Calculus 3 Summary and Formulas

One of the fundamental areas covered in Calculus 3 is performing calculus based operations on objects called vectors. Vectors are used to represent quantities that have both a direction and magnitude. They play a very important role in virtually all aspects of science and engineering. Therefore, our first series of lessons were used to develop the basic properties of vectors.

Basic Vector Definitions

A vector is used to express a quantity with both a magnitude and direction.

A vector, $\boldsymbol{v}$, is determined by a basepoint P and a terminal point Q as follows

$$\boldsymbol{v} = \overrightarrow{PQ} = \langle\, (Q_x - P_x), \left(Q_y - P_y \right), (\, Q_z - P_z) \,\rangle = \langle\, v_x, v_y, v_z \rangle$$

Where, v_x, v_y are called the components of the vector.

The magnitude, i.e. length, of a vector, $\boldsymbol{v}$, is referred to as the *norm* and is given by

$$\|\boldsymbol{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$$

A unit-vector is a vector that has a magnitude of one and can be expressed as follows:

$$\hat{\boldsymbol{v}} = \langle \cos\theta \,, \sin(\theta) \rangle$$

Furthermore, any vector, $\boldsymbol{v}$**,** can be scaled to be a unit vector as follows:

$$\hat{\boldsymbol{v}} = \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v}$$

Vector Operations Using Components

If $\boldsymbol{a} = \langle a_x,\ a_y \rangle$ and **b**= $\langle b_x,\ b_y \rangle$ then:

i.	*Addition*	$\boldsymbol{a} + \boldsymbol{b} = \langle a_x + b_x, a_y + b_y \rangle$
ii.	*Subtraction*	$\boldsymbol{a} - \boldsymbol{b} = \langle a_x - b_x, a_y - b_y \rangle$
iii.	*Scalar Multiplication*	$\lambda\boldsymbol{a} = \langle \lambda a_x, \lambda a_y \,\rangle$
iv.	*Addition Identity*	$\boldsymbol{a} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{a} = \boldsymbol{a}$

***Note: The operations are shown in 2 dimensions but apply equally in 3 dimensions.*

Basic Properties of Vector Algebra

For all vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and for all scalars, λ

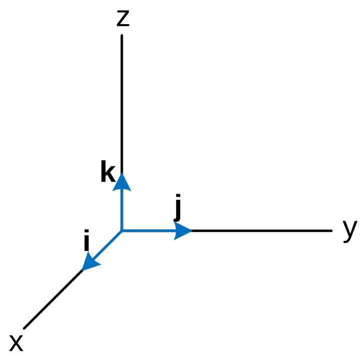
i.	Commutative Law	$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$
ii.	Associative Law	$\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{b} + \boldsymbol{a}) + \boldsymbol{c}$
iii.	Distributive law for Scalars	$\lambda(\boldsymbol{a} + \boldsymbol{b}) = \lambda\boldsymbol{b} + \lambda\boldsymbol{a}$

Standard Basis Vectors for Rectangular Coordinate System in 3-Dimensions

$\boldsymbol{i} = \langle 1, 0, 0 \rangle$

$\boldsymbol{j} = \langle 0, 1, 0 \rangle$

$\boldsymbol{k} = \langle 0, 0, 1 \rangle$



All vectors in can be written as a linear combination of the basic vectors.

$$\boldsymbol{v} = \langle a, b, c \rangle = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$$

Triangle Inequality

For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\|\boldsymbol{a} + \boldsymbol{b}\| \leq \|\boldsymbol{a}\| + \|\boldsymbol{b}\|$$

The equality holds when $\boldsymbol{a} = \boldsymbol{0}$ or $\boldsymbol{b} = \boldsymbol{0}$, or if $\boldsymbol{b} = \lambda\boldsymbol{a}$, where $\lambda > 0$.

The Dot Product

Given two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, as well as the angle, θ, between the two vectors.

The dot product can be equivalently be defined in the following two ways:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \, cos(\theta)$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = \left(a_x b_x + a_y b_y + a_z b_z\right)$$

Furthermore, if the angle is unknown it may be found as follows:

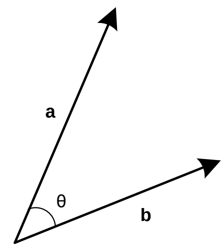
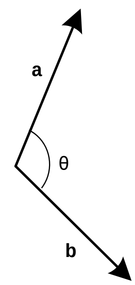
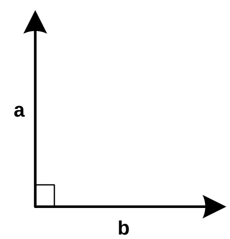
$$\theta = \cos^{-1}\left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\|\|\boldsymbol{b}\|}\right) = \cos^{-1}\left(\frac{a_x b_x + a_y b_y + a_z b_z}{\|\boldsymbol{a}\|\|\boldsymbol{b}\|}\right)$$

The angle between two vectors is chosen to satisfy $0 \leq \theta \leq \pi$

Properties of the Dot Product

1. *Commutative Property:* $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
2. *Zero Property:* $\boldsymbol{a} \cdot 0 = \boldsymbol{0}$
3. *Scalar Multiplication Property:* $\lambda(\boldsymbol{a} \cdot \boldsymbol{b}) = (\lambda\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (\lambda\boldsymbol{b})$
4. *Distributive Property:* $\boldsymbol{v} \cdot (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{v} \cdot \boldsymbol{b} + \boldsymbol{v} \cdot \boldsymbol{b}$
5. *Relation to Length:* $\boldsymbol{v} \cdot \boldsymbol{v} = \|\boldsymbol{v}\|^2$

Geometric Properties of The Dot Product

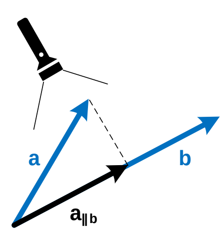
$\boldsymbol{a} \cdot \boldsymbol{b} > \mathbf{0}$		The angle between the two vectors is acute, i.e. $0^{\circ} \leq \theta < 90^{\circ}$
$\boldsymbol{a} \cdot \boldsymbol{b} < \mathbf{0}$		The angle between the two vectors is obtuse, i.e. $90^{\circ} < \theta \leq 180^{\circ}$
$\boldsymbol{a} \cdot \boldsymbol{b} = \mathbf{0}$		The angle between the two vectors is 90°. **Note: We use the word *orthogonal* to refer to vectors that form a $\mathbf{90}^{\circ}$ angle.**

Projection Vector

The projection of a vector $\boldsymbol{a}$ onto the vector $\boldsymbol{b}$ is the vector, $\boldsymbol{a}_{||\boldsymbol{b}}$ given by

$$\boldsymbol{a}_{||\boldsymbol{b}} = \left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\right) \boldsymbol{b} = \left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{||\boldsymbol{b}||^2}\right) \boldsymbol{b} = \left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{||\boldsymbol{b}||}\right) \widehat{\boldsymbol{b}}$$

The scalar, $\left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{||\boldsymbol{b}||}\right) = ||\boldsymbol{a}|| \, cos(\theta)$, is called the component of $\boldsymbol{a}$ along $\boldsymbol{b}$**.**



Vector Decomposition

Any vector, $\boldsymbol{a}$ can be decomposed into two orthogonal component vectors with respect to another vector, $\boldsymbol{b}$ as:

$$\boldsymbol{a} \; = \boldsymbol{a}_{\|\boldsymbol{b}} + \boldsymbol{a}_{\perp\boldsymbol{b}}$$

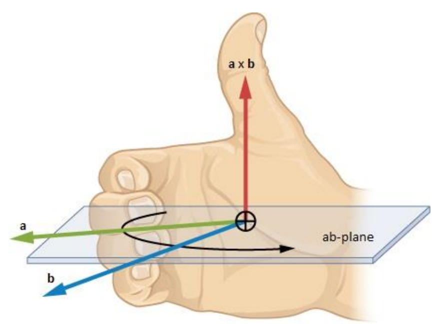
Where the parallel projection is given above, and the perpendicular projection is found as:

$$\boldsymbol{a}_{\perp\boldsymbol{b}} \; = \boldsymbol{a} - \boldsymbol{a}_{\|\boldsymbol{b}}$$

The Cross Product
The cross product of two vectors, $\boldsymbol{a} = \langle a_x, a_y, a_z \rangle$ and $\boldsymbol{b} = \langle b_x, b_y, b_z \rangle$ is a new vector $\boldsymbol{v}$, given as $$\boldsymbol{v} = \boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{\boldsymbol{\imath}} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{\boldsymbol{\jmath}} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{\boldsymbol{k}} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$$ $$= (a_y b_z - b_y a_z)\hat{\boldsymbol{\imath}} - (a_x b_z - b_x a_z)\hat{\boldsymbol{\jmath}} + (a_x b_y - b_x a_y)\hat{\boldsymbol{k}}$$

Geometric Interpretation of the Cross Product
Given two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, the cross product, $\boldsymbol{a} \times \boldsymbol{b}$ is a unique vector with the following properties. i. $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to $\boldsymbol{a}$ and $\boldsymbol{b}$**.** ii. The length of $\boldsymbol{a} \times \boldsymbol{b}$ is $\\|\boldsymbol{a}\\|\\|\boldsymbol{b}\\| \sin(\theta)$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ and is chosen to satisfy $0 \leq \theta \leq \pi$.

The Right-Hand Rule
The right-hand rule can be stated as: *The vector, $\boldsymbol{a} \times \boldsymbol{b}$, is orthogonal to a plane that is parallel to $\boldsymbol{a}$ and $\boldsymbol{b}$. Furthermore, when the fingers of your right hand curl from $\boldsymbol{a}$ to $\boldsymbol{b}$, your thumb points to the side of the plane for which the resulting vector points.*



Properties of the Cross Product
i. $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$
ii. $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$
iii. $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$ is and only if $\boldsymbol{a} = \lambda \boldsymbol{b}$ for some scalar λ or $\boldsymbol{b} = \boldsymbol{0}$
iv. $\lambda(\boldsymbol{a} \times \boldsymbol{b}) = (\lambda \boldsymbol{a}) \times \boldsymbol{b} = \boldsymbol{a} \times (\lambda \boldsymbol{b})$
v. $\boldsymbol{c} \times (\boldsymbol{a} + \boldsymbol{b}) = (\boldsymbol{c} \times \boldsymbol{a}) + (\boldsymbol{c} \times \boldsymbol{b})$**,** $(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a} \times \boldsymbol{c}) + (\boldsymbol{b} \times \boldsymbol{c})$

Properties of the Cross Product

vi. $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$

vii. $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$

viii. $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$ is and only if $\boldsymbol{a} = \lambda \boldsymbol{b}$ for some scalar λ or $\boldsymbol{b} = \boldsymbol{0}$

ix. $\lambda(\boldsymbol{a} \times \boldsymbol{b}) = (\lambda \boldsymbol{a}) \times \boldsymbol{b} = \boldsymbol{a} \times (\lambda \boldsymbol{b})$

x. $\boldsymbol{c} \times (\boldsymbol{a} + \boldsymbol{b}) = (\boldsymbol{c} \times \boldsymbol{a}) + (\boldsymbol{c} \times \boldsymbol{b})$, $(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a} \times \boldsymbol{c}) + (\boldsymbol{b} \times \boldsymbol{c})$

Cross Product of The Standard Basis Vectors

$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}$	$\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}} = \hat{\boldsymbol{\imath}}$	$\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}}$
$\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}} = -\hat{\boldsymbol{k}}$	$\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\jmath}} = -\hat{\boldsymbol{\imath}}$	$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{k}} = -\hat{\boldsymbol{\jmath}}$
$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}} = \boldsymbol{0}$	$\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}} = \boldsymbol{0}$	$\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} = \boldsymbol{0}$

Area and Cross Product

If $\mathcal{P}$ is the parallelogram formed by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the area, $A_{\mathcal{P}}$, can be found as

$$A_{\mathcal{P}} = \|\boldsymbol{a} \times \boldsymbol{b}\|$$

Volume and Cross Product

If $\mathcal{P}$ is the parallelepiped formed by the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, then the volume, $V_{\mathcal{P}}$, can be found as

$$V_{\mathcal{P}} = |(\boldsymbol{c}) \cdot (\boldsymbol{a} \times \boldsymbol{b})|$$

Where, $(\boldsymbol{c}) \cdot (\boldsymbol{a} \times \boldsymbol{b})$ is referred to as the *vector triple product* and can be represented as

$$(\boldsymbol{c}) \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \begin{vmatrix} c_x & c_y & c_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \boldsymbol{det}\begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}$$

Equation of a Line in R^3 (Point-Direction Form)

The line $\mathcal{L}$ through the point (x_0, y_0, z_0) in the directions of $\boldsymbol{v} = \langle a, b, c \rangle$ can be described in the following ways:

Vector Parameterization:

$$\boldsymbol{r}(t) = \boldsymbol{r_0} + t\boldsymbol{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

Parametric Equations:

$$x(t) = x_0 + at \qquad y(t) = y_0 + bt \qquad z(t) = z + ct$$

Where $(-\infty < t < \infty)$

Parallel, Perpendicular, and Intersecting Lines

Two lines are parallel when the cross product of their direction vectors is zero.

$$\boldsymbol{v_1} \times \boldsymbol{v_2} = \boldsymbol{0}$$

Two lines are perpendicular when the dot product of their direction vectors is zero.

$$\boldsymbol{v_1} \cdot \boldsymbol{v_2} = 0$$

The point of intersection between two lines can be found by setting the equations of the lines equal to one another for different values of the parameter t.

$$\boldsymbol{r_1}(t_1) = \boldsymbol{r_2}(t_2)$$

If an intersection point does not exist *and* the lines are not parallel, we refer to them as skewed.

Equation of a Plane in $\boldsymbol{R^3}$ (Point-Normal Form)

The plane $\mathcal{P}$ through the point (x_0, y_0, z_0) with a normal vector $\boldsymbol{n} = \langle a, b, c \rangle$ can be described in the following ways:

Vector Form:

$$\boldsymbol{n} \cdot \langle x, y, z \rangle = d$$

Scalar Form:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = d$$

Where, $d = ax_0 + by_0 + cz_0$

Parallel and Intersecting Planes

Two planes are parallel when the cross product of their normal vectors is zero.

$$\boldsymbol{n_1} \times \boldsymbol{n_2} = \boldsymbol{0}$$

Two planes are perpendicular when the dot product of their normal vectors is zero.

$$\boldsymbol{n_1} \cdot \boldsymbol{n_2} = 0$$

When two planes are not parallel, they intersect along a line, *Line of Intersection (LOS).* The direction vector of the LOS is given as

$$\boldsymbol{v} = \boldsymbol{n_1} \times \boldsymbol{n_2}$$

Parallel and Perpendicular Lines and Planes

A plane and a line are parallel when the dot product of the normal and direction vector is zero.

$$\boldsymbol{n} \cdot \boldsymbol{v} = 0$$

A plane and a line are perpendicular when the cross product of the normal and direction vector is zero.

$$\boldsymbol{n} \times \boldsymbol{v} = \boldsymbol{0}$$

Calculus 1 and 2 focused on single variable calculus. As you can see from the vector summary above, Calculus 3 studies multivariable functions also. Quadric surfaces are quadratic equations in three variables and are common in Calculus 3. In addition, other coordinates system are utilized in Calculus 3, usually to render a particular problem much easier

Quadric Surface

A quadric surface is defined by a quadratic equation in three variables. The general form is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + ax + by + cz + d = 0$$

When $D = E = F = a = b = c = 0$, the quadric axes are parallel to the coordinate axes and the surface is centered at $(0,0)$. When this is the case the equations are said to be in *standard form*.

Quadric Surfaces in Standard Form

1. **Sphere**: Centered at $(0,0,0)$ with a radius r.

$$x^2 + y^2 + z^2 = r^2$$

2. **Ellipsoid**: Centered at $(0,0,0)$ with x, y, and z 'radius' equal to a, b, and c respectively.

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

3. **Hyperboloid**:

One Sheet	$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 + 1$
Two Sheets	$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 - 1$
Elliptical Cone (limiting case of one sheet)	$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 + 0$

4. **Paraboloid:**

Elliptical (bowl)	$z = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$
Hyperbolic (Saddle)	$z = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$

Trace

A trace is the intersection of a surface with a given plane. A trace can be obtained by 'freezing' one of the three variables and sketching the resulting 2D equation. Traces can be used to help us to draw the graph of a surface.

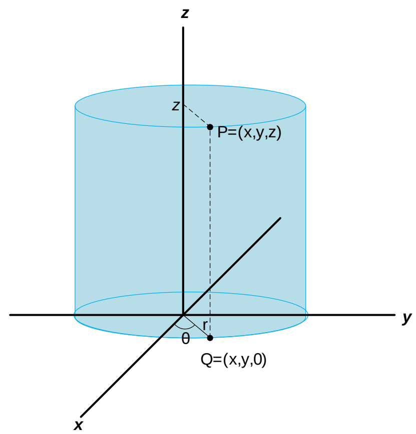
Horizontal Trace:

Setting $z = z_0$ results in a curve in an xy oriented plane.

Vertical Trace:

Setting $x = x_0$ results in a curve in a yz oriented plane.
Setting $y = y_0$ results in a curve in an xz oriented plane.

Cylindrical Coordinate System



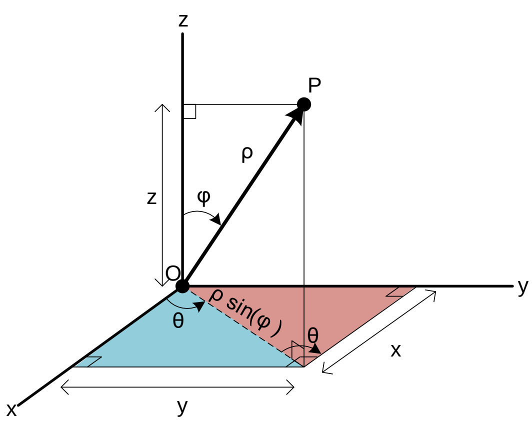
A point in cylindrical coordinates is described by: $P = (r, \theta, z)$

- r: The horizontal distance from the origin.
- θ: The polar angle measured from the positive x-axis.
- z: The vertical distance from the origin.

Rectangular and Cylindrical Coordinate Conversion Formulas

Cylindrical to Rectangular	***Rectangular to Cylindrical***
$x = r\,cos(\theta)$ $y = r\,sin(\theta)$ $z = z$	$r = \sqrt{x^2 + y^2}$ $\theta = tan^{-1}\left(\frac{y}{x}\right)$ $z = z$

Spherical Coordinate System



A point in spherical coordinates is described by: $P = (\rho, \theta, \phi)$

- ρ: The distance from the origin to the point, P, where $\rho \geq 0$
- θ: The angle of the projection for $\overrightarrow{0P}$ onto the x-y plane, where $-180^{\circ} \leq \theta \leq 180^{\circ}$
- ϕ: The angle of declination, which measures how much the vector, $\overrightarrow{0P}$, declines form the vertical, where $0^{\circ} \leq \phi \leq 180^{\circ}$

Rectangular and Spherical Coordinate Conversion Formulas

Spherical to Rectangular	***Rectangular to Spherical***
$x = \rho\,sin(\phi)\,cos(\theta)$ $y = \rho\,sin(\phi)\,sin(\theta)$ $z = \rho\,cos(\phi)$	$\rho = \sqrt{x^2 + y^2 + z^2}$ $\theta = tan^{-1}\left(\frac{y}{x}\right)$ $\phi = cos^{-1}\left(\frac{z}{\rho}\right)$

Level Surfaces

Level Surfaces are surfaces obtained by setting one of the coordinates to a constant.

Rectangular Coordinate System:

- $x = C$: Vertically aligned plane parallel to the y-z plane.
- $y = C$: Vertically aligned plane parallel to the x-z plane.
- $z = C$: Horizontally aligned plane parallel to the x-y plane

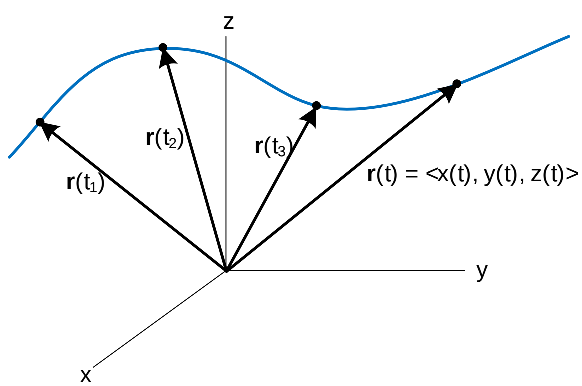
Cylindrical Coordinate System:

- $r = C$: Cylinder with radius, C.
- $\theta = C$: Vertical half plane oriented at an angle, C.
- $z = C$: Horizontally aligned plane parallel to the x-y plane

Spherical Coordinate System:

- $\rho = C$: Sphere with radius, C
- $\theta = C$: Vertical half plane oriented at an angle, C.
- $\phi = C$: *R*ight circular cone with an opening at an angle, C.

Once we understood the basics of vectors our next series of lessons focused on performing calculus on vectors. One of the main applications of vector calculus is the ability to study motion in 3 dimensions.

Vector-Valued Function
A vector-valued function is any function whose domain is a set of real number and whose range is a set of vectors. The variable t is called a parameter, which doesn't necessarily have to represent time, and the functions $x(t)$, $y(t)$ and $z(t)$ are called the components or coordinate functions. $$\boldsymbol{r}(t) = \langle x(t), y(t), z(t) \rangle$$ We can also represent the vector parameterization of a path as a curve with a set of parametric equations as $$c(t) = \big(x(t), y(t), z(t)\big)$$ Note: The curve is the set of all points, $x(t), y(t), z(t)$, as t varies over its domain. However, the path referred to by $\boldsymbol{r}(t)$ represents the particular way the curve is traversed, e.g. it may traverse the curve several times, reverse direction, move back and forth, etc. 
Projections
Projections of $\boldsymbol{r}(t)$ onto a plane can help us sketch the underlying curve. We project onto each plane by setting the third coordinate to zero. Projection onto x-y plane: Let $z(t) = 0$, $\boldsymbol{r}(t) = \langle x(t), y(t), 0 \rangle$ Projection onto x-z plane: Let $y(t) = 0$, $\boldsymbol{r}(t) = \langle x(t), 0, z(t) \rangle$ Projection onto y-z plane: Let $x(t) = 0$, $\boldsymbol{r}(t) = \langle 0, y(t), z(t) \rangle$

Derivative of Vector-Valued Function

The derivative of the vector-valued function $\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$, is computed component-wise as

$$\frac{d}{dt}\big(\boldsymbol{r}(t)\big) = \boldsymbol{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$$

Provided each component is differentiable.

Differentiation Rules for Vector-Valued Functions

Sum Rule: $$\frac{d}{dt}\big(\boldsymbol{r_1}(t) + \boldsymbol{r_2}(t)\big) = \frac{d}{dt}\big(\boldsymbol{r_1}(t)\big) + \frac{d}{dt}\big(\boldsymbol{r_2}(t)\big)$$

Constant Multiple Rule: $$\frac{d}{dt}\big(c\boldsymbol{r}(t)\big) = c\frac{d}{dt}\big(\boldsymbol{r}(t)\big)$$

Scaler Product Rule: $$\frac{d}{dt}\big(f(t)\boldsymbol{r}(t)\big) = f'(t)\boldsymbol{r}(t) + f(t)\boldsymbol{r}'(t)$$

Dot Product Rule: $$\frac{d}{dt}\big(\boldsymbol{r_1}(t)\cdot\boldsymbol{r_2}(t)\big) = \boldsymbol{r_1}'(t)\cdot\boldsymbol{r_2}(t) + \boldsymbol{r_1}(t)\cdot\boldsymbol{r_2}'(t)$$

Cross Product Rule: $$\frac{d}{dt}\big(\boldsymbol{r_1}(t)\times\boldsymbol{r_2}(t)\big) = \boldsymbol{r_1}'(t)\times\boldsymbol{r_2}(t) + \boldsymbol{r_1}(t)\times\boldsymbol{r_2}'(t)$$

Chain Rule: $$\frac{d}{dt}\Big(\boldsymbol{r}\big(f(t)\big)\Big) = \boldsymbol{r}'\big(f(t)\big)f'(t)$$

Derivative of Vector-Valued Function as a Tangent Vector

The derivative at t_0, $\boldsymbol{r}'(t_0)$, is a vector that is tangent to the path, $\boldsymbol{r}(t)$, at t_0.

The tangent line to the path, $\boldsymbol{r}(t)$, at t_0 can be written as

$$\boldsymbol{L}(t) = \boldsymbol{r}(t_0) + t\boldsymbol{r}'(t_0)$$

Orthogonality of $\boldsymbol{r}$ and $\boldsymbol{r}'$ when $\boldsymbol{r}$ has a Constant Length

If $\boldsymbol{r}(t)$ is a differentiable vector-valued function in R^2 or R^3, and if $\|\boldsymbol{r}(t)\|$ is constant for all t, then $\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t) = 0$. That is, $\boldsymbol{r}(t)$ and $\boldsymbol{r}'(t)$ are orthogonal vectors for all t.

Indefinite and Definite Integral of a Vector-Valued Function

The Indefinite Integral of a vector-valued function is defined as

$$\int \boldsymbol{r}(t)dt = \langle \int x(t)dt, \int y(t)dt, \int z(t)dt\rangle + \boldsymbol{c}$$

The Definite Integral of a vector-valued function is defined as

$$\int_a^b \boldsymbol{r}(t)dt = \langle \int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt\rangle$$

Arc Length - The length of a Path (Distance Traveled)

Assume $\boldsymbol{r}(t)$ is differentiable and $\boldsymbol{r}'(t)$ is continuous on $[a, b]$. Then the distance, s, a particle travels along the path, $\boldsymbol{r}(t)$, for $a \leq t \leq b$ is equal to

$$s = \int_a^b \|\boldsymbol{r}'(t)\| \, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt$$

The distance traveled as a function of t can also be written as

$$s(t) = \int_a^t \|\boldsymbol{r}'(u)\| \, du$$

Which, we sometimes refer to as the ***arc length function***.

Position, Velocity, Distance, and Speed Relationships

Given the following:

- $\boldsymbol{v}(t)$: The velocity of a particle at time t.
- $v(t)$: The speed of a particle at time t.
- $\boldsymbol{r}(t)$: The position of a particle at time t.
- $s(t)$: The distance a particle has traveled at time t.

We can write the following relationships:

The velocity is the time derivative of position: $\boldsymbol{v}(t) = \boldsymbol{r}'(t)$

The speed is the magnitude of velocity: $v(t) = \|\boldsymbol{v}(t)\| = \|\boldsymbol{r}'(t)\|$

The position is the time integral of velocity: $\boldsymbol{r}(t) = \int \boldsymbol{v}(t) \, dt + \boldsymbol{r}(a)$

The distance traveled, arc length, is the time integral of speed: $s(t) = \int_a^t \|\boldsymbol{r}'(u)\| \, du$

Arc Length (Unit Speed) Parameterization

The arc length parameterization of a curve is one in which the speed is unity, i.e. $\|\boldsymbol{v}(s)\| = 1$. This restriction, $\|\boldsymbol{v}(s)\| = 1$, allows for the creation of a unique parameterization that focusing on the shape of the curve only and not on the particular way in which it is traversed.

Starting with any parameterization, $\boldsymbol{r}(t)$, we proceed as follows:

Step 1: Find the arc length function.

$$s = g(t) = \int_a^t \|\boldsymbol{r}'(u)\| \, du$$

Step 2: Compute the following inverse function.

$$t = g^{-1}(s)$$

Step 3: Create the new unit speed parameterization as follows:

$$\boldsymbol{r}(s) = \boldsymbol{r}\big(g^{-1}(s)\big)$$

Curvature

Curvature is a positive numerically positive value that measures how a curve bends. It is defined based using the arc length parametrization of a curve as specified below.

Let $\boldsymbol{r}(s)$ be an arc length parameterization and $\boldsymbol{T} = \boldsymbol{T}(s)$ be the unit tangent vector. The curvature at $\boldsymbol{r}(s)$ is defined as follows:

$$\kappa(s) = \left\| \frac{d\boldsymbol{T}}{ds} \right\|$$

Where,

$$\boldsymbol{T} = \boldsymbol{T}(s) = \boldsymbol{r}'(s)$$

Note: This assumes $\boldsymbol{r}'(t) \neq 0$ for all t.

Curvature Defined for Arbitrary Parameterizations

Alternate forms for computing the curvature can be derived without using the arc length parametrization as shown below.

If $\boldsymbol{r}(t)$ is an arbitrary parameterization, the curvature can be computed with either of the two formulas:

$$\kappa(t) = \frac{1}{v(t)} \left\| \frac{d\boldsymbol{T}}{dt} \right\| \qquad \kappa(t) = \frac{\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\|}{\|\boldsymbol{r}'(t)\|^3}$$

Curvature of a Graph in a Plane

The curvature of the graph of $y = f(x)$ is equal to

$$\kappa(x) = \frac{|f''(x)|}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}}$$

Frenet Frame

A unit vector that is tangent to a space curve, $\boldsymbol{r}(t)$, for all t is called the ***unit tangent vector*** and is given as

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|}$$

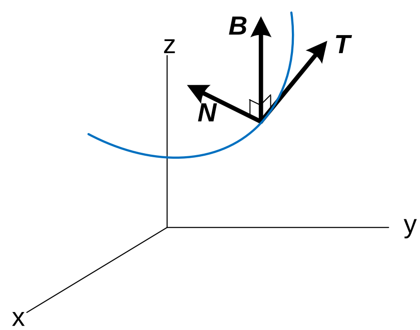
A unit normal vector that is orthogonal to $\boldsymbol{T}(t)$ for all t and points in the direction that the curve is turning is called the ***unit normal vector*** and is given as

$$\boldsymbol{N}(t) = \frac{\boldsymbol{T}'(t)}{\|\boldsymbol{T}'(t)\|}$$

A unit vector that is orthogonal to both $\boldsymbol{T}(t)$ and $\boldsymbol{N}(t)$ is called a ***unit binormal vector*** and is give as

$$\boldsymbol{B}(t) = \boldsymbol{T}(t) \times \boldsymbol{N}(t)$$

The three vectors, $(\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B})$**,** are mutually orthogonal and of unit length. Together they form an orthonormal set of vectors, which we refer to as the ***Frenet Frame****.* The Frenet frame is a function of the underlying curve and changes from point to point along the curve. As such, it is very useful in analyzing motion of objects in space

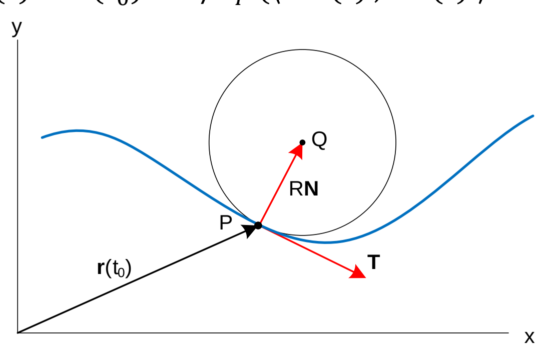


Osculating Circle

The *osculating circle* to a plane curve, $\boldsymbol{r}(t)$, at the point P is the circle that "best fits" the curve at P. The center of the circle lies in the direction of the normal vector, $\boldsymbol{N}$, to the curve, and the radius of the circle is called the *radius of curvature*, $R = 1/\kappa_P$.

The equation of the osculating circle to the plane curve, $\boldsymbol{r}(t)$, at $P = \boldsymbol{r}(t_0)$ is given as follows:

$$\boldsymbol{r_c}(t) = \boldsymbol{r}(t_0) + 1/\kappa_P\,(\langle cos(t)\,, sin(t)\,\rangle + \boldsymbol{N}_P)$$



Motion Describing Quantities

$\boldsymbol{r}(t)$	:	Position Vector – Represents the Position of an Object	:	$\boldsymbol{r}(t) = \langle x(t), y(t), z(t) \rangle$
$\boldsymbol{v}(t)$	:	Velocity Vector – Rate of change of Position	:	$\boldsymbol{v}(t) = \boldsymbol{r}'(t)$.
$v(t)$	:	Speed – Magnitude of Velocity	:	$v(t) = \lVert \boldsymbol{v}(t) \rVert$
$\boldsymbol{a}(t)$	:	Acceleration Vector - Rate of change of Velocity	:	$\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \boldsymbol{r}''(t)$

Acceleration Vector Decomposition

The acceleration vector for an object traveling along a path is given as

$$\boldsymbol{a}(t) = a_T \boldsymbol{T}(t) + a_N \boldsymbol{N}(t)$$

Where, $a_T = v'(t)$, *and* $a_N = \kappa v^2(t)$

- ***The Tangential Component "encodes" the change in the speed***
  - Since $a_T = v'(t)$ the tangential component is zero if the speed is constant.
- ***The Normal Component "encodes" the change in direction***
  - Since $a_N = \kappa v^2(t)$ the normal component is zero if $\kappa = 0$, which is the case when the path does not change direction.

The decomposition vectors can also be evaluated using the following formulas.

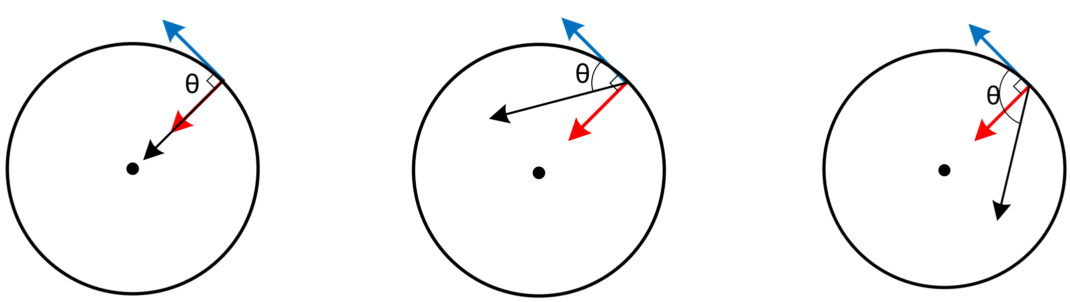
$$a_T \boldsymbol{T}(t) = \left(\frac{\boldsymbol{a}(t) \cdot \boldsymbol{v}(t)}{\lVert \boldsymbol{v}(t) \rVert^2} \right) \boldsymbol{v}(t)$$

$$\begin{aligned} a_N \boldsymbol{N}(t) &= \boldsymbol{a}(t) - a_T \boldsymbol{T}(t) \\ &= \boldsymbol{a}(t) - \left(\frac{\boldsymbol{a}(t) \cdot \boldsymbol{v}(t)}{\lVert \boldsymbol{v}(t) \rVert^2} \right) \boldsymbol{v}(t) \end{aligned}$$

Non-Uniform Circular Motion

$$v'(t) = a_T = \boldsymbol{a}(t) \cdot \boldsymbol{T}(t) = \lVert \boldsymbol{a}(t) \rVert \lVert \boldsymbol{T}(t) \rVert \, cos(\theta)$$

$(\boldsymbol{A})$ $(\boldsymbol{B})$ $(\boldsymbol{C})$



A. $\theta = 90^\circ$: Therefore, $cos(\theta) = 0$ and $v'(t) = 0$. The particles *speed is constant*, which results on uniform circular motion as shown in example 4.

B. $\theta < 90^\circ$: Therefore, $cos(\theta) > 0$ and $v'(t) > 0$. The particles *speed is increasing*.

C. $90^\circ < \theta < 180^\circ$: Therefore, $cos(\theta) < 0$ and $v'(t) < 0$. The particles *speed is decreasing*

Next series of lessons focused on differentiation of multivariable scalar functions. One way vectors play a role for multivariable differentiation is the derivative of multivariable functions are directional in the sense that the object has both a magnitude and direction.

Multivariable Functions

A multivariable function is one that takes n real variables as inputs, $(x_1, x_2, \ldots, x_n)$, and assigns a single value, y, to each n-tuple $(x_1, x_2, \ldots, x_n)$ in a domain in R^n. The range is the set of all y values for the $(x_1, x_2, \ldots, x_n)$ in the domain.

- $(x_1, x_2, \ldots, x_n)$ are called the independent variables.
- y is the dependent variable.

The function is represented as

$$y = f(x_1, x_2, \ldots, x_n)$$

Traces, Level Curves, and Contour Maps

- *Vertical Trace*
  - The intersection of the graph with a vertical plane obtained by setting x or y to a.
    - Vertical trace parallel with the y-z plane: Consists of all points $(a, y, f(a, y)\,)$.
    - Vertical trace parallel with the x-z plane: Consists of all points $(x, a, f(x, a)\,)$.
- *Horizontal Trace*
  - The intersection of the graph with a horizontal plane obtained by setting $f(x, y)$ to c.
    - Horizontal traces are parallel to the x-y plane and consist of all points $(x, y, c\,)$.
- *Level Curve*
  - The projection of a horizontal trace in the x-y plane.
    - The curve $f(x, y) = c$ in the x-y plane.
- *Contour Map*
  - A plot in the x-y plane showing level curves $f(x, y) = c$ for equally spaced values of c.
- *Contour Interval*
  - The interval, m, between the level curves in a contour map.
  - When moving from one level curve to the next, the value of $f(x, y)$ changes by $\pm m$.

Contour Maps and Rate of Change

- The level curves on a contour map are drawn at equally spaced changes in $f(x, y)$.
- The spacing between level curves on a contour map indicates the "steepness" of the change in $f(x, y)$.
- The average rate of change from a point P to a point Q on a contour map, $A\Delta_{P\to Q}$,is

$$A\Delta_{P\to Q} = \frac{\Delta\, Functuion\ value}{\Delta\, Horizonatl\ Distance}$$

When the function represents the physical height of an area, we usually say

$$A\Delta_{P\to Q} = \frac{\Delta\, Altitude}{\Delta\, Horizonatl\ Distance}$$

Partial Derivatives
Partial derivatives are defined for multivariable functions. They are derivatives with respect to one of the variables. Specifically, when computing a partial derivative for a generic multivariable function, e.g. $f(x_1, x_2, \dots, x_n)$, with respect to a specific variable, e.g. x_1, we treat all other variables, e.g. $x_2, \dots x_n$, as if they are constant values.

Partial Derivatives for Two Variable Functions	
The partial derivative of $f(x, y)$ with respect to x is defined $f_x(x,y) = \lim_{h \to 0} \left\{ \frac{f(x+h,y) - f(x,y)}{h} \right\}$ Equivalent Notations $f_x(x,y) = f_x = \frac{\partial}{\partial x} f(x,y) = \frac{\partial f}{\partial x}$	The partial derivative of $f(x, y)$ with respect to y is defined $f_y(x,y) = \lim_{h \to 0} \left\{ \frac{f(x,y+h) - f(x,y)}{h} \right\}$ Equivalent Notations $f_y(x,y) = f_y = \frac{\partial}{\partial y} f(x,y) = \frac{\partial f}{\partial y}$

Partial Differentiation Algebraic Rules	
Sum Rule:	$\frac{\partial}{\partial x}(f \pm g) = \frac{\partial f}{\partial x} \pm \frac{\partial g}{\partial x}$
Product Rule:	$\frac{\partial}{\partial x}(fg) = \frac{\partial f}{\partial x} g + f \frac{\partial g}{\partial x}$
Quotient Rule:	$\frac{\partial}{\partial x}\left(\frac{f}{g}\right) = \frac{\frac{\partial f}{\partial x} g - f \frac{\partial g}{\partial x}}{g^2}$

Higher Order Partial Derivatives and Clairaut's Theorem
Similar to single variable derivatives, higher order partial derivatives are derivatives of derivatives. For example, the second order partial derivative with respect to x is $f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ For multivariable functions we also have what are called mixed partials, e.g. f_{xy} and f_{yx}. ***Clairaut's Theorem*** states that the order in which we choose to perform the derivatives does not matter, provided the mixed partials are continuous functions. In other words, for two variable functions the theorem guarantees the following: $f_{xy} = f_{yx}$

Equation of the Tangent Plane and Normal Line
The tangent plane to the surface, $f(x,y)$, at the point (x_0, y_0, z_0) is given by $$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + z_0$$ The normal line to the surface is given as $$\boldsymbol{n}(t) = \langle (x_0 + f_x(x_0, y_0)t), \left(y_0 + f_y(x_0, y_0)t\right), (z_0 - t) \rangle$$
Linear Approximation and Differentials
The linear approximation of $f(x,y)$ around the point $\left(a, b, f(a,b)\right)$ is given by the equation of the tangent plane at that point. $$L(x,y) = f_x(a,b)(x - a) + f_y(a,b)(y - b) + f(a,b)$$ The value of a function, $f(x,y)$, at $(a + \Delta x, b + \Delta y)$ can be approximated by this linearization, $L(x,y)$, as $$f(a + \Delta x, b + \Delta y) \cong f_x(a,b)\Delta x + f_y(a,b)\Delta y + f(a,b)$$ Note: This can be extended to any number variables. In three variables we have: $$f(a + \Delta x, b + \Delta y, c + \Delta z) \cong f_x(a,b,c)\Delta x + f_y(a,b,c)\Delta y + f_z(a,b,c)\Delta z + f(a,b)$$ If Δx and Δy are sufficiently small, then we can approximate Δf as $$\Delta f \cong f_x(a,b)\Delta x + f_y(a,b)\Delta y$$ The differential of $f(x,y)$ is defined as $$\begin{aligned} df &= f_x(x,y)dx + f_y(x,y)dy \\ &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \end{aligned}$$

Multivariable Chain Rule

Let $f(x_1, \ldots, x_n)$ be a differentiable function of n variables. Suppose that each of the variables, $x_1, \ldots, x_n$, is a differentiable function of m independent variables, $t_1, \ldots, t_m$.
Then for $k = 1, \ldots, m$

$$\frac{\partial f}{\partial t_k} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_k} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_k} + \ldots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_k}$$

Note: Since x_i is assumed to be a function of more than one variable, the partial derivative notation is required, $\frac{\partial x_i}{\partial t_k}$. If $m = 1$ then $\frac{dx_i}{dt}$ could be used.

Multivariable Implicit Differentiation

Suppose we have the equation $F(x, y) = 0$, and that $F(x, y)$ is differentiable. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Provided $F_y \neq 0$

Suppose we have the equation $F(x, y, z) = 0$, and that $F(x, y, z)$ is differentiable. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \textit{and} \qquad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

Provided $F_z \neq 0$

The Directional Derivative
Let $f(x,y)$ be a function of two variables and let $\boldsymbol{u}$ denote a unit vector. Then the derivative of $f(x,y)$ in the direction of $\boldsymbol{u}$ is called the *directional derivative,* $D_{\boldsymbol{u}}f$. $$D_{\boldsymbol{u}}f = \nabla f \cdot \boldsymbol{u}$$ Where, $$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \quad \textit{and} \quad \boldsymbol{u} = \langle u_x, u_y \rangle$$ The definition can be extended to three or more dimensions as follows where $$\nabla f = \langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle \quad \textit{and} \quad \boldsymbol{u} = \langle u_{x_1}, \dots, u_{x_n} \rangle$$

Algebraic Properties of the Gradient Vector
If $f(x,y,z)$ and $g(x,y,z)$ are differentiable functions and c is a constant, then *i.* $\nabla(f+g) = \nabla f + \nabla g$ *ii.* $\nabla(cf) = c\nabla f$ *iii.* **Product Rule for Gradients***:* $\nabla(fg) = \nabla f g + f \nabla g$ iv. **Chain Rule for Gradients***:* If $F(t)$ is a differentiable function of one variable, then $$\nabla\Big(F\big(f(x,y,z)\big)\Big) = F'\big(f(x,y,z)\big)\nabla f$$

Gradient Vector as the Direction of Maximum Increase
Let f be a differentiable function at a fixed point, P, with $\nabla f\|_P \neq 0$. • ∇f points in the direction of the maximum rate of ***increase*** of f at P, and the maximum rate of ***increase*** is $\lVert \nabla f \rVert$. • $-\nabla f$ points in the direction of the maximum rate of ***decrease*** of f at P, and the maximum rate of ***decrease*** is $\lVert \nabla f \rVert$.

Gradient Vector as a Normal Vector
Let P be a point on a level curve, $f(x,y) = c$, or on a level surface, $f(x,y,z) = c$, and assume that $\nabla f\|_P \neq 0$. Then $\nabla f\|_P$ is a vector that is normal to the tangent line/plane to the curve/surface at the point P. Moreover, the tangent line/plane to the curve/surface at the point P has the equation *Tangent Line* : $f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = 0$ *Tangent Plane* : $f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0$

Critical Points Definition for Two Variable Functions

A point $P = (a, b)$ in the domain of $f(x, y)$ is called a **critical point** if:

- $f_x(a, b) = 0$ or $f_x(a, b)$ does not exists, AND
- $f_y(a, b) = 0$ or $f_y(a, b)$ does not exists.

Fermat's Theorem of Local Extrema for Two Variable Functions

If $f(a, b)$ is a local minimum or maximum, then $P = (a, b)$ is a critical point of $f(x, y)$.

Note: This theorem does not claim that all critical points are local extreme values, but rather that all local extreme values are critical points.

Second Derivative Test for Two Variable Functions

Let $P = (a, b)$ be a critical point of the function, $f(x, y)$ and assume f_{xx}, f_{yy} and f_{xy} are continuous near P. Then:

1. If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
2. If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
3. If $D < 0$ then $f(a, b)$ is a saddle point.
4. If $D = 0$ then the test is inconclusive.

Where D is called the discriminate

$$D = f_{xx}(a, b)f_{yy}(a, b) - {f_{xy}}^2(a, b)$$

Existence and Location of Absolute Extrema

Let $f(x, y)$ be a continuous function on a closed domain D in R^2. Then:

1. $f(x, y)$ takes on both a minimum and maximum value on D.
2. The extreme values occur either at critical points in the interior of D or at points on the boundary of D.

Optimizing with Constraints

Optimizing with constraints involves finding the minimum or maximum value of a function, e.g. $f(x_1, \ldots, x_n)$ subject to the fact that the independent variables are related in some fashion, e.g. $g(x_1, \ldots, x_n) = 0$. The terminology used is as follows:

Objective Function $f(x_1, \ldots, x_n)$	Expresses the quantity we would like to optimize in terms of n independent variables.
Constraint Function $g(x_1, \ldots, x_n) = 0$	Expresses a relationship between the independent that must be satisfied within the context of optimizing the objective function.

Lagrange Multiplier Theorem

Assume $f(x, y)$ and $g(x, y)$ are differentiable functions. If $f(x, y)$ has a local extremum on the constraint curve, $g(x, y) = 0$, at $P = (a, b)$ and if $\nabla g_P \neq 0$, then there is a scalar, λ, such that

$$\nabla f_P = \lambda \nabla g_P$$

Lagrange Multipliers Technique Applied to Optimization with Constraints

The above Lagrange Multiplier Theorem can be applied to optimization problems with constraints. The theorem can be generalized to any number of variables and any number of constraints functions as follows:

Given an n variable differentiable objective function, $f(x_1, \ldots, x_n)$, and m differentiable constraint functions, $\{g_1(x_1, \ldots, x_n) = 0, \ldots, g_m(x_1, \ldots, x_n) = 0\}$. The Lagrange condition is written as follows:

$$\nabla f_P = \sum_{i=1}^{m} \lambda_i \nabla g_{i,P}$$

Expanding this expression creates n equations that can then be used to find the extreme values of $f(x_1, \ldots, x_n)$ subject to $\{g_1(x_1, \ldots, x_n) = 0, \ldots, g_m(x_1, \ldots, x_n) = 0\}$.

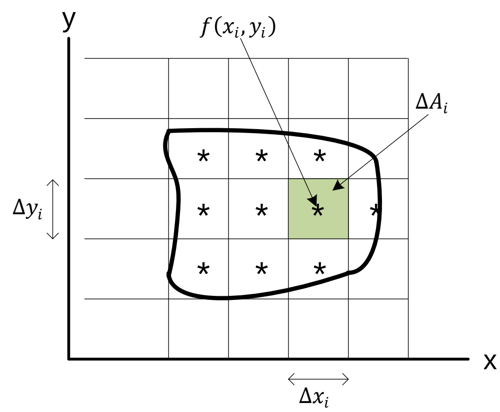
Next series of lessons focused on integration of multivariable scalar functions. Although vectors are not prominent in these lessons, multivariable integration is used to compute many different quantities on science and engineering. Multivariable integration is a natural extension of single variable integration. Alternate coordinate systems also become important is some multivariable integration problems.

Double Integral over a Rectangular Region

The definite double integral of $f(x, y)$ over a rectangular region, R, is the limit of the Riemann Sum.

$$\iint_R f(x,y)dA = \lim_{\|P\|\to 0}\left\{\sum_{i=1}^{N} f(x_i,y_i)\Delta A_i\right\} = \lim_{\|P\|\to 0}\left\{\sum_{i=1}^{N} f(x_i,y_i)\Delta x_i \Delta y_i\right\}$$

When this limit exists, we say $f(x, y)$ is integrable over R.



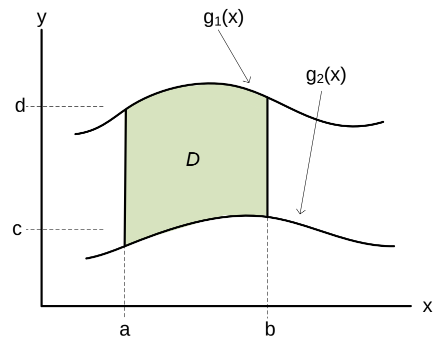
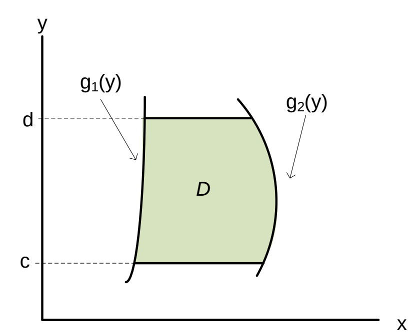
Fubini's Theorem

The double integral of a continuous function $f(x, y)$ over the rectangular region, $R = \{(x, y)|a \le x \le b, c \le y \le d\}$, is equal to the iterated single integral (in either order).

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dx\,dy = \int_a^b \int_c^d f(x,y)dy\,dx$$

When $f(x, y) = g(x)h(y)$, the double integral can be expressed as the product of two integrals as shown below.

$$\int_c^d \int_a^b f(x,y)dx\,dy = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right)$$

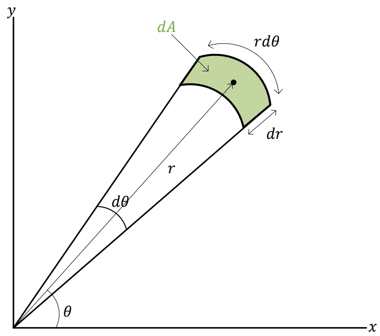
Double Integral over Vertically Simple Regions
A vertically simple region is defined as: $$D = (x, y) \mid \; a \le x \le b, \qquad g_1(x) \le y \le g_2(x)$$ And the double integral of $f(x, y)$ over D is $$\iint_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$ 
Double Integral over Horizontally Simple Regions
A horizontally simple region is defined as: $$D = \{(x, y) \mid \; g_1(y) \le x \le g_2(y), \qquad c \le y \le d\}$$ And the double integral of $f(x, y)$ over D is $$\iint_D f(x,y)dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)dx\,dy$$ 
Volume Between Two Surfaces
Assuming the integrable functions, $f_1(x, y) \ge f_2(x, y)$, for all points in D, then the *volume* between the *surfaces* is given as $$V = \iint_D (\, f_1(x,y) - \, f_2(x,y))dA$$

Triple Integral Over a Boxed Region
The triple integral of a continuous function $f(x, y, z)$ over a box, R is: $$\iiint_R f(x,y,z)dV = \int_{x=a}^{b}\int_{y=c}^{d}\int_{z=p}^{q} f(x,y,z)dzdydx$$ Where, $$R = (x,y,z)\mid\ a \le x \le b, c \le y \le d, p \le z \le q$$ Furthermore, the integral can be evaluated in any order.
Triple Integral Over a General Region
For a general region, the triple integral is best written as follows: $$\iiint_D zdV = \iint_R \left(\int_{q_1}^{q_2} f(x,y,z)dq\right)dA$$ Where, the inner integral is with respect to any one of the three variables, i.e. we choose q to be one of the elements of the set $\{x, y, z\}$. The region, R, is the projection of the solid object in the plane defined by the two remaining variables. We can then express the dA in two different orders for each of the 3 possible projections as shown below.
Area and Volume
The *area* of a general region can be found using the double integral of $f(x, y) = 1$ over a region, R. For example $$R = \{(x,y)\mid\ a \le x \le b, c \le y \le d\}$$ $$\iint_R 1dA = \int_c^d\int_a^b 1dxdy = (b-a)\cdot(d-c) = \ Area\ of\ Region$$ The *volume* of a general region can be found using the triple integral of $f(x, y, z) = 1$ over a region, R. For example $$R = \{(x,y,z)\mid\ a \le x \le b, c \le y \le d, e \le z \le f\}$$ $$\iiint_R 1dV = \int_e^f\int_c^d\int_a^b 1dxdydz = (b-a)\cdot(d-c)\cdot(f-e) = \ Volume\ of\ Region$$

Double Integral in Polar Coordinates

For a continuous function, f, on the domain, $D = \{(r,\theta)|\ r_1 \le r \le r_2, \theta_1 \le \theta \le \theta_2\}$

$$\iint\limits_D f(x,y)dA = \int_{\theta=\theta_1}^{\theta_2}\int_{r=r_1}^{r_2} f(r\,cos(\theta)\,, r\,sin(\theta))\,rdrd\theta$$

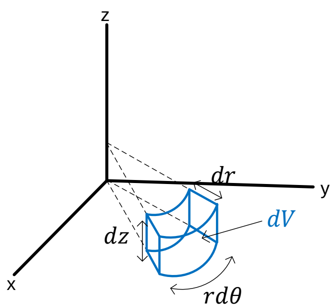


$$dA = rdrd\theta$$

Triple Integral in Cylindrical Coordinates

For a continuous function, f, on the domain, $D = \{(r,\theta,z)|\ r_1 \le r \le r_2, \theta_1 \le \theta \le \theta_2, z_1 \le z \le z_2\ \}$

$$\iiint\limits_D f(x,y,z)dV = \int_{\theta=\theta_1}^{\theta_2}\int_{r=r_1}^{r_2}\int_{z=z_1}^{z_2} f(r\,cos(\theta)\,, r\,sin(\theta)\,, z)\,rdzdrd\theta$$



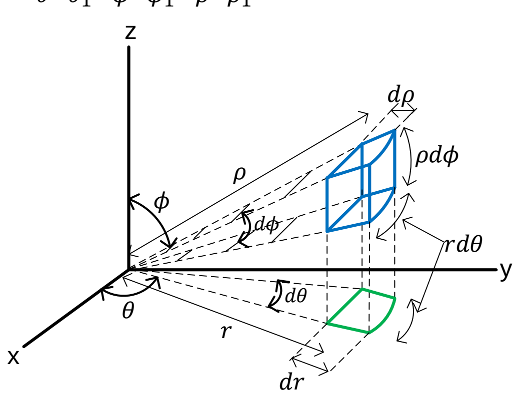
$$dV\ = rdzdrd\theta$$

Triple Integral in Spherical Coordinates

For a continuous function, f, on the domain, $D = \{(\rho,\phi,\theta)|\ \rho_1 \le \rho \le \rho_2, \phi_1 \le \phi \le \phi_2, \theta_1 \le \theta \le \theta_2, \}$

$$\iiint\limits_D f(x,y,z)dV$$

$$= \int_{\theta=\theta_1}^{\theta_2}\int_{\phi=\phi_1}^{\phi_2}\int_{\rho=\rho_1}^{\rho_2} f(\rho\,sin(\phi)\,cos(\theta)\,, \rho\,sin(\phi)\,sin(\theta)\,, \rho\,cos(\phi))\,\rho^2\,sin(\phi)\,d\rho d\phi d\theta$$



$$dV\ = \rho^2\,sin(\phi)\,d\rho d\phi d\theta$$

Total Amount Using Density

One Dimension

$$Total\ Amount\ = \int_R \delta(x)dx$$

Where, $\delta(x)$ is the amount per unit length and R is the interval of integration

Two Dimensions

$$Total\ Amount\ = \iint_R \delta(x,y)dA$$

Where, $\delta(x,y)$ is the amount per unit area and R is the region of integration.

Three Dimensions

$$Total\ Amount = \iiint_R \delta(x,y,z)dV$$

Where, $\delta(x,y,z)$ is the amount per unit volume and R is the region of integration.

Center of Mass

One Dimension

$$x_{com} = \frac{\int_R x\delta(x)dx}{\int_R \delta(x)dx}$$

Where, $\delta(x)$ is mass density per unit length and R is the interval of integration

Two Dimensions

$$x_{com} = \frac{\iint_R x\delta(x,y)dA}{\iint_R \delta(x,y)dA} \qquad y_{com} = \frac{\iint_R y\delta(x,y)dA}{\iint_R \delta(x,y)dA}$$

Where, $\delta(x,y)$ is the mass density per unit area and R is the region of integration.

Three Dimensions

$$x_{com} = \frac{\iiint_R x\delta(x,y)dV}{\iiint_R \delta(x,y)dV} \qquad y_{com} = \frac{\iiint_R y\delta(x,y)dV}{\iiint_R \delta(x,y)dV} \qquad z_{com} = \frac{\iiint_R z\delta(x,y)dV}{\iiint_R \delta(x,y)dV}$$

Where, $\delta(x,y,z)$ is the mass density per unit volume and R is the region of integration.

Rotational Inertia (2nd Moment)

One Dimension

$$I = \int_R x^2 \delta(x) dx$$

Where, $\delta(x)$ is mass density per unit length and R is the interval of integration

Two Dimensions

$$I_x = \iint_R x^2 \delta(x, y) dA \qquad I_y = \iint_R y^2 \delta(x, y) dA \qquad I_z = \iint_R (x^2 + y^2) \delta(x, y) dA$$

Where, $\delta(x, y)$ is the mass density per unit area, R is the region of integration, and $I_{x,y,z}$ is the rotational inertia with respect to the x, y, z-axis respectively.

Three Dimensions

$$I_x = \iiint_R (y^2 + z^2) \delta(x, y, z) dV \qquad I_y = \iiint_R (x^2 + z^2) \delta(x, y, z) dV \qquad I_z = \iiint_R (x^2 + y^2) \delta(x, y, z) dV$$

Where, $\delta(x, y, z)$ is the mass density per unit volume, R is the region of integration, and $I_{x,y,z}$ is the rotational inertia with respect to the x, y, z-axis respectively.

Probability Density Functions

One Random Variable

$$P(a \leq X \leq b) = \int_{x=a}^{b} p(x) dx$$

Two Random Variables

$$P(a \leq X \leq b;\ c \leq Y \leq d) = \int_{y=c}^{d} \int_{x=a}^{b} p(x, y) dx dy$$

Three Random Variables

$$P(a \leq X \leq b;\ c \leq Y \leq d;\ e \leq Z \leq f) = \int_{z=e}^{f} \int_{y=c}^{d} \int_{x=a}^{b} p(x, y, z) dx dy dz$$

The Jacobian Determinant

Given the transformation $T: R^2 \to R^2$, where T is defined as

$$T(u,v) = \big(x(u,v), y(u,v)\big)$$

The Jacobian of T, $Jac(T)$, is given as

$$Jac(T) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

The Jacobian generalizes to n dimensions. For example, with three variables we have $T: R^3 \to R^3$, where T is defined as

$$T(u,v,w) = \big(x(u,v,w), y(u,v,w), z(u,v,w)\big)$$

$$Jac(T) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

- The Jacobian of T is also denoted as $\frac{\partial(x,y)}{\partial(u,v)}, \frac{\partial(x,y,z)}{\partial(u,v,w)}$
- The Jacobian is sometimes meant to express the matrix only and not its determinant. In these cases, we refer to the above as the *Jacobian Determinant*.

Change of Variable Formula in

Let $T: (u,v) \to (x,y)$ be a mapping from u-v space to x-y space that is one-to-one. If $f(x,y)$ is continuous, then

$$\iint_D f(x,y)\,dxdy = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

Where, D is some region in x-y space and S is the corresponding region in u-v space.

Note: The Change of Variables Formula as stated above turns an xy integral into a uv integral, but the map, T, goes from the uv domain to the xy domain, i.e. $T(u,v) = \big(x(u,v), y(u,v)\big)$

In R^3 we have:

$$\iiint_D f(x,y,z)\,dxdydz = \iiint_S f\big(x(u,v,w), y(u,v,w), z(u,v,w)\big) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$

The next series of lessons pushed integration even further to includes integration over curves and surfaces. In addition, we introduced integration over vector fields.

Vector Field

A *vector field* is a function that assigns a vector to each point, $P = \langle x, y, z\rangle$, in space. In three dimensions it is denoted as

$$\boldsymbol{F}(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\rangle$$

A unit vector field, $\boldsymbol{e}_F$, is defined as

$$\boldsymbol{e}_F = \frac{\boldsymbol{F}(x,y,z)}{\|\boldsymbol{F}(x,y,z)\|}$$

An important example is a unit radial vector.

Two dimensional unit radial vector	***Three dimensional unit radial vector***
$\boldsymbol{e}_r = \langle \frac{x}{r}, \frac{y}{r}\rangle$ Where, $r = \sqrt{x^2 + y^2}$	$\boldsymbol{e}_r = \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\rangle$ Where, $r = \sqrt{x^2 + y^2 + z^2}$

Divergence of a Vector Field

The divergence of a vector field, $\boldsymbol{F}$, results in a scalar function. In three dimensions it is defined as

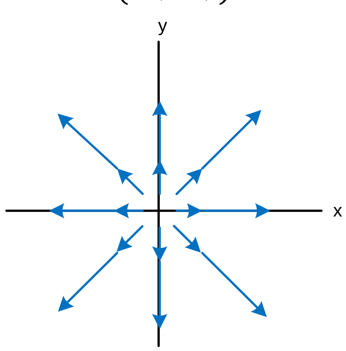
$$div(\boldsymbol{F}) = \nabla \cdot \boldsymbol{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \cdot \langle F_1, F_2, F_3\rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

The divergence generalizes to an arbitrary number of dimensions.

Divergence Intuition – Assume F is a fluid velocity vector field

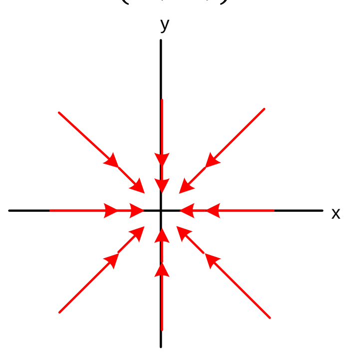
The divergence of a vector field represents the degree to which the fluid is flowing in towards or away from each point in space.

$div(\boldsymbol{F}(0,0)) > 0$



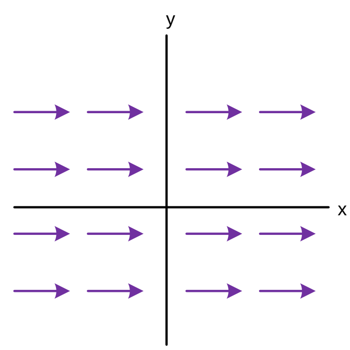
There is a net liquid flow *outward* from the origin.

$div(\boldsymbol{F}(0,0)) < 0$



There is a net liquid flow *inward* from the origin.

$div(\boldsymbol{F}) = 0$



There is a net flow of zero at any point in space.

Curl of a Vector Field

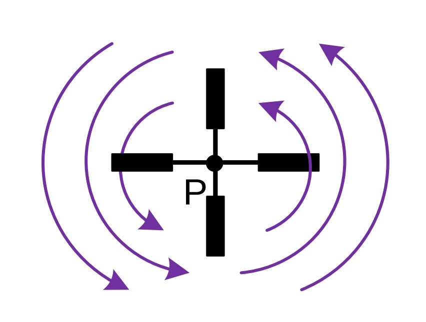
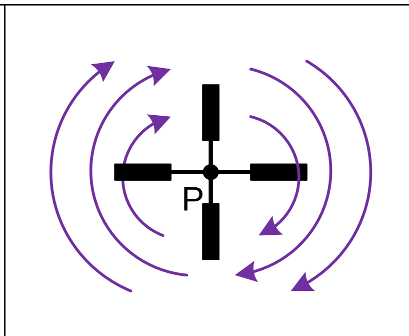
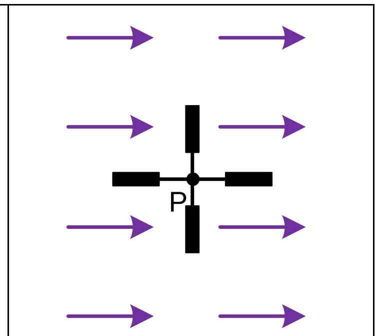
The curl of a vector field, $\boldsymbol{F}$, results in a vector function. It is defined as

$$curl(\boldsymbol{F}) = \nabla \times \boldsymbol{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left\langle \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \right\rangle$$

The curl is defined on three dimensions only.

Curl Intuition – Assume $\boldsymbol{F}$ is a fluid velocity vector field

The curl measures the amount to which the fluid circulates around a fixed axis at each point in space.

		
The paddle wheel rotates counterclockwise and the resulting vector points out of the page.	The paddle wheel rotates clockwise and the resulting vector points into the page.	The paddle wheel does not rotate.

Conservative Vector Fields

- If $\boldsymbol{F} = \nabla f$, then f is called the potential function for $\boldsymbol{F}$.
- $\boldsymbol{F}$ is called conservative it has a potential function.
- Potential functions are unique up to a constant, C.

The vector field, $\boldsymbol{F}$, is conservative if

$$curl(\boldsymbol{F}) = 0$$

Or equivalently,

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \qquad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}, \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

Scalar Line Integral

The scalar line integral of the function $f(x, y, z)$ over the curve, C, is given as

$$\int_C f(x, y, z)ds$$

Let $\boldsymbol{r}(t)$ be a parameterization of a curve, C, for $a \leq t \leq b$, then the scalar line integral is also given as

$$\int_C f(x, y, z)ds = \int_a^b f\big(\boldsymbol{r}(t)\big)\|\boldsymbol{r}'(t)\|dt$$

Vector Line Integral

The vector line integral of the vector field $\boldsymbol{f}(x, y, z)$ over the curve, C, is given as

$$\int_C \boldsymbol{f}(x, y, z) \cdot d\boldsymbol{r}$$

Let $\boldsymbol{r}(t)$ be a parameterization of a curve, C, for $a \leq t \leq b$, then the vector line integral is also given as by the two equivalent expressions

$$\int_a^b \Big(\boldsymbol{f}\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{r}'(t)\Big)\, dt = \int_a^b \left(f_1\big(\boldsymbol{r}(t)\big)\frac{dx}{dt} + f_2\big(\boldsymbol{r}(t)\big)\frac{dy}{dt} + f_3\big(\boldsymbol{r}(t)\big)\frac{dz}{dt}\right) dt$$

Work done by a Vector Force Field

The work done on a particle moving along curve parameterized by $\boldsymbol{r}(t)$ in the presence of a vector force field, $\boldsymbol{F}$, is given as

$$W = \int_a^b \Big(\boldsymbol{F}\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{r}'(t)\Big)\, dt$$

Flux Across a Plane Curve

The flux across a plane curve parameterized by $\boldsymbol{r}(t)$ in the presence of a vector field, $\boldsymbol{v}$, is given as

$$\Phi = \int_a^b \Big(\boldsymbol{v}\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{N}(t)\Big)\, dt$$

Where, $\boldsymbol{N}(t) = \langle y'(t), -x'(t)\rangle$ and $\boldsymbol{r}'(t) = \langle x'(t), y'(t)\rangle$

The Fundamental Theorem for Conservative Vector Fields
Assume $\boldsymbol{F} = \nabla f$ on a domain D. 1. If $\boldsymbol{r}$ is a path along a curve C from A to B in D, then $$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b \frac{d}{dt} f(\boldsymbol{r}(t))\, dt = f(\boldsymbol{r}(b)) - f(\boldsymbol{r}(a)) = f(B) - f(A)$$ In other words, $\boldsymbol{F}$ is path-independent 2. The circulation around a closed curve C, (i.e. $A = B$) is zero $$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$$
Conservative Vector Field Criteria
The vector field $\boldsymbol{F}$ is conservative on a simply connected domain, D, if $\boldsymbol{F}$ satisfies the cross-partials conditions derived from the fact that the curl is zero. $curl(\boldsymbol{F}) = 0 \quad \rightarrow \quad \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$
Conservative Fields in Physics
The gravitational force and the electrostatic forces are conservative forces. They are both governed by the inverse square law.

Inverse Square Law Force and its Potential Function
$\boldsymbol{F_C}(x, y, z) = \frac{C}{r^2} \boldsymbol{e}_r \quad \rightarrow \quad f(x, y, z) = \frac{C}{r}$

Specifically,

Gravitational Force	*Electrostatic Force*
$\boldsymbol{F}_G = -\frac{GMm}{r^2} \boldsymbol{e}_r$	$\boldsymbol{F_E} = \frac{kQq}{r^2} \boldsymbol{e}_r$

Where, $G = 6.67E^{-11}$, $k = 8.9E^9$, m_1 and m_2 are the two masses in kilograms, q_1 and q_2 are the two charges in Coulombs, and $\boldsymbol{e}_r = \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \rangle$.

Scalar Line Integral
Let $\boldsymbol{r}(t)$ be a parameterization of a curve, C, for $a \leq t \leq b$, then the scalar line integral is also given as $$\int_C f(x,y,z)ds = \int_a^b f(\boldsymbol{r}(t))\|\boldsymbol{r}'(t)\|dt$$
Vector Line Integral
Let $\boldsymbol{r}(t)$ be a parameterization of a curve, C, for $a \leq t \leq b$, then the vector line integral is also given as by the two equivalent expressions $$\int_C \boldsymbol{F}(x,y,z)\cdot d\boldsymbol{s} \int_a^b \left(\boldsymbol{F}(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)\right)dt$$ ***Work Along a Curve*** $$W = \int_a^b \left(\boldsymbol{F}(\boldsymbol{r}(t))\cdot\boldsymbol{r}'(t)\right)dt$$ ***Flux Across a Curve*** $$\Phi = \int_a^b \left(\boldsymbol{v}(\boldsymbol{r}(t))\cdot\boldsymbol{N}(t)\right)dt$$
Scalar Surface Integral
Let $\boldsymbol{G}(u,v)$ be a parameterization of a surface, $\mathcal{S}$, on the domain. The scalar surface integral of the function $f(x,y,z)$ over the surface on the given domain is $$\iint_{\mathcal{S}} f(x,y,z)dS = \iint_D f(\boldsymbol{G}(u,v))\,\|\boldsymbol{N}(u,v)\|dudv$$ For $f(x,y,z) = 1$, we obtain the surface area on the domain D. $$Area(\mathcal{S}) = \iint_D \|\boldsymbol{N}(u,v)\|dudv$$
Scalar Surface Integral over a Surface $z = g(x,y)$
The scalar surface integral of the function $f(x,y,z)$ over a portion of a surface that can be represented as $z = g(x,y)$, is given as $$\iint_{\mathcal{S}} f(x,y,z)dS = \iint_D f(x,y,g(x,y))\left(\sqrt{{g_x}^2 + {g_y}^2 + 1}\right)dxdy$$
Vector Surface Integral
Let $\boldsymbol{G}(u,v)$ be a parameterization of a surface, $\mathcal{S}$, on the domain, D. The vector surface integral, also called the flux, of the vector field $\boldsymbol{F}(x,y,z)$ over the surface on the given domain is $$\iint_{\mathcal{S}} (\boldsymbol{F}\cdot\boldsymbol{n})dS = \iint_D \left(\boldsymbol{F}(\boldsymbol{G}(u,v))\cdot\boldsymbol{N}(u,v)\right)dudv$$

The final series of lessons introduced the Fundamental Theorems of Vector Calculus. These theorems are also shown to follow directly from the Fundamental Theorem of Single Variable Calculus. These theorems are extremely important and can be used as a gateway for more advanced studies in various science and engineering applications.

Green's Theorem

Let D be a domain in R^2 whose boundary is a simple closed curve, C, oriented counterclockwise. Then

$$\iint_D \left(curl_z(\mathbf{F})\right)dA = \oint_C \boldsymbol{F}\cdot d\boldsymbol{r}$$

Where, $curl_z(\boldsymbol{F}) = \left(\frac{\partial F_2(x,y)}{\partial x} - \frac{\partial F_1(x,y)}{\partial y}\right)$

With $\boldsymbol{F} = \langle F_1(x,y), F_2(x,y)\rangle$ and $d\boldsymbol{r} = \langle dx, dy\rangle$, we can also express the line integral as

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \oint_C F_1(x,y)dx + F_2(x,y)dy$$

Area of Region Using Green's Theorem

There are three equivalent formulas we can use for the area of a region, D, enclosed by a simple curve, C.

$$Area\ Enclosed\ by\ C = \left(\oint_C xdy\right) = \left(\oint_C -ydx\right) = \left(\frac{1}{2}\oint_C xdy - ydx\right)$$

Green's Theorem Using Normal Vector – Flux

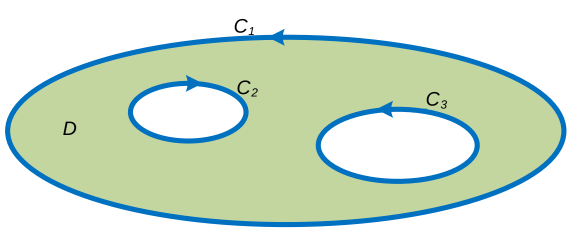
Using the *normal* vector to the curve, Green's Theorem can be used to express the flux across the curve as follows

$$\iint_D div(\boldsymbol{F})dA = \oint_C (\boldsymbol{F}\cdot\boldsymbol{N})dt$$

Where, $\boldsymbol{N}(t) = \langle y'(t), -x'(t)\rangle$ and $div(\boldsymbol{F}) = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right)$

General Form of Green's Theorem

Green's theorem can also be applied to non-simple regions as long as we keep in mind the fact that the region to be considered always lies to the left of the curve according to its orientation.



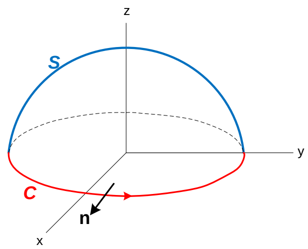
In this example the region, D, is represented as

$$D = C_1 + C_2 - C_3$$

Surfaces and Surface Boundaries

Different surfaces may have different types of boundaries. For example, the surface below has a single simple closed curve as its boundary. We define the orientation of the curve as follows:

- When you walk around the curve with your body pointing out in the direction of the normal vector, you should be walking in such a way that the surface is to your left side.



Stokes' Theorem

Let S be an oriented smooth surface that is bounded by a single simple closed curve, C, and let $\boldsymbol{F}$ be a vector field. Then

$$\iint_S curl(\boldsymbol{F}) \cdot d\boldsymbol{S} = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

Where, $curl(\boldsymbol{F}) = \nabla \times \boldsymbol{F}$

Surface Independence

The ***surface integral*** of a vector field, $\boldsymbol{F}$, with an associated ***vector potential*** function, $\boldsymbol{A}$, (where $\boldsymbol{F} = curl(\boldsymbol{A})$), is ***surface independent***. It depends only on the ***boundary curve,*** C.

$$\iint_{S_x} \boldsymbol{F} \cdot d\boldsymbol{S}_{\boldsymbol{x}} = \iint_{S_x} curl(\boldsymbol{A}) \cdot d\boldsymbol{S}_{\boldsymbol{x}} = \oint_C \boldsymbol{A} \cdot d\boldsymbol{r}$$

Divergence Theorem

Let S be a closed surface that encloses a region, W, in R^3. Assume that S is piecewise smooth and is oriented by a normal vector pointing to the outside of W. Let $\boldsymbol{F}$ be a vector field whose domain contains W. Then

$$\iiint_W div(\boldsymbol{F}) dV = \iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

Gauss's Law

The electric flux through a closed surface is proportional to the total charge enclosed within the surface.

$$\iint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{q_T}{\epsilon_0}$$

Where, $\boldsymbol{E} = \left(\frac{q}{4\pi\epsilon_0}\right)\left(\frac{\hat{r}}{r^2}\right)$ and $\hat{\boldsymbol{r}} = \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \rangle$.

Relationship Between Fundamental Theorems
In a general sense the theorems *relate the integral of some type of derivative of some function over some region to the values of that function along the boundary of the region.*
Fundamental Theorem of Single Variable Calculus
$$\int_a^b f'(t)dt \;=\; f(b) - f(a)$$ Relates the integral of the *derivative* of a scalar function over a one-dimensional region to the values of the function at the *endpoints of the region*.
Gradient Theorem
$$\int_C \nabla f \cdot d\boldsymbol{s} = f(b) - f(a)$$ Relates the integral of the *gradient* of a scalar function over a curve in three dimensions, C, to the values of that function at the *endpoints of the curve*.
Green's Theorem
$$\iint_D curl_z(\boldsymbol{F})dA = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$$ Relates the double integral of the *two dimensional curl* of a vector field over a region, D, to the value of the line integral of that vector field along the *boundary curve for that region*, C.
Stokes' Theorem
$$\iint_S curl(\boldsymbol{F}) \cdot d\boldsymbol{S} = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$$ Relates the surface integral of the *three dimensional curl* of a vector field over a surface, S, to the value of the line integral of that vector field along the *boundary curve for that surface*, C.
Divergence Theorem
$$\iiint_W div(\boldsymbol{F})dV = \iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$$ Relates the triple integral of the *divergence* of a vector field over a 3D region, W, to the value of the surface integral of that vector field over the *boundary surface for the region*, S.

By: ferrantetutoring