Line and Surface Integrals – Vector Fields

In the past few lessons, we extended single variable integration to multiple integration. In the next few lessons, we seek to generalize integration even further. This will include integrating not just scalar functions but also objects called *vector fields*. Vector fields are enormously important in science and engineering and are used to describe things such as wind velocity, ocean currents, gravity, and electromagnetism. In this lesson we formally introduce the vector field as a foundation for upcoming lessons.

Vector Fields

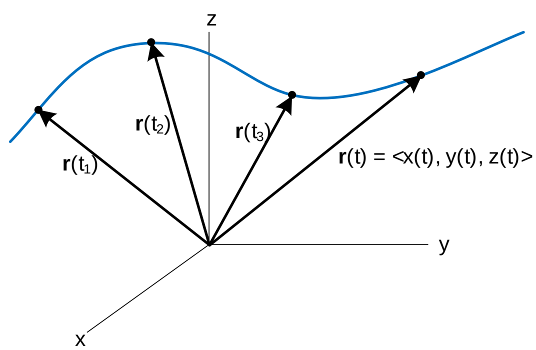
A *vector field* is a function that assigns a vector to each point, $P = \langle x, y, z \rangle$, in space. In three dimensions it is denoted as

$$\boldsymbol{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) \rangle$$

You may notice that this seems very similar to vector-valued functions we studied earlier. They are indeed similar concepts. For our purposes the best way to distinguish between the two relates to the physical phenomena for which we generally use them to represent.

A *vector-valued function* can be used to represents the position of a particle in space as a function of time. The position of the particle at time t lies at the endpoint of the vector, $\boldsymbol{r}$, which is refenced to a fixed location, e.g. the origin.

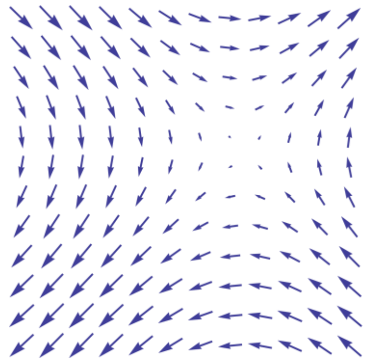
$$\boldsymbol{r}(t) = \langle x(t), y(t), z(t) \rangle$$



A *vector field* is generally used to represent a physical ‘object’ that exists in all space, (or some subset of space). Moreover, this ‘object’ requires description at each point by a vector. One example of such an object is ‘wind’, for which we can describe its velocity, a vector quantity, at each point in space. Contrary to a vector-valued function the velocity vector at each point is referenced to that point and not some fixed location.

The figure below illustrates a generic vector field in two dimensions.

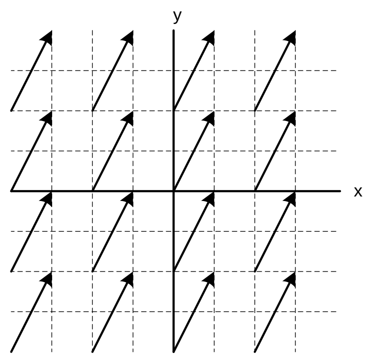
$$\boldsymbol{F}(x,y) = \langle F_1(x,y), F_2(x,y)\rangle$$



Vector Field Examples

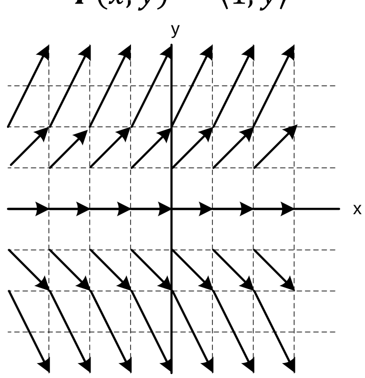
The simplest vector field is a constant, an example of which is shown below.

$$\boldsymbol{F}(x,y) = \langle 1,2\rangle$$

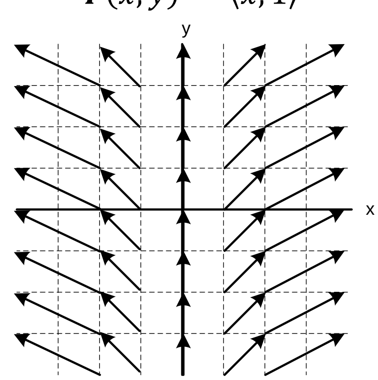


Each point in space is associated with a vector of equal length and pointing the same direction. If, for example, the vector field represented the velocity of wind this may be true over some small area. Of greater interest are vector fields that are not constant. Two examples are shown below for illustrative purposes. For the one on the left the x component of the vector field is constant while the y component is equal to the y coordinate value. The one on the right switches the roles of the x and y component.

$$\boldsymbol{F}(x,y) = \langle 1,y\rangle$$

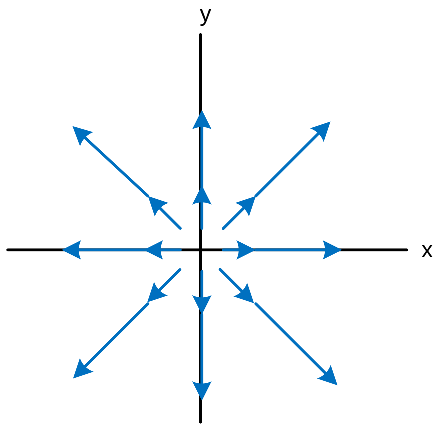


$$\boldsymbol{F}(x,y) = \langle x,1\rangle$$

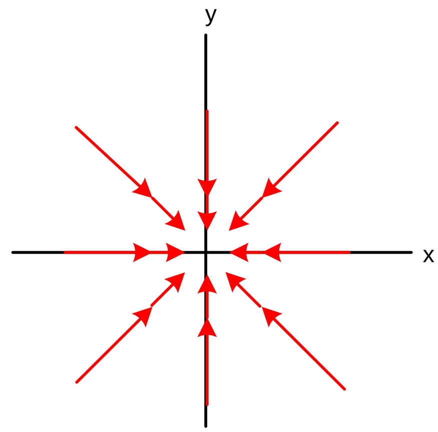


Another interesting vector field is called a *radial vector field* – A vector field where all vectors point straight towards or away from the origin.

$\boldsymbol{F}(x,y) = \langle x,y\rangle$



$\boldsymbol{F}(x,y) = -\langle x,y\rangle$



We can also define a unit vector field, which is one where $\|\boldsymbol{F}\| = 1$. We can create a unit vector field by dividing the given vector field by its magnitude at each point.

$$\boldsymbol{e}_F = \frac{\boldsymbol{F}(x,y,z)}{\|\boldsymbol{F}(x,y,z)\|}$$

An important example is a unit radial vector.

Two dimensional unit radial vector	***Three dimensional unit radial vector***
$\boldsymbol{e}_r = \langle\frac{x}{r},\frac{y}{r}\rangle$ Where, $r = \sqrt{x^2+y^2}$	$\boldsymbol{e}_r = \langle\frac{x}{r},\frac{y}{r},\frac{z}{r}\rangle$ Where, $r = \sqrt{x^2+y^2+z^2}$

One of the most well-known radial vector fields is that of the gravitational force. Isaac Newton discovered that the gravitational force is an attractive radially directed force, which we can described by a force vector field. The magnitude of the gravitational force that is exerted on a particle of mass, m, located a distance r from another object of mass, M is given by

$$\|\boldsymbol{F}_G\| = \frac{GMm}{r^2}$$

Where, G is the gravitational constant.

We can describe the vector nature of this force using the unit radial vector as follows.

$$\boldsymbol{F}_G(x,y,z) = -\frac{GMm}{r^2}\boldsymbol{e}_r = -\frac{GMm}{r^2}\langle\frac{x}{r},\frac{y}{r},\frac{z}{r}\rangle = -GMm\,\langle\frac{x}{r^3},\frac{y}{r^3},\frac{z}{r^3}\rangle$$

Operations on Vector Fields

Just as with other functions we can perform operations on vector fields. Two that are particularly useful in a variety of contexts are the *divergence* and the *curl*. In this section we'll show how these quantities are calculated and also provide an intuitive description of what they tell us about the vector field on which they operate. We'll have to wait for later lessons before we can make this intuitive description precise.

Divergence of a Vector Field

The divergence of a vector field, $\boldsymbol{F}$, results in a scalar function. In three dimensions it is defined as

$$div(\boldsymbol{F}) = \nabla \cdot \boldsymbol{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle F_1, F_2, F_3 \rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

The divergence generalizes to an arbitrary number of dimensions.

Curl of a Vector Field

The curl of a vector field, $\boldsymbol{F}$, results in a vector function. It is defined as

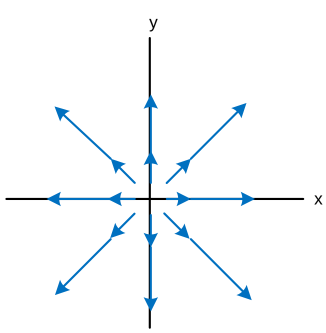
$$curl(\boldsymbol{F}) = \nabla \times \boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \langle \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \rangle$$

The curl is defined on three dimensions only.

Divergence

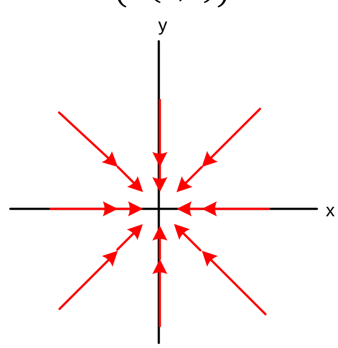
To help illustrate the divergence we can imagine a vector field that describes the velocity of a fluid in space. The divergence of this vector field represents the degree to which the fluid is flowing in towards or away from each point in space.

$div\big(\boldsymbol{F}(0,0)\big) > 0$



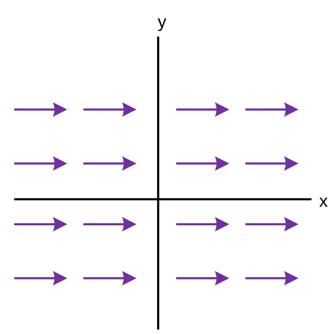
There is a net liquid flow *outward* from the origin.

$div\big(\boldsymbol{F}(0,0)\big) < 0$



There is a net liquid flow *inward* to the origin.

$div(\boldsymbol{F}) = 0$

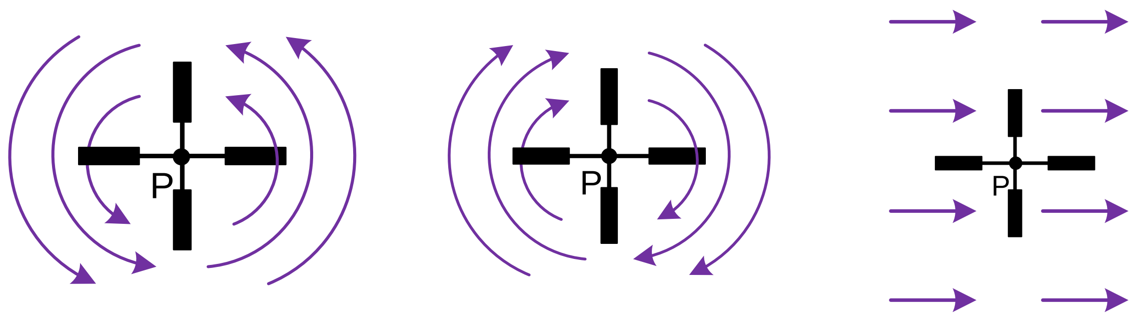


There is a net liquid flow of zero at any point in space.

Curl

We'll use the same fluid velocity vector field to illustrate the curl. In this case, the curl measures the amount to which the fluid circulates around a fixed axis at each point in space. In the figures below, we assume we are looking down on the x-y plane into a plane of fluid. Now imagine inserting a paddle wheel into the fluid. The magnitude of the curl represents the amount by which the paddle circulates. The direction is determined by the right-hand rule.

$$curl(\boldsymbol{F}(P))$$



The paddle wheel rotates counter clockwise and the resulting curl vector points out of the page.

The paddle wheel rotates clockwise and the resulting curl vector points into the page.

The paddle wheel does not rotate.

Conservative Vector Fields

Conservative vector fields make up an extremely important class of vector fields. In this section we'll state criteria for and introduce some of the properties of conservative vector fields. However, similar to the divergence and curl, we'll have to wait for later lessons to gain a more comprehensive understanding of all of the implications.

We begin by recalling the definition of the gradient vector.

$$\boldsymbol{F} = \nabla f = \langle \frac{\partial f(x,y,z)}{\partial x}, \frac{\partial f(x,y,z)}{\partial y}, \frac{\partial f(x,y,z)}{\partial z} \rangle$$

Although we did not explicitly describe the gradient vector as a vector field when we first introduced it, we are now in a position to recognize that the gradient is indeed a *vector field*. It's simply a special case of the general vector field definition where,

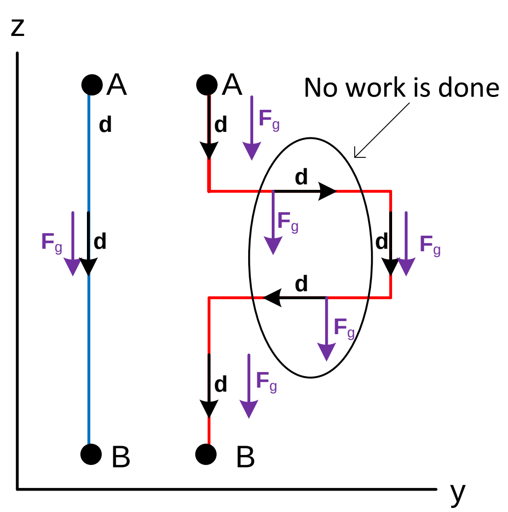
$$F_1(x,y,z) = \frac{\partial f(x,y,z)}{\partial x} \qquad F_2(x,y,z) = \frac{\partial f(x,y,z)}{\partial y} \qquad F_3(x,y,z) = \frac{\partial f(x,y,z)}{\partial z}$$

The function, $f(x, y, z)$, is referred to a *potential function*, (or a scalar potential function) and is the key for identifying conservative vector fields. Meaning if a potential function exists for a given vector field then the vector field is considered conservative, otherwise it's a non-conservative vector field. Therefore, a gradient vector field is, by definition, a conservative vector field.

Conservative vector fields have critically important properties that we will discover as we progress through the next few lessons. One of these properties is called *Path Independence.* Although we are not yet in a position to formally prove this property, it's implications can be presented, which will in turn give us an intuitive way to think about conservative vector fields.

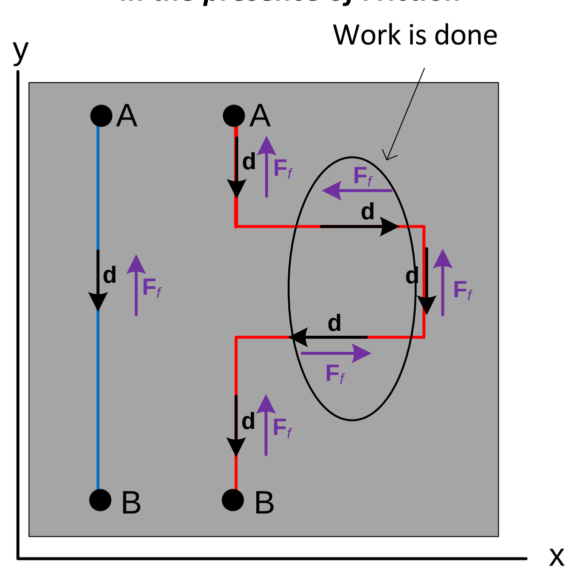
In physics conservative forces appear naturally as a consequence of the Law of The Conservation of Energy. One such conservative force is the gravitational force. Conversely, an example of a non-conservative force is the force of friction. You may recall from basic physics that the work done by a force is equal to the dot product of the force vector and a distance vector, i.e. $W = \boldsymbol{F} \cdot \boldsymbol{d}$. The path independence property states that the work done by a conservative force as an object travels from point A to B is *independent* of the path taken. The figure on the left illustrates the fact that no work is done when the object moves horizontally against the gravitational force field. Therefore, the same amount of work is done for the blue and red paths. Conversely, the figure on the right illustrates the fact that the work is done *regardless* of the direction of motion and therefore there is more work done traveling along the red path compared to the blue path.

Moving an object vertically down in the presence of Gravity



No work is done when object moves horizontally. Therefore, same amount of work done for the red and blue path, i.e. ***Conservative Force***.

Moving an object horizontally across a table in the presence of Friction



Work is done regardless of direction the object moves. Therefore, more work done for red path compared to the blue path, i.e. ***Non-Conservative Force***.

We can find another, more computational, way to determine whether a vector field is conservative by computing the curl of the gradient vector.

$$curl(\nabla f) = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \langle \left(\frac{\partial^2 f}{\partial yz} - \frac{\partial^2 f}{\partial zy}\right), \left(\frac{\partial^2 f}{\partial xz} - \frac{\partial^2 f}{\partial zx}\right), \left(\frac{\partial^2 f}{\partial xy} - \frac{\partial^2 f}{\partial yx}\right) \rangle$$

Clairaut's Theorem states that higher order mixed partials derivatives are equivalent regardless of the order in which they are carried out, e.g. $f_{xy} = f_{yx}$. Therefore, each component above is equal to zero. Furthermore, since the gradient vector field is conservative we can state the following.

Curl of a Conservative Vector Field
The vector field, $\boldsymbol{F}$, is conservative if $$curl(\boldsymbol{F}) = 0$$ Or equivalently, $$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \qquad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}, \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

Our final observation relates to potential functions. The fact that the curl of a conservative vector field is zero can be used to show that potential functions, similar to antiderivatives, are unique to within an additive constant, i.e. if $curl(\boldsymbol{F}) = 0$, the $curl(\boldsymbol{F} + \boldsymbol{C}) = 0$. This can be shown directly with the definition of the gradient vector field, $\boldsymbol{F} = \nabla f$.

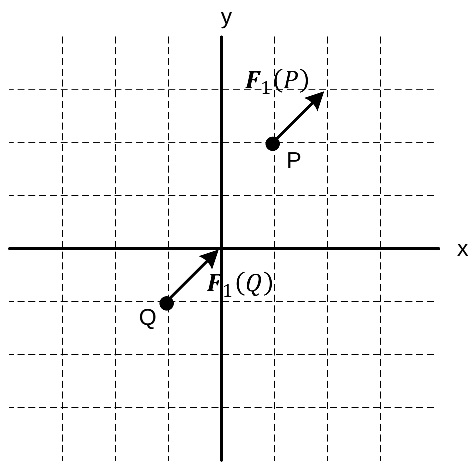
$$\nabla f = \nabla(f + C)$$
$$\nabla f = \nabla(f) + \nabla(C)$$
$$\nabla f = \nabla(f) + 0$$
$$\nabla f = \nabla(f)$$

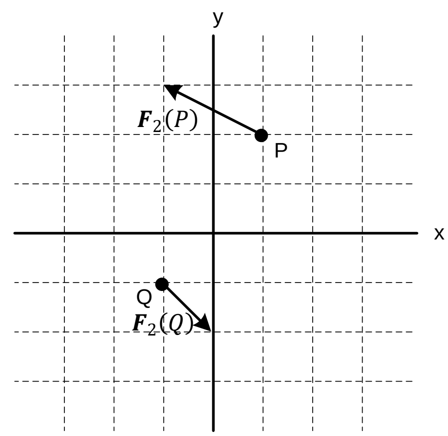
Example 1: Compute and sketch the vectors assigned to the points $P = (1,2)$ and $Q = (-1,-1)$ by the vector fields $\boldsymbol{F}_1 = \langle x^2, x\rangle$, and $\boldsymbol{F}_2 = \langle -y, x\rangle$.

Solution:

$$\boldsymbol{F}_1(P) = \langle x^2, x\rangle|_{(1,2)} = \langle 1,1\rangle \qquad \boldsymbol{F}_2(P) = \langle -y, x\rangle|_{(1,2)} = \langle -2,1\rangle$$

$$\boldsymbol{F}_1(Q) = \langle x^2, x\rangle|_{(-1,-1)} = \langle 1,1\rangle \qquad \boldsymbol{F}_2(Q) = \langle -y, x\rangle|_{(-1,-1)} = \langle 1,-1\rangle$$





Example 2: Compute the divergence and curl of the following vector fields.

1. $\boldsymbol{F} = \langle xy, yz, y^2 - x^3\rangle$ *2.* $\boldsymbol{F} = \langle z - y^2, x + z^3, y + x^2\rangle$ *3.* $\boldsymbol{F} = \langle e^y, sin(x), cos(x)\rangle$

Solution:

1. $\boldsymbol{F} = \langle xy, yz, y^2 - x^3\rangle$

$$\begin{aligned} div(\boldsymbol{F}) &= \nabla \cdot \boldsymbol{F} \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \\ &= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(y^2 - x^3) \\ &= y + z + 0 \\ &= y + z \end{aligned}$$

$$curl(\boldsymbol{F}) = \nabla \times \boldsymbol{F}$$

$$= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \langle \left(\frac{\partial}{\partial y}(y^2 - x^3) - \frac{\partial}{\partial z}(yz) \right), \left(\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(y^2 - x^3) \right), \left(\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xy) \right) \rangle$$

$$= \langle (2y - y), (0 - -3x^2), (0 - x) \rangle$$

$$= \langle y, 3x^2, -x \rangle$$

2. $\boldsymbol{F} = \langle z - y^2, x + z^3, y + x^2 \rangle$

$$div(\boldsymbol{F}) = \nabla \cdot \boldsymbol{F}$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(z - y^2) + \frac{\partial}{\partial y}(x + z^3) + \frac{\partial}{\partial z}(y + x^2)$$

$$= 0 + 0 + 0$$

$$= 0$$

$$curl(\boldsymbol{F}) = \nabla \times \boldsymbol{F}$$

$$= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \langle \left(\frac{\partial}{\partial y}(y + x^2) - \frac{\partial}{\partial z}(x + z^3) \right), \left(\frac{\partial}{\partial z}(z - y^2) - \frac{\partial}{\partial x}(y + x^2) \right), \left(\frac{\partial}{\partial x}(x + z^3) - \frac{\partial}{\partial y}(z - y^2) \right) \rangle$$

$$= \langle (1 - 3z^2), (1 - 2x), (1 - -2y) \rangle$$

$$= \langle 1 - 3z^2, 1 - 2x, 1 + 2y \rangle$$

3. $\boldsymbol{F} = \langle e^y, sin(x), cos(x) \rangle$

$$div(\boldsymbol{F}) = \nabla \cdot \boldsymbol{F}$$
$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$
$$= \frac{\partial}{\partial x}(e^y) + \frac{\partial}{\partial y}(sin(x)) + \frac{\partial}{\partial z}(\ cos(x))$$
$$= 0 + 0 + 0$$
$$= 0$$

$$curl(\boldsymbol{F}) = \nabla \times \boldsymbol{F}$$
$$= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$
$$= \left\langle \left(\frac{\partial}{\partial y}(cos(x)) - \frac{\partial}{\partial z}(sin(x)) \right), \left(\frac{\partial}{\partial z}(e^y) - \frac{\partial}{\partial x}(cos(x)) \right), \left(\frac{\partial}{\partial x}(sin(x)) - \frac{\partial}{\partial y}(e^y) \right) \right\rangle$$
$$= \langle (0 - 0), (0 - - sin(x)), (cos(x) - e^y) \rangle$$
$$= \langle 0, sin(x), cos(x) - e^y \rangle$$

Example 3: Find a potential function for the following vector fields by inspection or show that one does not exist.

1. $\boldsymbol{F} = \langle x, y \rangle$ *2.* $\boldsymbol{F} = \langle ye^{xy}, xe^{xy} \rangle$ *3.* $\boldsymbol{F} = \langle xy, x^2/2\,, zy \rangle$

Solution: If a given vector field has a potential function it is a conservative vector field. This statement can be written mathematically using the gradient vector field equation.

$$\boldsymbol{F} = \nabla f$$

Where, f, is the potential function for the vector field $\boldsymbol{F}$.

However, finding f by inspection may not be obvious. We also know that the curl of a conservative vector field is equal to zero.

$$curl(\boldsymbol{F}) = 0$$

We can use this to check if a vector field is non-conservative, thereby not wasting time looking for a potential function.

$$curl(\boldsymbol{F}) = 0$$

1. $\boldsymbol{F} = \langle x, y \rangle$

Assuming $\boldsymbol{F}$ is a gradient vector field it has a potential function, f, that satisfies

$$\boldsymbol{F} = \nabla f = \langle \frac{\partial}{\partial x} f(x,y), \frac{\partial}{\partial y} f(x,y) \rangle$$

Therefore, we need to find $f(x,y)$ such that

$$\frac{\partial}{\partial x} f(x,y) = x, \quad \frac{\partial}{\partial y} f(x,y) = y$$

Thinking of integration, it should be clear that the following function satisfies the above criteria.

$$f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + C$$

2. $\boldsymbol{F} = \langle ye^{xy}, xe^{xy} \rangle$

Assuming $\boldsymbol{F}$ is a gradient vector field it has a potential function, f, that satisfies

$$\boldsymbol{F} = \nabla f = \langle \frac{\partial}{\partial x} f(x, y), \frac{\partial}{\partial y} f(x, y) \rangle$$

Therefore, we need to find $f(x, y)$ such that

$$\frac{\partial}{\partial x} f(x, y) = ye^{xy}, \ \frac{\partial}{\partial y} f(x, y) = xe^{xy}$$

Thinking again of integration, the following function satisfies the above criteria.

$$f(x, y) = e^{xy} + C$$

3. $\boldsymbol{F} = \langle xy, x^2/2, zy \rangle$

Assuming $\boldsymbol{F}$ is a gradient vector field it has a potential function, f, that satisfies

$$\boldsymbol{F} = \nabla f = \langle \frac{\partial}{\partial x} f(x, y, z), \frac{\partial}{\partial y} f(x, y, z), \frac{\partial}{\partial z} f(x, y, z) \rangle$$

Therefore, we need to find $f(x, y, z)$ such that

$$\frac{\partial}{\partial x} f(x, y, z) = xy, \ \frac{\partial}{\partial y} f(x, y, z) = x^2/2, \ \frac{\partial}{\partial z} f(x, y, z) = zy$$

In this case a potential function is not as obvious so instead we check whether the vector field is non-conservative by checking if the curl is non-zero. If the curl of a vector field is zero all three equations below must be satisfied.

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \qquad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}, \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

Using only the first relation we already see the vector field is not conservative, and therefore a potential function does not exist.

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$$

$$\frac{\partial}{\partial y}(zy) = \frac{\partial}{\partial z}(x^2/2)$$

$$z \neq 0$$

Final Summary for Line and Surface Integrals – Vector Fields

Vector Field

A *vector field* is a function that assigns a vector to each point, $P = \langle x, y, z\rangle$, in space.
In three dimensions it is denoted as

$$\boldsymbol{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)\rangle$$

A unit vector field, $\boldsymbol{e}_F$, is defined as

$$\boldsymbol{e}_F = \frac{\boldsymbol{F}(x, y, z)}{\|\boldsymbol{F}(x, y, z)\|}$$

An important example is a unit radial vector.

Two dimensional unit radial vector	*Three dimensional unit radial vector*
$\boldsymbol{e}_r = \langle \frac{x}{r}, \frac{y}{r}\rangle$ Where, $r = \sqrt{x^2 + y^2}$	$\boldsymbol{e}_r = \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\rangle$ Where, $r = \sqrt{x^2 + y^2 + z^2}$

Divergence of a Vector Field

The divergence of a vector field, $\boldsymbol{F}$, results in a scalar function. In three dimensions it is defined as

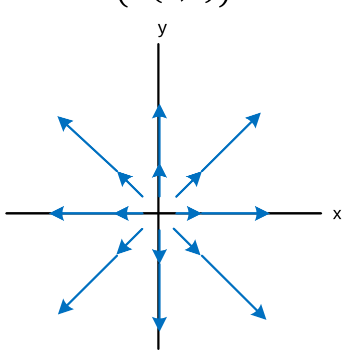
$$div(\boldsymbol{F}) = \nabla \cdot \boldsymbol{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \cdot \langle F_1, F_2, F_3\rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

The divergence generalizes to an arbitrary number of dimensions.

Divergence Intuition – Assume $\boldsymbol{F}$ is a fluid velocity vector field

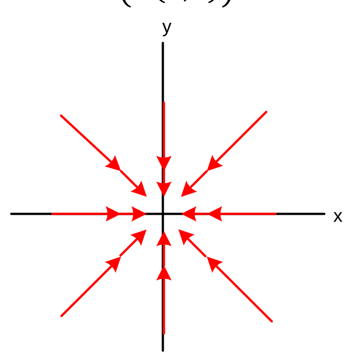
The divergence of a vector field represents the degree to which the fluid is flowing in towards or away from each point in space.

$div(\boldsymbol{F}(0,0)) > 0$



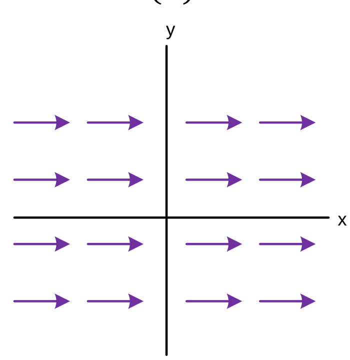
There is a net liquid flow *outward* from the origin.

$div(\boldsymbol{F}(0,0)) < 0$



There is a net liquid flow *inward* from the origin.

$div(\boldsymbol{F}) = 0$



There is a net flow of zero at any point in space.

Curl of a Vector Field

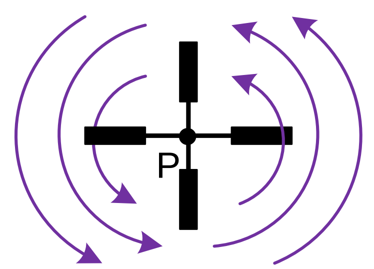
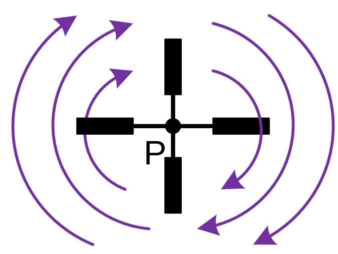
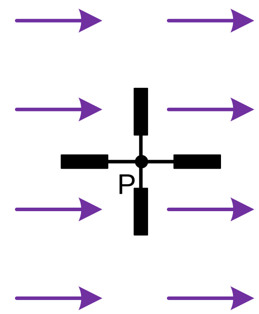
The curl of a vector field, $\boldsymbol{F}$, results in a vector function. It is defined as

$$curl(\boldsymbol{F}) = \nabla \times \boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left\langle \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \right\rangle$$

The curl is defined on three dimensions only.

Curl Intuition – Assume $\boldsymbol{F}$ is a fluid velocity vector field

The curl measures the amount to which the fluid circulates around a fixed axis at each point in space.

		
The paddle wheel rotates counterclockwise and the resulting vector points out of the page.	The paddle wheel rotates clockwise and the resulting vector points into the page.	The paddle wheel does not rotate.

Conservative Vector Fields

- If $\boldsymbol{F} = \nabla f$, then f is called the potential function for $\boldsymbol{F}$.
- $\boldsymbol{F}$ is called conservative it has a potential function.
- Potential functions are unique up to a constant, C.

The vector field, $\boldsymbol{F}$, is conservative if

$$curl(\boldsymbol{F}) = 0$$

Or equivalently,

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \qquad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}, \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

By: [ferrantetutoring]