the curve in the line integral and the region in the double integral both exist in three dimensions. The intuition for the theorem is therefore the same as Green's Theorem, i.e. it measures the amount of overall flow of a vector field that is in the direction of the boundary curve. The theorem is stated without proof below.

Stokes' Theorem
Let S be an oriented smooth surface that is bounded by a single simple closed curve, C, and let $\boldsymbol{F}$ be a vector field. Then $$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S curl(\boldsymbol{F}) \cdot d\boldsymbol{S}$$ Where, $curl(\boldsymbol{F}) = \nabla \times \boldsymbol{F}$

Let's start, as we did with Green's Theorem, by doing an example to verify Stokes' Theorem.

Example 1: Verify Stokes' Theorem to evaluate the circulation of the vector field, $\boldsymbol{F}$, around the boundary of the upper hemisphere with an outward pointing normal vector.

S: $x^2 + y^2 + z^2 = 1, z \geq 0$ $\qquad\qquad$ $\boldsymbol{F} = \langle -y, 2x, x + z \rangle$



Solution: We start with the left side of Stokes' Theorem.

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S curl(\boldsymbol{F}) \cdot d\boldsymbol{S}$$

The curve, C, can be parameterized using polar coordinates as a circle in the xy plane.

$$\boldsymbol{r}(\theta) = \langle cos(\theta), sin(\theta), 0 \rangle$$

Therefore,

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_0^{2\pi} \boldsymbol{F}(\boldsymbol{r}(\theta))\cdot \boldsymbol{r}'(\theta)d\theta$$

$$= \int_0^{2\pi} \langle -\, sin(\theta), 2\,cos(\theta), cos(\theta) + 0\rangle \cdot \langle - sin(\theta), cos(\theta), 0\rangle d\theta$$

$$= \int_0^{2\pi} sin^2(\theta) + \ 2\,cos^2(\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 1 - cos(2\theta)\, d\theta + \int_0^{2\pi} 1 + cos(2\theta)\, d\theta$$

$$= \frac{1}{2}(2\pi + 0) + \ (2\pi + 0)$$

$$= \ 3\pi$$

Next, we compute the surface integral on the right-hand side. We start by computing the curl.

$$curl(\boldsymbol{F}) = \nabla \times \boldsymbol{F}$$

$$= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & 2x & x + z \end{vmatrix}$$

$$= \langle \left(\frac{\partial}{\partial y}(x+z) - \frac{\partial}{\partial z}(2x)\right), \left(\frac{\partial}{\partial z}(-y) - \frac{\partial}{\partial x}(x+z)\right), \left(\frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial y}(-y)\right) \rangle$$

$$= \langle 0, -1, 3\ \rangle$$

For the surface component, $d\boldsymbol{S}$**,** we start by parameterizing the surface using spherical coordinates as follows.

$$\boldsymbol{G}(\theta,\phi) = \ \langle sin(\phi)\,cos(\theta), sin(\phi)\,sin(\theta), cos(\phi)\rangle, \qquad 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2}$$

Next, you may recall in our earlier lesson on surface integrals we learned that $d\boldsymbol{S} = \boldsymbol{N}(u,v)dudv$. In that lesson we also found the normal vector for a sphere was

$$\boldsymbol{N}(\theta,\phi) = \langle sin^2(\phi)\,cos(\theta), sin(\theta)\,sin^2(\phi), cos(\phi)\,sin(\phi)\rangle$$

The integral is then evaluated as follows

$$\iint_S curl(\boldsymbol{F}) \cdot d\boldsymbol{S} = \iint_D curl(\boldsymbol{F}) \cdot \boldsymbol{N}(\theta,\phi) d\theta d\phi$$

$$= \iint_D \langle 0,-1,3 \rangle \cdot \langle sin^2(\phi)\, cos(\theta)\, , sin(\theta)\, sin^2(\phi)\, , cos(\phi)\, sin(\phi)\rangle d\theta d\phi$$

$$= \int_0^{\pi/2}\int_0^{2\pi} 3\, cos(\phi)\, sin(\phi) - sin(\theta)\, sin^2(\phi)\; d\theta d\phi$$

$$= 3\int_0^{\pi/2}\left(cos(\phi)\, sin(\phi)\int_0^{2\pi} d\theta\right) d\phi - \int_0^{\pi/2} sin^2(\phi)\left(\int_0^{2\pi} sin(\theta)\, d\theta\right) d\phi$$

$$= 6\pi\int_0^{\pi/2}(cos(\phi)\, sin(\phi)) d\phi - 0$$

$$= 6\pi\int_0^{1}(u) du$$

$$= 6\pi\left(\frac{1}{2}\right)$$

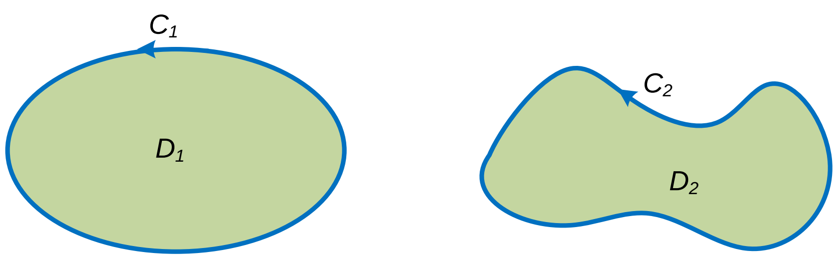
$$= 3\pi$$

Surface Independence

Take a look at Green's Theorem as stated below.

$$\oint_C \boldsymbol{F} \cdot \boldsymbol{r}'(t) dt = \iint_D curl_z(\boldsymbol{F}) dA$$

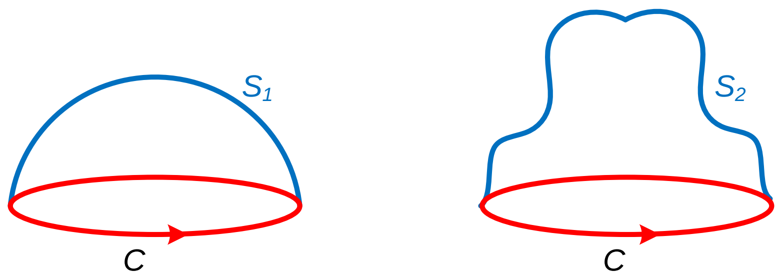
Note that $\boldsymbol{r}(t)$ parameterizes the curve, C, and D represents the *planar region* enclosed by the curve. In this case, changing the region, D, will in turn change the shape of the curve, C, that enclosed the region. As such, both integrals will change in order to maintain the equality.



As a contrast let's now look at Stokes' Theorem as stated below.

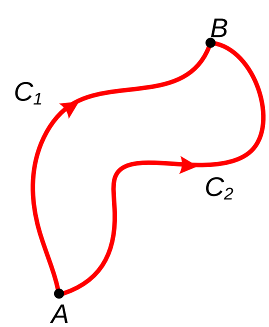
$$\oint_C \boldsymbol{F}\cdot\boldsymbol{r}'(t)dt = \iint_S curl(\boldsymbol{F})\cdot d\boldsymbol{S}$$

The curve, C, is again parameterized by $\boldsymbol{r}(t)$. The difference in Stokes' Theorem is that the region enclosed by the curve is not necessarily planar, but rather a *surface* in three dimensions. For Green's Theorem, if we change the region over which the double integral is evaluated the boundary curve, by definition, also changes. The very interesting phenomenon inherent in Stokes' theorem is that *the same boundary curve can be associated with an innumerable number of shapes*. Therefore, as long as the boundary curve remains the same, the shape of the surface can change but the value of the surface integral of $curl(\boldsymbol{F})$ cannot change so that the equality is maintained.



$$\iint_{S_1} curl(\boldsymbol{F})\cdot d\boldsymbol{S_2} = \iint_{S_2} curl(\boldsymbol{F})\cdot d\boldsymbol{S_2} = \oint_C \boldsymbol{F}\cdot d\boldsymbol{r}$$

In this case we can say that the surface integral of $curl(\boldsymbol{F})$ is *surface independent*. This may bring to mind the notion of *path independence*, which we saw in an earlier lesson. Recall that if $\boldsymbol{F}$ is conservative, i.e. $\boldsymbol{F} = \nabla f$, then the line integral of $\boldsymbol{F}$ is *path independent*.



$$\int_{C_1} \boldsymbol{F}\cdot d\boldsymbol{r} = \int_{C_1} \boldsymbol{F}\cdot d\boldsymbol{r} = \big(f(B) - f(A)\big)$$

Where, f is called the scalar potential of $\boldsymbol{F}$ and $\boldsymbol{F} = \nabla f$**.** Written with the potential function only we have.

$$\int_{C_1} \nabla f\cdot d\boldsymbol{r} = \int_{C_1} \nabla f\cdot d\boldsymbol{r} = \big(f(B) - f(A)\big)$$

By direct analogy we can express Stokes' Theorem as

$$\iint_{S_1} \boldsymbol{F} \cdot d\boldsymbol{S_2} = \iint_{S_2} \boldsymbol{F} \cdot d\boldsymbol{S_2} = \oint_C \boldsymbol{A} \cdot d\boldsymbol{r}$$

Where, $\boldsymbol{A}$ is called the *vector potential* of $\boldsymbol{F}$ and $\boldsymbol{F} = curl(\boldsymbol{A})$**.** Once again, written with the vector potential function only we have.

$$\iint_{S_1} curl(\boldsymbol{A}) \cdot d\boldsymbol{S_2} = \iint_{S_2} curl(\boldsymbol{A}) \cdot d\boldsymbol{S_2} = \oint_C \boldsymbol{A} \cdot d\boldsymbol{r}$$

We summarize the two concepts below to highlight the similarities.

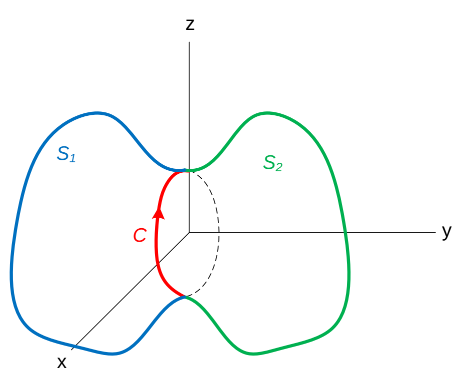
The ***line integral*** of a vector field, $\boldsymbol{F}$, with an associated ***scalar potential*** function, f, (where $\boldsymbol{F} = \nabla f$), is path independent. It depends only on the ***boundary points***, A and B.

$$\int_{C_x} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C_x} \nabla f \cdot d\boldsymbol{r} = \big(f(B) - f(A)\big)$$

The ***surface integral*** of a vector field, $\boldsymbol{F}$, with an associated ***vector potential*** function, $\boldsymbol{A}$, (where $\boldsymbol{F} = curl(\boldsymbol{A})$), is ***surface independent***. It depends only on the ***boundary curve,*** C.

$$\iint_{S_x} \boldsymbol{F} \cdot d\boldsymbol{S_x} = \iint_{S_x} curl(\boldsymbol{A}) \cdot d\boldsymbol{S_x} = \oint_C \boldsymbol{A} \cdot d\boldsymbol{r}$$

Example 2: Let $\boldsymbol{F} = curl(\boldsymbol{A})$, where $\boldsymbol{A} = \langle y + z, sin(xy), e^{xyz} \rangle$. Find the flux of $\boldsymbol{F}$ outward through the surfaces S_1 and S_2 shown below with a common boundary, C, of a unit circle.



Solution: Recall, the flux of a vector field is given by the surface integral

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

We are not explicitly given the vector field, $\boldsymbol{F}$, nor a mathematical expression for the surfaces. Fortunately, we are given the vector potential of $\boldsymbol{F}$, i.e. $\boldsymbol{A}$, and a boundary curve, C. With this we can write Stokes' theorem, which states.

$$\iint_S curl(\boldsymbol{A}) \cdot d\boldsymbol{S} = \oint_C \boldsymbol{A} \cdot d\boldsymbol{r}$$

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \oint_C \boldsymbol{A} \cdot d\boldsymbol{r}$$

Since the surfaces share the same boundary curve, the value of the line integral does not change for the different surfaces, i.e. the flux of $\boldsymbol{F}$ is surface independent! Therefore, we need only to evaluate the line integral to find the flux for either surface, (or any other surface that shares the same boundary curve).

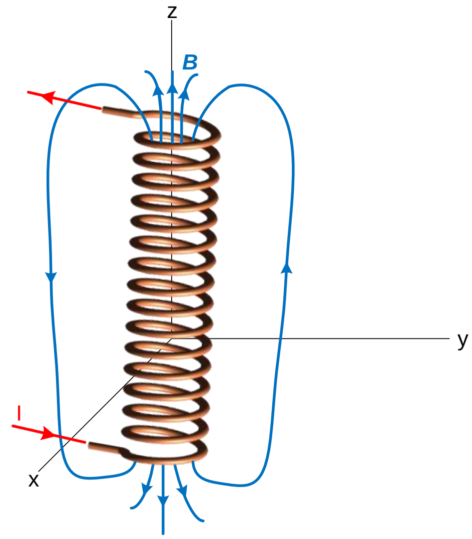
$$\begin{aligned}
\oint_C \boldsymbol{A} \cdot d\boldsymbol{r} &= \int_0^{2\pi} \boldsymbol{A}\big(\boldsymbol{r}(\boldsymbol{\theta})\big) \cdot \boldsymbol{r}'(\theta) d\theta \\
&= \int_0^{2\pi} \langle 0 + sin(\theta)\,, sin(0)\,, e^0 \rangle \cdot \langle - sin(\theta)\,, 0, cos(\theta) \rangle d\theta \\
&= \int_0^{2\pi} - sin^2(\theta) + cos(\theta)\, d\theta \\
&= -\frac{1}{2} \int_0^{2\pi} (1 - cos(2\theta)) d\theta + \int_0^{2\pi} cos(\theta)\, d\theta \\
&= -\frac{1}{2} 2\pi + 0 \\
&= -\pi
\end{aligned}$$

There is, however, one subtle difference. When we traverse the curve, C, S_1 lies to the left and S_2 to the right. Therefore, the flux of $\boldsymbol{F}$ through S_1 is π, while the flux through and S_2 is $-\pi$.

Example 3 – Solenoid: A solenoid is a tightly wound spiral of wire. When an electric current is made to flow through a solenoid it creates a magnetic field, $\boldsymbol{B}$, as shown. If we assume the solenoid is infinitely long with radius, R, then

$$\boldsymbol{B} = \begin{cases} \langle 0,0,0\rangle, & r > R \\ \langle 0,0,B\rangle, & r < R \end{cases}$$

Where, $r = \sqrt{x^2 + y^2}$, and B is a constant that depends on the strength of the current, I.



a) Show that the vector potential for the magnetic field, $\boldsymbol{B}$, is

$$\boldsymbol{A}(r) = \begin{cases} \frac{1}{2}R^2B\,\langle -\frac{y}{r^2}, \frac{x}{r^2}, 0\rangle, & r > R \\ \frac{1}{2}B\langle -y, x, 0\rangle, & r < R \end{cases}$$

b) Calculate the flux of $\boldsymbol{B}$ through a hemispherical surface, (with an upward pointing normal), whose boundary is a circle of radius r in the xy-plane.

Solution:

a) We need to show that the curl of the magnetic vector potential, $\boldsymbol{A}$, is equal to the magnetic field, $\boldsymbol{B}$. We'll evaluate the curl for each region separately. However, we first notice that the curl has a specific form for vector fields with no dependence on the z component.

$$curl(\langle f(x,y), g(x,y), 0\rangle) = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(x,y) & g(x,y) & 0 \end{vmatrix} = \langle 0, 0, \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\rangle$$

Which, in both regions matches the fact that $\boldsymbol{B}$ does not have an x or y component. Therefore, we need only evaluate the quantity $\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)$.

- For $r > R$

$$\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) = \frac{1}{2}R^2B\left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)\right)$$

$$= \frac{1}{2}R^2B\left(\left(\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right) + \left(\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right)\right)$$

$$= \frac{1}{2}R^2B\left(\frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2}\right) = 0$$

- For $r < R$

$$\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) = \frac{1}{2}B\left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right)$$

$$= \frac{1}{2}B\big((1)+(1)\big) = B$$

Using the vector potential given we find the associated vector field is

$$\boldsymbol{B} = \begin{cases} \langle 0,0,0\rangle, & r > R \\ \langle 0,0,B\rangle, & r < R \end{cases}$$

b) The flux of $\boldsymbol{B}$ through a hemispherical surface is $\iint_S \boldsymbol{B}\cdot d\boldsymbol{S}$. However, since $curl(\boldsymbol{A}) = \boldsymbol{B}$ we can use Stokes' theorem to write the following.

$$\iint_S \boldsymbol{B}\cdot d\boldsymbol{S} = \iint_S curl(\boldsymbol{A})\cdot d\boldsymbol{S} = \oint_C \boldsymbol{A}\cdot d\boldsymbol{r}$$

We start with a circle outside the solenoid where, $r > R$. Parameterizing the circle, we have

$$\boldsymbol{r}(\theta) = \langle r\cos(\theta), r\sin(\theta), 0\rangle \qquad \boldsymbol{r}'(\theta) = \langle -r\sin(\theta), r\cos(\theta), 0\rangle$$

And

$$\boldsymbol{A}\big(\boldsymbol{r}(\theta)\big) = \frac{1}{2}R^2B\,\langle -\frac{r\sin(\theta)}{r^2}, \frac{r\cos(\theta)}{r^2}, 0\rangle$$

$$= \frac{R^2B}{2}\langle \frac{-\sin(\theta)}{r}, \frac{\cos(\theta)}{r}, 0\rangle$$

With this we can evaluate the line integral to find the flux of $\boldsymbol{B}$.

$$\iint_S \boldsymbol{B}\cdot d\boldsymbol{S} = \oint_C \boldsymbol{A}\cdot d\boldsymbol{r}$$

$$= \oint_C \boldsymbol{A}\big(\boldsymbol{r}(\theta)\big)\cdot \boldsymbol{r}'(\theta)d\theta$$

$$= \int_0^{2\pi} \frac{R^2B}{2}\langle\frac{-sin(\theta)}{r},\frac{cos(\theta)}{r},0\rangle\cdot\langle -rsin(\theta),rcos(\theta),0\rangle\, d\theta$$

$$= \frac{R^2B}{2}\int_0^{2\pi}(sin^2(\theta)+cos^2(\theta))\,d\theta$$

$$= \frac{1}{2}R^2B\int_0^{2\pi} d\theta$$

$$= \pi R^2 B$$

Next, we repeat the procedure for $r < R$. In this case

$$\boldsymbol{A}\big(\boldsymbol{r}(\theta)\big) = \frac{1}{2}B\langle -rsin(\theta),rcos(\theta),0\rangle$$

Therefore,

$$\iint_S \boldsymbol{B}\cdot d\boldsymbol{S} = \oint_C \boldsymbol{A}\cdot d\boldsymbol{r}$$

$$= \oint_C \boldsymbol{A}\big(\boldsymbol{r}(\theta)\big)\cdot \boldsymbol{r}'(\theta)d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}B\langle -rsin(\theta),rcos(\theta),0\rangle\cdot\langle -sin(\theta),cos(\theta),0\rangle\, d\theta$$

$$= \frac{1}{2}B\int_0^{2\pi} r(sin^2(\theta)+cos^2(\theta))\,d\theta$$

$$= \frac{1}{2}Br\int_0^{2\pi} d\theta$$

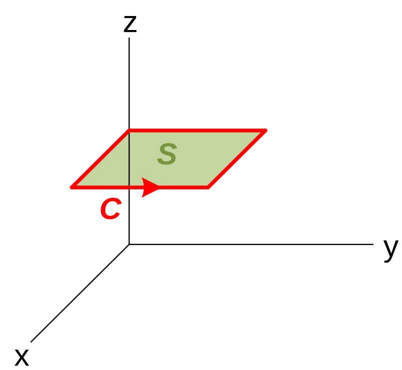
$$= \pi Br$$

Finally, we can write the magnetic flux, Φ_B, for a solenoid as

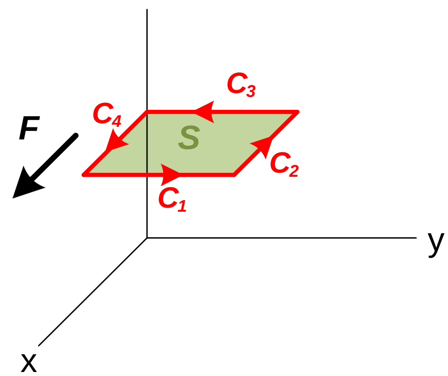
$$\Phi_B = \begin{cases} \pi R^2 B, & r > R \\ \pi Br, & r < R \end{cases}$$

We finish this section with additional examples so that you become more familiar with Stokes' Theorem.

Example 4: Verify Stokes' Theorem using a unit square in the xy plane at $z = 1$ as shown below, in a vector field $\boldsymbol{F} = \langle e^{y-z}, 0, 0 \rangle$.



Solution: Stokes' Theorem is given as $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S curl(\boldsymbol{F}) \cdot d\boldsymbol{S}$. We'll start with the line integral on the left-hand side. The curve is made of 4 line segments as shown



Therefore,

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \cancel{\oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r_1}} + \oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r_2} + \cancel{\oint_{C_3} \boldsymbol{F} \cdot d\boldsymbol{r_3}} + \oint_{C_4} \boldsymbol{F} \cdot d\boldsymbol{r_4}$$

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r_2} + \oint_{C_4} \boldsymbol{F} \cdot d\boldsymbol{r_4}$$

Where, the line integrals for C_1 and C_3 are zero since the vector field and the path are orthogonal. Parameterizing the other two curves we have

$$\boldsymbol{r}_2(t) = \langle t, 1, 1 \rangle, t: 1 \to 0 \qquad \boldsymbol{r}_4(t) = \langle t, 0, 1 \rangle, t: 0 \to 1$$

The line integrals can then be solved as shown.

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \oint_{C_2} \boldsymbol{F}\cdot d\boldsymbol{r_2} + \oint_{C_4} \boldsymbol{F}\cdot d\boldsymbol{r_4}$$

$$= \oint_{C_2} \boldsymbol{F}\big(\boldsymbol{r}_2(t)\big)\cdot \boldsymbol{r}_2{}'(t)dt + \oint_{C_4} \boldsymbol{F}\big(\boldsymbol{r}_4(t)\big)\cdot \boldsymbol{r}'{}_4(t)dt$$

$$= \int_1^0 \langle e^0, 0, 0\rangle \cdot \langle 1, 0, 0\rangle dt + \int_0^1 \langle e^{-1}, 0, 0\rangle \cdot \langle 1, 0, 0\rangle dt$$

$$= 1\int_1^0 dt + e^{-1}\int_0^1 dt$$

$$= 1(-1) + e^{-1}(1)$$

$$= (e^{-1} - 1)$$

Next, we evaluate the surface integral on the right-hand side.

$$\iint_S curl(\boldsymbol{F})\cdot d\boldsymbol{S} = \iint_D curl\Big(\boldsymbol{F}\big(G(x,y)\big)\Big)\cdot \boldsymbol{N}(x,y)dxdy$$

The surface is parameterized as $G(x,y) = \langle x, y, 1\rangle$, the normal vector is $\boldsymbol{N} = \langle 0, 0, 1\rangle$, and $\boldsymbol{F}\big(G(x,y)\big) = \langle e^{y-1}, 0, 0\rangle$. We start by computing the curl.

$$curl\Big(\boldsymbol{F}\big(G(x,y)\big)\Big) = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{-1}e^y & 0 & 0 \end{vmatrix} = \langle 0, 0, -e^{-1}e^y\rangle$$

The surface integral can then be evaluated as shown

$$\iint_S curl(\boldsymbol{F})\cdot d\boldsymbol{S} = \iint_D \langle 0, 0, -e^{-1}e^y\rangle \cdot \langle 0, 0, 1\rangle dxdy$$

$$= -e^{-1}\int_0^1 e^y\left(\int_0^1 dx\right)dy$$

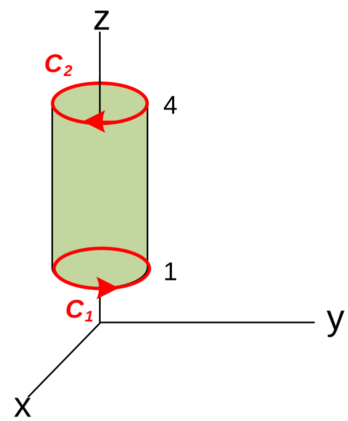
$$= -e^{-1}\int_0^1 e^y dy$$

$$= -e^{-1}(e^1 - 1)$$

$$= (e^{-1} - 1)$$

Example 5: Use a line integral to compute the flux of $curl(\boldsymbol{F})$ with $\boldsymbol{F} = \langle yz, xz, xy \rangle$ for a part of a cylinder with radius 1 that lies between the planes $z = 1$ and $z = 4$ and has an outward pointing normal.

Solution: The surface is closed by the two unit circle curves as shown below.



The surface has two boundaries and therefore the line integral is

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}$$

The curves can be represented with cylindrical coordinates as follows

$\boldsymbol{r}_1(t) = \langle cos(\theta), sin(\theta), 1 \rangle, \theta: 0 \to 2\pi$ $\quad\quad$ $\boldsymbol{r}_2(t) = \langle cos(\theta), sin(\theta), 4 \rangle, \theta: 2\pi \to 0$

Therefore,

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_{C_1} \boldsymbol{F}(\boldsymbol{r}_1(t)) \cdot \boldsymbol{r}_1'(t)dt + \oint_{C_2} \boldsymbol{F}(\boldsymbol{r}_2(t)) \cdot \boldsymbol{r}_1'(t)dt$$

$$= \int_0^{2\pi} \langle sin(\theta), cos(\theta), cos(\theta)\, sin(\theta) \rangle \cdot \langle -sin(\theta), cos(\theta), 0 \rangle dt$$

$$+ \int_{2\pi}^{0} \langle 4\, sin(\theta), 4\, cos(\theta), cos(\theta)\, sin(\theta) \rangle \cdot \langle -sin(\theta), cos(\theta), 0 \rangle dt$$

$$= \int_0^{2\pi} (-\, sin^2(\theta) + cos^2(\theta))dt - 4\int_0^{2\pi} (-\, sin^2(\theta) + cos^2(\theta))dt$$

$$= -3\int_0^{2\pi} (cos^2(\theta) - sin^2(\theta))dt$$

$$= -3\int_0^{2\pi} (cos(2\theta))dt$$

$$= 0$$

Example 6: Use Stokes' Theorem to compute $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where $\boldsymbol{F} = \langle 3y, -2x, 3y \rangle$ and C is the circle, $x^2 + y^2 = 9,\ z = 2$.

Solution: The surface integral in Stokes' Theorem is $\iint_S curl(\boldsymbol{F}) \cdot d\boldsymbol{S}$

To evaluate this integral we can start by computing $curl(\boldsymbol{F})$.

$$curl(\boldsymbol{F}) = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3y & -2x & 3y \end{vmatrix} = \langle 3, 0, -5 \rangle$$

Next, we parameterize the surface, a circle, with polar coordinates.

$$\boldsymbol{G}(r, \theta) = \langle r\,cos(\theta)\,, r\,sin(\theta)\,, 2 \rangle$$

Substituting $d\boldsymbol{S} = \boldsymbol{N}dS = \boldsymbol{N}drd\theta$ we can write

$$\iint_S curl(\boldsymbol{F}) \cdot d\boldsymbol{S} = \int_0^{2\pi} \int_0^3 curl(\boldsymbol{F}) \cdot \boldsymbol{N}drd\theta$$

The normal vector is found using $\boldsymbol{G}(r, \theta)$ as follows

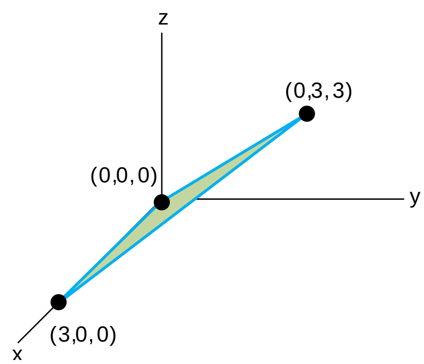
$$\frac{\partial}{\partial r}\boldsymbol{G}(r, \theta) = \langle cos(\theta)\,, sin(\theta)\,, 0 \rangle \qquad \frac{\partial}{\partial \theta}\boldsymbol{G}(r, \theta) = \langle -r\,sin(\theta)\,, r\,cos(\theta)\,, 0 \rangle$$

$$\boldsymbol{N}(u, v) = \frac{\partial}{\partial r}\boldsymbol{G}(r, \theta) \times \frac{\partial}{\partial \theta}\boldsymbol{G}(r, \theta) = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ cos(\theta) & sin(\theta) & 0 \\ -r\,sin(\theta) & r\,cos(\theta) & 0 \end{vmatrix} = \langle 0, 0, r \rangle$$

Finally, we evaluate the surface integral as shown.

$$\begin{aligned} \iint_S curl(\boldsymbol{F}) \cdot d\boldsymbol{S} &= \int_0^{2\pi} \int_0^3 curl(\boldsymbol{F}) \cdot \boldsymbol{N}drd\theta \\ &= \int_0^{2\pi} \int_0^3 \langle 3, 0, -5 \rangle \cdot \langle 0, 0, r \rangle drd\theta \\ &= -5 \int_0^{2\pi} \left(\int_0^3 rdr \right) d\theta \\ &= -5 \int_0^{2\pi} \left(\frac{9}{2} \right) d\theta \\ &= -45\pi \end{aligned}$$

Example 6: Use Stokes' Theorem to compute $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where $\boldsymbol{F} = \langle y, z, x \rangle$ and C is a triangle as shown below.



Solution: Stokes' Theorem states the following.

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S curl(\boldsymbol{F}) \cdot d\boldsymbol{S}$$

In this case the surface integral can be expressed as

$$\iint_S (curl(\boldsymbol{F}) \cdot \boldsymbol{n})\, dS$$

The unit normal for the surface can be found with the cross product of the two vectors starting at the origin. A normal vector is

$$\boldsymbol{N} = \langle 3,0,0 \rangle \times \langle 0,3,3 \rangle = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ 3 & 0 & 0 \\ 0 & 3 & 3 \end{vmatrix} = \langle 0, -9, 9 \rangle$$

The unit normal is then

$$\boldsymbol{n} = \frac{\boldsymbol{N}}{||\boldsymbol{N}||} = \frac{1}{\sqrt{2}} \langle 0, -1,1 \rangle$$

Next, we can compute $curl(\boldsymbol{F})$ as

$$curl(\boldsymbol{F}) = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \langle -1, -1, -1 \rangle$$

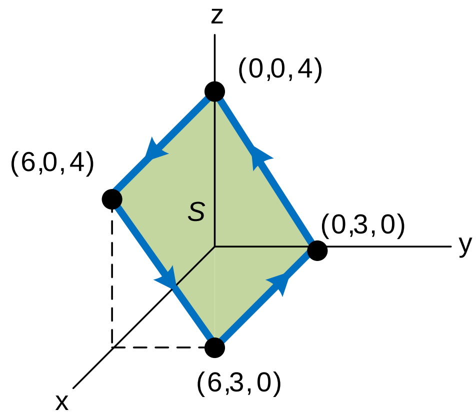
Finally, we evaluate the surface integral as shown.

$$\iint_S curl(\boldsymbol{F}) \cdot d\boldsymbol{S} = \frac{1}{\sqrt{2}} \iint_S \langle -1, -1, -1 \rangle \cdot \langle 0, -1, 1 \rangle dS = \iint_S 0 dS = 0$$

Example 7: A uniform magnetic field is given as $\boldsymbol{B} = \langle 0, 0, b \rangle$.

a) Verify that $\boldsymbol{A} = \frac{1}{2}\boldsymbol{B} \times \boldsymbol{r}$, is a vector potential for $\boldsymbol{B}$, where $\boldsymbol{r} = \langle x, y, 0 \rangle$.

b) Calculate the flux of $\boldsymbol{B}$ through the rectangle shown below.



Solution:

a) If $\boldsymbol{B} = curl(\boldsymbol{A})$, then $\boldsymbol{A}$ is a vector potential of $\boldsymbol{B}$. To verify this is true above we can start by evaluating the expression given for $\boldsymbol{A}$.

$$\boldsymbol{A} = \frac{1}{2}\boldsymbol{B} \times \boldsymbol{r} = \frac{1}{2}\begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 0 & 0 & b \\ x & y & 0 \end{vmatrix} = \frac{1}{2}\langle -by, bx, 0 \rangle$$

Now let's compute the curl of $\boldsymbol{A}$.

$$curl(\boldsymbol{A}) = \frac{b}{2}\begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \frac{b}{2}\langle 0,0,2 \rangle = \langle 0, 0, b \rangle = \boldsymbol{B}$$

Therefore, $\boldsymbol{A}$ is a vector potential of $\boldsymbol{B}$**.**

b) We can find the flux using the following representation of the surface integral

$$\iint_S \boldsymbol{B} \cdot d\boldsymbol{S} = \iint_S (\boldsymbol{B} \cdot \boldsymbol{n}) dS$$

A normal vector for the surface is found in the same was it was in example 6.

$$\begin{aligned}\boldsymbol{N} &= \langle(0,3,0) - (6,3,0)\rangle \times \langle(6,0,4) - (6,3,0)\rangle \\ &= \langle -6,0,0\rangle \times \langle 0,-3,4\rangle \\ &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -6 & 0 & 0 \\ 0 & -3 & 4 \end{vmatrix} \\ &= \langle 0,24,18\rangle\end{aligned}$$

Therefore, the unit normal is

$$\boldsymbol{n} = \frac{\boldsymbol{N}}{||\boldsymbol{N}||} = \frac{1}{5}\langle 0,4,3\rangle$$

The integral is then evaluated as shown.

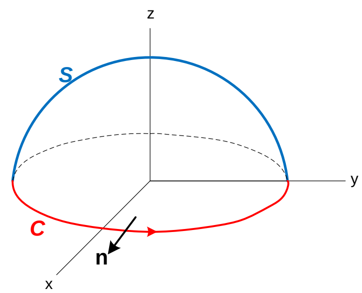
$$\begin{aligned}\iint_S (\boldsymbol{B}\cdot\boldsymbol{n})dS &= \frac{1}{5}\iint_S (\langle 0,0,b\rangle \cdot \langle 0,4,3\rangle)dS \\ &= \frac{3b}{5}\iint_S dS \\ &= \frac{3b}{5}(6\cdot 5) \\ &= 18b\end{aligned}$$

Final Summary for Theorem of Vector Calculus – Stokes' Theorem

Surfaces and Surface Boundaries

Different surfaces may have different types of boundaries. For example, the surface below has a single simple closed curve as its boundary. We define the orientation of the curve as follows:

- When you walk around the curve with your body pointing out in the direction of the normal vector, you should be walking in such a way that the surface is to your left side.



Stokes' Theorem

Let S be an oriented smooth surface that is bounded by a single simple closed curve, C, and let $\boldsymbol{F}$ be a vector field. Then

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S curl(\boldsymbol{F}) \cdot d\boldsymbol{S}$$

Where, $curl(\boldsymbol{F}) = \nabla \times \boldsymbol{F}$

Stokes' Theorem relates the circulation of a vector field, $\boldsymbol{F}$, via the vector line along the surface boundary curve, to a surface integral of the curl of $\boldsymbol{F}$ over the given surface.

Surface Independence

The ***surface integral*** of a vector field, $\boldsymbol{F}$, with an associated ***vector potential*** function, $\boldsymbol{A}$, (where $\boldsymbol{F} = curl(\boldsymbol{A})$), is ***surface independent***. It depends only on the ***boundary curve,*** C.

$$\iint_{S_x} \boldsymbol{F} \cdot d\boldsymbol{S_x} = \iint_{S_x} curl(\boldsymbol{A}) \cdot d\boldsymbol{S_x} = \oint_C \boldsymbol{A} \cdot d\boldsymbol{r}$$

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