Fundamental Theorems of Vector Calculus – Divergence Theorem

At the beginning of this series of lessons we said there were 4 fundamental theorems that are generally associated with vector calculus. The first was the *Gradient Theorem*, which we introduced in an earlier series when discussing conservative vector fields. The second was *Green's Theorem*, which was the first theorem introduced in this series of lessons. The third theorem we introduced was *Stokes' Theorem*, which is a direct extension of Green's Theorem to three dimensions. The fourth theorem, which we introduce in this lesson is called the *Divergence Theorem*. In the first lesson in this series we mentioned how these theorems can be seen as generalizations of the Fundamental Theorem of Single Variable Calculus.

$$\int_a^b f'(t)dt = f(b) - f(a)$$

In a general sense this theorem *relates the integral of some type of derivative of some function over some region to the values of that function along the boundary of the region*. Specifically, FTC relates the integral of the derivative of a scalar function over a one-dimensional region to the values of the function at the endpoints of the region.

Using this generalization for the Gradient Theorem

$$\int_C \nabla f \cdot d\boldsymbol{s} = f(b) - f(a)$$

We can say that the Gradient Theorem relates the integral of the gradient of a scalar function over a curve in three dimensions, C, to the values of that function at the endpoints of the curve.

Using the generalization for Green's Theorem

$$\iint_D curl_z(\boldsymbol{F})dA = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

We can say that the Green's Theorem relates the double integral of the two dimensional curl of a vector field over a region, D, to the value of the line integral of that vector field along the boundary curve for that region, C.

And for Stokes' Theorem

$$\iint_S curl(\boldsymbol{F}) \cdot d\boldsymbol{S} = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

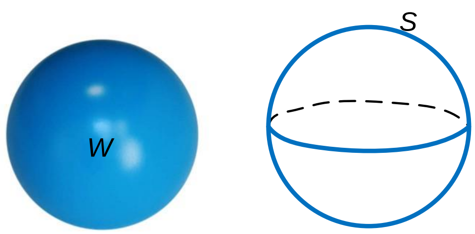
We can say that the Stokes' Theorem relates the surface integral of the three dimensional curl of a vector field over a surface, S, to the value of the line integral of that vector field along the boundary curve for that surface, C.

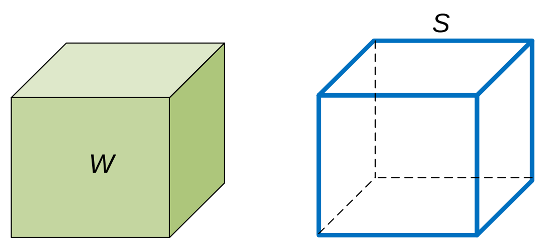
Finally, in this lesson we introduce the Divergence Theorem, which we state below without proof.

$$\iiint_W div(\boldsymbol{F})dV = \iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

In this case, the Divergence Theorem relates the triple integral of the divergence of a vector field over a 3D region, W, to the value of the surface integral of that vector field over the boundary surface for the region, S.

We assume S is a closed surface that encloses a three dimensional region, W. Recall from the previous lesson that a closed surface does not have a boundary. Examples are spheres and cubes.



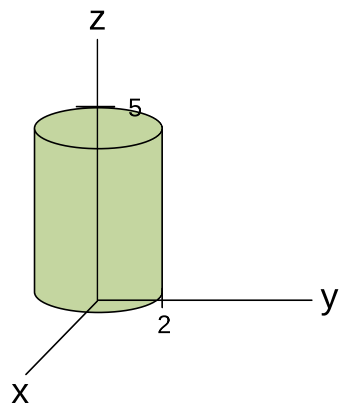


The theorem is formally stated below.

Divergence Theorem
Let S be a closed surface that encloses a region, W, in R^3. Assume that S is piecewise smooth and is oriented by a normal vector pointing to the outside of W. Let $\boldsymbol{F}$ be a vector field whose domain contains W. Then $$\iiint_W div(\boldsymbol{F})dV = \iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

Let's again start, as we did with the other theorem, by doing an example to verify the Divergence Theorem.

Example 1: Verify the Divergence Theorem for $\boldsymbol{F} = \langle y, yz, z^2\rangle$ using the cylindrical region below.



Solution: The 3D region, W, is a solid cylinder with radius 2 and height 5. The Divergence Theorem is given as

$$\iiint_W div(\boldsymbol{F})dV = \iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$$

For the right-hand side we need to determine the closed surface that encloses the region, W. In this case, the region is enclosed by three piecewise smooth surfaces. The top and bottom of the cylinder are circular surfaces, and the sides are enclosed by the surface of the cylinder itself. The right-hand side of the Divergence theorem is then

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_{S_{top}} \boldsymbol{F}\cdot d\boldsymbol{S}_{top} + \iint_{S_{bottom}} \boldsymbol{F}\cdot d\boldsymbol{S}_{bottom} + \iint_{S_{side}} \boldsymbol{F}\cdot d\boldsymbol{S}_{side}$$

Top and Bottom

Both surfaces can be parameterized as follows

$$\boldsymbol{G}(r,\theta) = \langle r\,cos(\theta)\,, r\,sin(\theta)\,, z\rangle$$

With the bottom surface having $z = 0$ and the top with $z = 5$.

Starting with the top surface we have

$$\iint_{S_{top}} \boldsymbol{F}\cdot d\boldsymbol{S}_{top} = \iint_{S_{top}} \big(\boldsymbol{F}\big(\boldsymbol{G}(r,\theta)\big)\cdot \boldsymbol{n}\big)\, dS_{top}$$

$$= \iint_{S_{top}} (\langle\, r\,sin(\theta)\,, 5\,r\,sin(\theta)\,, 25\rangle\; \cdot \langle 0,0,1\rangle)\; rdrd\theta$$

$$= 25\int_0^{2\pi}\left(\int_0^2 rdr\right)d\theta$$

$$= 100\pi$$

For the bottom surface we have

$$\iint_{S_{bottom}} \boldsymbol{F} \cdot d\boldsymbol{S}_{top} = \iint_{S_{bottom}} \big(\boldsymbol{F}\big(\boldsymbol{G}(r,\theta)\big) \cdot \boldsymbol{n}\big)\, dS_{bottom}$$

$$= \iint_{S_{top}} (\langle\, r\, sin(\theta)\,, 0, 0\rangle\ \cdot \langle 0,0,1\rangle)\, dS_{bottom}$$

$$= \iint_{S_{top}} (0)\, dS_{bottom}$$

$$= 0$$

Next, for the side surface we parameterize as follows

$$\boldsymbol{G}(z,\theta) = \langle 2\,cos(\theta)\,, 2\,sin(\theta)\,, z\rangle$$

The normal to the surface can then be found using the cross product of the partials.

$$\frac{\partial}{\partial z}\big(\boldsymbol{G}(z,\theta)\big) = \langle 0,0,1\rangle \qquad \frac{\partial}{\partial \theta}\big(\boldsymbol{G}(z,\theta)\big) = \langle -2\,sin(\theta)\,, 2\,cos(\theta)\,, 0\rangle$$

$$\boldsymbol{N} = \left(\frac{\partial}{\partial \theta}\big(\boldsymbol{G}(z,\theta)\big) \times \frac{\partial}{\partial z}\big(\boldsymbol{G}(z,\theta)\big)\right) = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ -2\,sin(\theta) & 2\,cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2\,cos(\theta)\,, 2\,sin(\theta)\,, 0\rangle$$

$$\iint_{S_{side}} \boldsymbol{F} \cdot d\boldsymbol{S}_{side} = \iint_{S_{side}} \boldsymbol{F} \cdot \boldsymbol{N} dS_{side}$$

$$= \iint_{S_{side}} \langle\, 2\,sin(\theta)\,, 2z\,sin(\theta)\,, z^2\rangle \cdot \langle 2\,cos(\theta)\,, 2\,sin(\theta)\,, 0\rangle dS_{side}$$

$$= 4\int_0^{2\pi}\int_0^5 sin(\theta)\,cos(\theta) + z\,sin^2(\theta)\,dzd\theta$$

$$= 4\int_0^{2\pi} sin(\theta)\,cos(\theta)\left(\int_0^5 dz\right)d\theta + 4\int_0^{2\pi} sin^2(\theta)\left(\int_0^5 zdz\right)d\theta$$

$$= 20\int_0^{2\pi} sin(\theta)\,cos(\theta)\,d\theta + 50\int_0^{2\pi} sin^2(\theta)\,d\theta$$

$$= 0 + 25\left(\int_0^{2\pi} 1d\theta - \int_0^{2\pi} cos(2\theta)\,d\theta\right)$$

$$= 25(2\pi - 0)$$

$$= 50\pi$$

Therefore,

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_{S_{top}} \boldsymbol{F}\cdot d\boldsymbol{S}_{top} + \iint_{S_{bottom}} \boldsymbol{F}\cdot d\boldsymbol{S}_{bottom} + \iint_{S_{side}} \boldsymbol{F}\cdot d\boldsymbol{S}_{side}$$

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = 100\pi + 0 + 50\pi$$

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = 150\pi$$

Next, we evaluate the triple integral staring by computing the divergent of $\boldsymbol{F}$.

$$div(\boldsymbol{F}) = \nabla\cdot\boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(z^2) = 3z$$

To evaluate the triple integral, we use cylindrical coordinates

$$\iiint_W div(\boldsymbol{F})dV = \int_0^5\int_0^{2\pi}\int_0^2 (3z)r\,dr\,d\theta\,dz$$

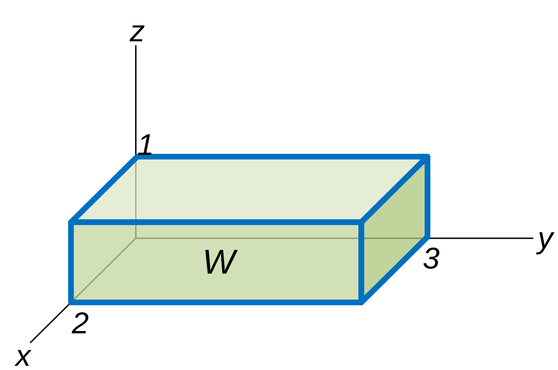
$$= \left(\int_0^5 3z\,dz\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 r\,dr\right)$$

$$= \left(\frac{3}{2}\cdot 25\right)(2\pi)(2)$$

$$= 150\pi$$

As you may notice, the surface integral on the right-hand side of the Divergence Theorem represents the flux of $\boldsymbol{F}$ over through the surface, S. In many applications the triple integral on the left-hand side is much easier to use to compute the flux. The next example illustrates one such case.

Example 2: Find the flux of $\boldsymbol{F} = \langle x^2, z^4, e^z\rangle$ through the rectangular prism shown below.



Solution: The flux through the surface is given by

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

The surface, however, is piecewise smooth and the integral would be quite time consuming to compute. However, the Divergence Theorem says that

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_W div(\boldsymbol{F})dV$$

So, we can instead evaluate the triple integral. We start by computing then divergence of $\boldsymbol{F}$.

$$div(\boldsymbol{F}) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z^4) + \frac{\partial}{\partial z}(e^z) = 2x + e^z$$

The flux can then be computed as shown

$$\begin{aligned}\iiint_W div(\boldsymbol{F})dV &= \iiint_W (2x + e^z)dV \\ &= \int_0^1 \int_0^3 \left(\int_0^2 (2x + e^z)dx \right) dydz \\ &= \int_0^1 \left(\int_0^3 (4 + 2e^z)dy \right) dz \\ &= \int_0^1 (12 + 6e^z)\, dz \\ &= 12 + 6e^1 - 6 \\ &= 6e^1 + 6\end{aligned}$$

Electrostatics Application

The Divergence Theorem can be used to show an extremely fundamental result relating to the flux of an electrostatic field. The electric field due to a point charge is given by

$$\boldsymbol{E} = \left(\frac{q}{4\pi\epsilon_0 r^2} \right) \hat{\boldsymbol{r}}$$

Where, q is the charge in Coulombs, $\epsilon_0 = 8.85E^{-12}\, C^2/N - m^2$ is the permittivity constant, r is the distance in meters from the charge, and $\hat{\boldsymbol{r}} = \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \rangle$ is the unit radial vector where $r = \sqrt{x^2 + y^2 + z^2}$

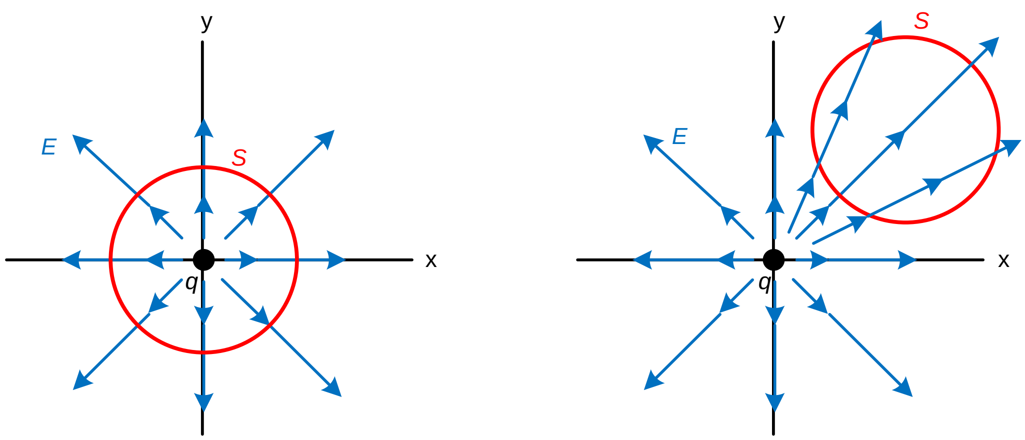
The electric field is what we refer to as an inverse square field, which we can write in general as

$$\boldsymbol{F}_{IS} = A\left(\frac{\hat{\boldsymbol{r}}}{r^2}\right)$$

Where A is a constant.

Various phenomena in physics can be accurately described using an inverse square field, e.g. gravity, hence it is an important vector field to study.

Let's find the electric flux through a spherical surface with radius R for a point charge placed at the origin. For illustrative purposes, the figure below shows the field in 2 dimensions. Additionally, for the figure on the left the 'sphere' is centered at the origin, whereas on the right the sphere is shifted away from the origin.



In both cases the flux is given by $\Phi = \iint_S \boldsymbol{E} \cdot d\boldsymbol{S}$. Furthermore, using the Divergence Theorem we can write

$$\iint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \iiint_W div(\boldsymbol{E})dV$$

We'll start by computing the electric flux for the sphere that is shifted away from the origin. In this case, we can use the triple integral for the calculation. The electric vector field can be written as

$$\boldsymbol{E} = \left(\frac{q}{4\pi\epsilon_0}\right)\left(\frac{\hat{\boldsymbol{r}}}{r^2}\right)$$

$$= A\left\langle\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right\rangle$$

Where, $A = \frac{q}{4\pi\epsilon_0}$, $\hat{\boldsymbol{r}} = \langle\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\rangle$, and $r = \sqrt{x^2+y^2+z^2}$

The divergence is given as

$$div(\boldsymbol{E}) = \nabla \cdot \boldsymbol{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

The individual terms will all be similar since the components are similar. Therefore, we'll compute the first term only as an illustration.

$$\begin{aligned}
\frac{\partial E_x}{\partial x} &= \frac{\partial}{\partial x}\left((x)(x^2+y^2+z^2)^{-3/2}\right) \\
&= \left((1)(x^2+y^2+z^2)^{-3/2} + (x)\left(-\frac{3}{2}(x^2+y^2+z^2)^{-5/2}\cdot 2x\right)\right) \\
&= \left(\frac{1}{(r^2)^{3/2}} - \frac{3x^2}{(r^2)^{5/2}}\right) \\
&= \left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right) \\
&= \left(\frac{r^2 - 3x^2}{r^5}\right)
\end{aligned}$$

The remaining terms follow the same pattern. The divergence is then

$$\begin{aligned}
div(\boldsymbol{E}) &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \\
&= \left(\frac{r^2 - 3x^2}{r^5}\right) + \left(\frac{r^2 - 3y^2}{r^5}\right) + \left(\frac{r^2 - 3z^2}{r^5}\right) \\
&= \left(\frac{3r^2 - 3x^2 - 3y^2 - 3z^2}{r^5}\right) \\
&= 3\left(\frac{r^2 - (x^2+y^2+z^2)}{r^5}\right) = 3\left(\frac{r^2 - r^2}{r^5}\right) = 0
\end{aligned}$$

Therefore, the electric flux through a sphere that is shifted away from the origin is zero.

$$\Phi_E = \iint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \iiint_W div(\boldsymbol{E}) dV = 0$$

Next, we look at the electric flux for a sphere centered at the origin.

In this case, since $\boldsymbol{E}$ is undefined at the origin we cannot use the triple integral from the divergence theorem to compute the flux. Instead we'll compute the surface integral using a spherical coordinate parameterization of the surface.

$$\iint_S \boldsymbol{E}\cdot d\boldsymbol{S} = \iint_S \Big(\boldsymbol{E}\big(\boldsymbol{G}(\theta,\phi)\big)\cdot \boldsymbol{N}(\theta,\phi)\Big)\, d\theta d\phi$$

Where, $\boldsymbol{G}(\theta,\phi) = r\langle sin(\phi)\, cos(\theta)\,, sin(\phi)\, sin(\theta)\,, cos(\phi)\rangle,\ 0\le\theta\le 2\pi, 0\le\phi\le\pi$

The normal vector was computed in an earlier lesson and is given as

$$\begin{aligned}\boldsymbol{N}(\theta,\phi) &= \frac{\partial \boldsymbol{G}(\theta,\phi)}{\partial\phi}\times\frac{\partial \boldsymbol{G}(\theta,\phi)}{\partial\theta}\\ &= r^2\langle sin^2(\phi)\, cos(\theta)\,, sin(\theta)\, sin^2(\phi)\,, cos(\phi)\, sin(\phi)\rangle\end{aligned}$$

Next, we'll compute the dot product

$$\boldsymbol{E}\big(\boldsymbol{G}(\theta,\phi)\big)\cdot\boldsymbol{N}(\theta,\phi)$$

$$\begin{aligned}&= \frac{A}{r^2}\langle sin(\phi)\, cos(\theta)\,, sin(\phi)\, sin(\theta)\,, cos(\phi)\rangle\\ &\cdot r^2\langle sin^2(\phi)\, cos(\theta)\,, sin(\theta)\, sin^2(\phi)\,, cos(\phi)\, sin(\phi)\rangle\\ &= A(sin^3(\phi)\, cos^2(\theta) + sin^3(\phi)\, sin^2(\theta) + cos^2(\phi)\, sin(\phi))\\ &= A(sin^3(\phi)\,(cos^2(\theta) + sin^2(\theta)) + cos^2(\phi)\, sin(\phi))\\ &= A(sin^2(\phi)\, sin(\phi) + cos^2(\phi)\, sin(\phi))\\ &= A\big(sin(\phi)\,(sin^2(\phi) + cos^2(\phi))\big)\\ &= A\, sin(\phi)\end{aligned}$$

The surface integral is computed as follows

$$\begin{aligned}\iint_S \boldsymbol{E}\cdot d\boldsymbol{S} &= \int_0^{\pi}\int_{\mathbf{0}}^{\mathbf{2\pi}} (A\, sin(\phi))\, d\theta d\phi\\ &= A\int_0^{\pi} sin(\phi)\left(\int_{\mathbf{0}}^{\mathbf{2\pi}} d\theta\right) d\phi\\ &= 2\pi A\int_0^{\pi} sin(\phi)\, d\phi\\ &= 4\pi A\end{aligned}$$

Substituting for A we have

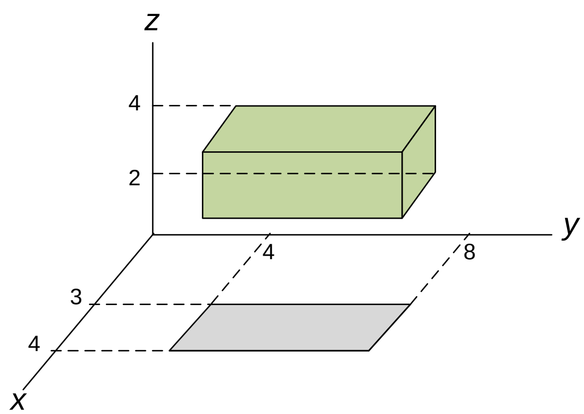
$$\iint_S \boldsymbol{E}\cdot d\boldsymbol{S} = 4\pi A = 4\pi\frac{q}{4\pi\epsilon_0} = \frac{q}{\epsilon_0}$$

As it turns out the above results hold regardless of the shape of the surface and is known more formally as Gauss's Law. It also holds regardless of how the charge is distributed throughout the surface.

Gauss's Law
The electric flux through a closed surface is proportional to the total charge enclosed within the surface. $$\iint_S \boldsymbol{E}\cdot d\boldsymbol{S} = \frac{q_T}{\epsilon_0}$$

We'll finish this lesson with some additional examples.

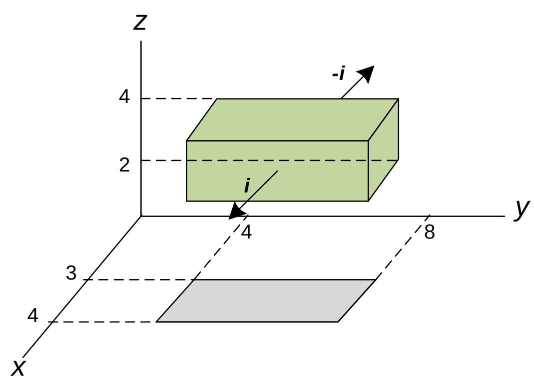
Example 3: Verify the Divergence Theorem for the region shown in a vector field, $\boldsymbol{F} = \langle x, y, z\rangle$.



Solution: The surface is piecewise smooth. Therefore, to compute the surface integral we need to split the surface into 6 parts (faces of box). We can formally write the surface integral as follows.

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \sum_{i=1}^{6}\iint_{S_i} \boldsymbol{F}\big(\boldsymbol{G_i}(u,v)\big)\cdot \boldsymbol{N_i}(u,v)dudv$$

We'll illustrate the procedure starting with the two sides, we can call them S_1 and S_2, oriented in the yz-plane. The other sides are calculated in a similar manner.



The two sides are parameterized as follows

$$\boldsymbol{G}_1(y,z) = \langle 4, y, z\rangle \qquad\qquad \boldsymbol{G}_2(y,z) = \langle 3, y, z\rangle$$

The corresponding normal vectors are

$$\boldsymbol{N}_1 = \langle 1, 0, 0\rangle \qquad\qquad \boldsymbol{N}_2 = \langle -1, 0, 0\rangle$$

The surface integrals are then computed as follows

$$\begin{aligned}\iint_{S_1} \boldsymbol{F}\big(\boldsymbol{G}_1(y,z)\big)\cdot \boldsymbol{N}_1 dydz &= \int_2^4\int_4^8 \langle 4, y, z\rangle\cdot\langle 1,0,0\rangle dydz \\ &= 4\int_2^4\int_4^8 dydz \\ &= 4\cdot Area(S_1) \\ &= 4\cdot 8 \\ &= 32\end{aligned}$$

$$\begin{aligned}\iint_{S_2} \boldsymbol{F}\big(\boldsymbol{G}_2(y,z)\big)\cdot \boldsymbol{N}_2 dydz &= \int_2^4\int_4^8 \langle 3, y, z\rangle\cdot\langle -1,0,0\rangle dydz \\ &= -3\int_2^4\int_4^8 dydz \\ &= -3\cdot Area(S_2) \\ &= -3\cdot 8 \\ &= -24\end{aligned}$$

As mentioned, the remaining sides are similar.

$$\iint_{S_3} \boldsymbol{F}(\boldsymbol{G}_3(x,z)) \cdot \boldsymbol{N}_3 dxdz$$
$$= \int_2^4 \int_3^4 \langle x, 8, z\rangle \cdot \langle 0,1,0\rangle dxdz$$
$$= 8 \cdot Area(S_3)$$
$$= 8 \cdot 2$$
$$= 16$$

$$\iint_{S_4} \boldsymbol{F}(\boldsymbol{G}_4(x,z)) \cdot \boldsymbol{N}_4 dxdz$$
$$= \int_2^4 \int_3^4 \langle x, 4, z\rangle \cdot \langle 0,-1,0\rangle dxdz$$
$$= -4 \cdot Area(S_4)$$
$$= -4 \cdot 2$$
$$= -8$$

$$\iint_{S_5} \boldsymbol{F}(\boldsymbol{G}_5(x,y)) \cdot \boldsymbol{N}_5 dxdy$$
$$= \int_4^8 \int_3^4 \langle x, y, 4\rangle \cdot \langle 0,0,1\rangle dxdy$$
$$= 4 \cdot Area(S_5)$$
$$= 4 \cdot 4$$
$$= 16$$

$$\iint_{S_6} \boldsymbol{F}(\boldsymbol{G}_6(x,y)) \cdot \boldsymbol{N}_6 dxdy$$
$$= \int_4^8 \int_3^4 \langle x, y, 2\rangle \cdot \langle 0,0,-1\rangle dxdy$$
$$= -2 \cdot Area(S_6)$$
$$= -2 \cdot 4$$
$$= -8$$

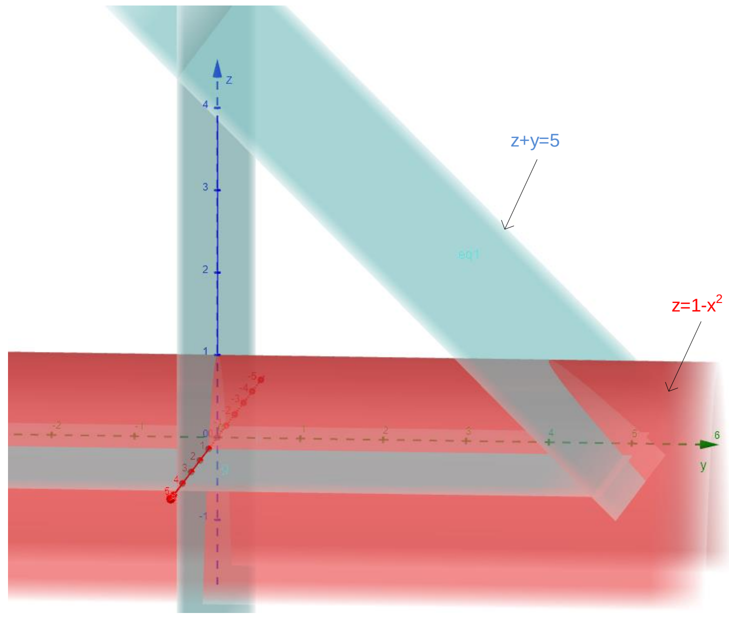
Summing the individual results from above we have

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \sum_{i=1}^{6} \iint_{S_i} \boldsymbol{F}(\boldsymbol{G_i}(u,v)) \cdot \boldsymbol{N_i}(u,v)dudv$$
$$= (32) + (-24) + (16) + (-8) + (16) + (-8)$$
$$= 24$$

Verifying with the triple integral we have

$$\iiint_W div(\boldsymbol{F})dV = \iiint_W \left(\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right) dV$$
$$= 3 \iiint_W dV$$
$$= 3 \cdot Volume(W)$$
$$= 3 \cdot 8$$
$$= 24$$

Example 4: Use the Divergence Theorem to evaluate the flux, $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F} = \langle x + z^2, xz + y^2, xz - y \rangle$, and S is the surface that bounds the solid region given by the parabolic cylinder $z = 1 - x^2$, and the planes $z = 0, y = 0$, and $z + y = 5$.



Solution: Using the Divergence Theorem we can find the flux with the triple integral

$$\iiint_W div(\boldsymbol{F}) dV$$

The divergence is computed as

$$div(\boldsymbol{F}) = \frac{\partial}{\partial x}(x + z^2) + \frac{\partial}{\partial y}(xz + y^2) + \frac{\partial}{\partial z}(xz - y) = 1 + 2y + x$$

The region, W, can be described as

$$-1 \leq x \leq 1$$
$$0 \leq z \leq 1 - x^2$$
$$0 \leq y \leq 5 - z$$

Therefore,

$$\iiint_W div(\boldsymbol{F})dV = \int_{-1}^{1}\int_{0}^{1-x^2}\left(\int_{0}^{5-z}(1+2y+x)dy\right)dzdx$$

$$= \int_{-1}^{1}\int_{0}^{1-x^2}\left(5-z+(5-z)^2+x(5-z)\right)dzdx$$

$$= \int_{-1}^{1}\left(\int_{0}^{1-x^2}(z^2-(11+x)z+30+5x)dz\right)dx$$

$$= \int_{-1}^{1}\left(\frac{1}{3}(1-x^2)^3-\frac{11+x}{2}(1-x^2)^2+(30+5x)(1-x^2)\right)dx$$

Although the final integral can be solved it is quite cumbersome and using a computer algebra system is recommended. Doing so results in the following

$$\iiint_W div(\boldsymbol{F})dV = \frac{3616}{105}$$

Example 5: Let W be the region between a sphere of radius 4 and a cube of side 1, both centered at the origin. What is the flux through the boundary of this region for a vector field, $\boldsymbol{F}$, whose divergence is equal to -4?

Solution: Using the Divergence Theorem we can find the flux using the triple integral.

$$\iiint_W div(\boldsymbol{F})dV = \iiint_W -4dV = -4\cdot Volume(W)$$

The volume of the region can be found without integration as follows

$$Volume(W) = Volume(Sphere) - Volume(Cube)$$
$$= \frac{4}{3}\pi\cdot 4^3 - 1$$

Therefore,

$$\iiint_W div(\boldsymbol{F})dV = -4\cdot Volume(W)$$

$$= -4\left(\frac{4}{3}\pi\cdot 64-1\right)$$

$$\cong -1068.33$$

Final Summary for Theorem of Vector Calculus – Divergence Theorem

Divergence Theorem
Let S be a closed surface that encloses a region, W, in R^3. Assume that S is piecewise smooth and is oriented by a normal vector pointing to the outside of W. Let $\boldsymbol{F}$ be a vector field whose domain contains W. Then $$\iiint_W div(\boldsymbol{F})dV = \iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$$
Gauss's Law
The electric flux through a closed surface is proportional to the total charge enclosed within the surface. $$\iint_S \boldsymbol{E}\cdot d\boldsymbol{S} = \frac{q_T}{\epsilon_0}$$ Where, $\boldsymbol{E} = \left(\frac{q}{4\pi\epsilon_0}\right)\left(\frac{\hat{\boldsymbol{r}}}{r^2}\right)$ and $\hat{\boldsymbol{r}} = \langle\frac{x}{r},\frac{y}{r},\frac{z}{r}\rangle$.

By: ferrantetutoring