Calculus 2 Summary and Formulas

Calculus 2 begins with the study of more advanced techniques for solving integrals. Integrals in general are much more difficult to evaluate compared to derivatives. In fact, a good number of integrals cannot be evaluated analytically. Nevertheless, integration is an important operation that is used in a variety of applications and the more advanced integration techniques allow us to solve a much wider array of integrals.

Integration by Parts Formula

$$\int u dv = uv - \int v du$$

Integration by Parts Formula for Definite Integrals

$$\int_a^b u dv = uv\Big|_a^b - \int_a^b v du = \left(uv - \int v du\right)\Big|_a^b$$

Guidelines for Choosing u and dv

There are no hard and fast rules for how to choose u and dv for IBP, however the following guidelines can sometimes be useful.

1. Choose dv so that $v = \int dv$ can be evaluated.
2. Choose u so that $\frac{du}{dx}$ is "simpler" that u itself.

Another guideline is referred to as "LIATE", and is stated as follows:

Choose u to be the function that comes first in the list:

L:	Logarithmic Function
I:	Inverse Trigonometric Function
A:	Algebraic Function
T:	Trigonometric Function
E:	Exponential Function

Note: The last two, (T and E), can actually be in either order.

Integration Techniques for Sine and Cosine

Integrals of the following form

$$\int cos^n(x)\,dx \qquad\qquad \int sin^n(x)\,dx$$

Can be evaluated using the following reduction formulas

$$\int cos^n(x)\,dx = \frac{1}{n}cos^{n-1}(x)\,sin(x) + \frac{n-1}{n}\int cos^{n-2}(x)\,dx$$

$$\int sin^n(x)\,dx = -\frac{1}{n}sin^{n-1}(x)\,cos(x) + \frac{n-1}{n}\int sin^{n-2}(x)\,dx$$

Integral of the following form

$$\int sin^m(x)\,cos^n(x)\,dx$$

- If either m or n are odd:
  - Apply the following steps depending on which power is odd.

1. Split off one power from function with the odd power
   - m odd: $sin^{2k+1}(x) = sin^{2k}(x)\,sin(x)$.
   - n odd: $cos^{2k+1}(x) = cos^{2k}(x)\,cos(x)$
2. Pull out the power of k.
   - m odd: $sin^{2k}(x)\,sin(x) = (sin^2(x))^k\,sin(x)$
   - n odd: $cos^{2k}(x)\,cos(x) = (cos^2(x))^k\,cos(x)$
3. Apply the Pythagorean identity
   - m odd: $(sin^2(x))^k\,sin(x) = (1 - cos^2(x))^k\,sin(x)$
   - n odd: $(cos^2(x))^k\,cos(x) = (1 - sin^2(x))^k\,cos(x)$
4. Use the following substitution
   - m odd: $u = cos(x) \rightarrow du = -sin(x)\,dx$
   - n odd: $u = sin(x) \rightarrow du = cos(x)dx$
5. Rewrote the integral using the substitutions, expand the integrand and evaluate.
   - m odd: $\int cos^{2l}(x)\,(1 - cos^2(x))^k\,sin(x)\,dx = -\int u^{2l}(1-u^2)^k\,du$
   - n odd: $\int sin^{2l}(x)\,(1 - sin^2(x))^k\,cos(x)\,dx = \int u^{2l}(1-u^2)^k\,du$

- If both m or n are even:
  - We can use the Pythagorean Identity to write the integrand as a sum of powers of cosine or powers of sine, as the example below shows, then use the reduction formula

$$\int sin^{2l}(x)\,cos^{3k}(x)\,dx = \int (sin^2(x))^l\,cos^n(x)\,dx$$
$$= \int (1 - cos^2(x))^l\,cos^n(x)\,dx$$

  - We can also use the double angle formula shown in method 2 of example 5.

Table of Trigonometric Integrals

$$\int sin(x)\,dx = -cos(x) + C$$

$$\int csc(x)\,dx = -ln|csc(x) + cot(x)| + C$$

$$\int cos(x)\,dx = sin(x) + C$$

$$\int sec(x)\,dx = ln|sec(x) + tan(x)| + C$$

$$\int tan(x)\,dx = ln|sec(x)| + C$$

$$\int cot(x)\,dx = ln|sin(x)| + C$$

$$\int sin^n(x)\,dx = -\frac{1}{n} sin^{n-1}(x)\,cos(x) + \frac{n-1}{n}\int sin^{n-2}(x)\,dx$$

$$\int cos^n(x)\,dx = \frac{1}{n} cos^{n-1}(x)\,sin(x) + \frac{n-1}{n}\int cos^{n-2}(x)\,dx$$

$$\int tan^n(x)\,dx = \frac{1}{n-1} tan^{n-1}(x) - \int tan^{n-2}(x)\,dx$$

$$\int cot^n(x)\,dx = -\frac{1}{n-1} cot^{n-1}(x) - \int cot^{n-2}(x)\,dx$$

$$\int sec^n(x)\,dx = \frac{1}{n-1} sec^{n-2}(x)\,tan^n(x) + \frac{n-2}{n-1}\int sec^{n-2}(x)\,dx$$

$$\int csc^n(x)\,dx = \frac{1}{n-1} csc^{n-2}(x)\,cot(x) + \frac{n-2}{n-1}\int csc^{n-2}(x)\,dx$$

$$\int sin^m(x)\,cos^n(x)\,dx = \frac{sin^{m+1}(x)\,cos^{n-1}(x)}{m+n} + \frac{n-1}{m+n}\int sin^m(x)\,cos^{n-2}(x)\,dx$$

$$\int sin(mx)\,sin(nx)\,dx = \frac{sin((m-n)x)}{2(m-n)} - \frac{sin((m+n)x)}{2(m+n)} + C, \qquad (m \neq n)$$

$$\int cos(mx)\,cos(nx)\,dx = -\frac{cos((m-n)x)}{2(m-n)} - \frac{cos((m+n)x)}{2(m+n)} + C, \qquad (m \neq n)$$

$$\int sin(mx)\,cos(nx)\,dx = \frac{sin((m-n)x)}{2(m-n)} + \frac{sin((m+n)x)}{2(m+n)} + C, \qquad (m \neq n)$$

Trigonometric Substitution

Trigonometric substitution may be used on the class of functions involving square root expressions of the form $\sqrt{\pm a^2 \pm x^2}$. The substitution involves the following three steps.

1. Substitute to eliminate the square root and convert to a trigonometric integral in θ.
2. Evaluate the trigonometric integral.
3. Convert the solution back to the original variable using the appropriate right triangle.

The general procedure is shown below for the three different cases.

General Quadratic Functions

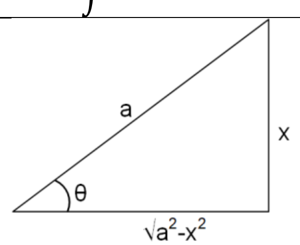
If the expression contains instead a general quadratic function as follows:

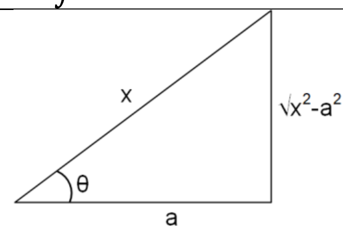
$$\sqrt{ax^2 + bx + c}$$

We can convert the standard form quadradic into vertex form by completing the square and then use u-substitution. The example below illustrates the procedure when $a = 1$.

$$\sqrt{\left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)} \quad \rightarrow \quad u = x + \frac{b}{2} \rightarrow \sqrt{u^2 + \left(c - \frac{b^2}{4}\right)}$$

Once in this form we return to the original three steps from above to evaluate.

Case 1: $\sqrt{a^2 - x^2}$	
Original Integral	$\int \sqrt{a^2 - x^2}\, dx$
Substitution	$x = a\,sin(\theta) \rightarrow dx = a\,cos(\theta)\, d\theta$
New Integral	$\begin{aligned} \int \sqrt{a^2 - x^2}\, dx &= \int \sqrt{a^2 - a^2\,sin^2(\theta)} \cdot a\,cos(\theta)\, d\theta \\ &= \int \sqrt{a^2 \Big(\underbrace{1 - sin^2(\theta)}_{cos^2(\theta)}\Big)} \cdot a\,cos(\theta)\, d\theta \\ &= \int a\,cos(\theta) \cdot a\,cos(\theta)\, d\theta \\ &= a^2 \int cos^2(\theta)\, d\theta \end{aligned}$
Associated Right Triangle	

<table>
<tr><th colspan="2">Case 2: $\sqrt{x^2 - a^2}$</th></tr>
<tr><td>Original Integral</td><td>$$\int \sqrt{x^2 - a^2}\,dx$$</td></tr>
<tr><td>Substitution</td><td>$$x = a\,sec(\theta) \;\rightarrow\; dx = a\,sec(\theta)\,tan(\theta)\,d\theta$$</td></tr>
<tr><td>New Integral</td><td>$$\begin{aligned} \int \sqrt{x^2 - a^2}\,dx &= \int \sqrt{a^2\,sec^2(\theta) - a^2} \cdot a\,sec(\theta)\,tan(\theta)\,d\theta \\ &= \int \sqrt{a^2\Big(\underbrace{sec^2(\theta) - 1}_{tan^2(\theta)}\Big)} \cdot a\,sec(\theta)\,tan(\theta)\,d\theta \\ &= \int a\,tan(\theta) \cdot a\,sec(\theta)\,tan(\theta)\,d\theta \\ &= a^2 \int sec(\theta)\,tan^2(\theta)\,d\theta \end{aligned}$$</td></tr>
<tr><td>Associated Right Triangle</td><td>
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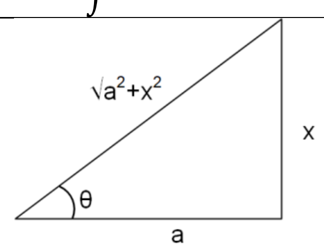
<table>
<tr><th colspan="2">Case 3: $\sqrt{a^2 + x^2}$</th></tr>
<tr><td>Original Integral</td><td>$$\int \sqrt{a^2 + x^2}\,dx$$</td></tr>
<tr><td>Substitution</td><td>$$x = a\,tan(\theta) \;\rightarrow\; dx = a\,sec^2(\theta)\,d\theta$$</td></tr>
<tr><td>New Integral</td><td>$$\begin{aligned} \int \sqrt{a^2 + x^2}\,dx &= \int \sqrt{a^2 + a^2\,tan^2(\theta)} \cdot a\,sec^2(\theta)\,d\theta \\ &= \int \sqrt{a^2\Big(\underbrace{1 + tan^2(\theta)}_{sec^2(\theta)}\Big)} \cdot a\,sec^2(\theta)\,d\theta \\ &= \int a\,sec(\theta) \cdot a\,sec^2(\theta)\,d\theta \\ &= a^2 \int sec^3(\theta)\,d\theta \end{aligned}$$</td></tr>
<tr><td>Associated Right Triangle</td><td>
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The Method of Partial Fractions

The method of partial fractions may be applied when the integrand is a rational function.

$$f(x) = \frac{N(x)}{D(x)}$$

We may identify the following two categories:

1. Proper Rational Functions
   - The degree of the numerator, $Deg(N)$, is less than the degree of the denominator, $Deg(D)$. i.e. $Deg(N) < Deg(D)$
   - In this case we perform a partial fraction decomposition on the integrand before we attempt to evaluate the integral.
2. Improper Rational Functions
   - $Deg(N) \geq Deg(D)$
   - In this case we need to perform long division first, which will result in two terms: a non-fraction term and a *proper* rational function term.
     - We then perform partial fraction decomposition on the second term if required before attempting to evaluate.

Partial Fraction Decomposition

The partial fraction decomposition for 4 different cases is illustrated below.

1. $D(x)$ contains distinct linear terms.

$$\frac{N(x)}{(x-a_1)(x-a_2)\dots(x-a_N)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \cdots + \frac{A_N}{(x-a_N)}$$

2. $D(x)$ contains repeated linear terms.

$$\frac{N(x)}{(x-a_1)^M} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_1)^2} + \cdots + \frac{A_M}{(x-a_1)^M}$$

3. $D(x)$ contains irreducible quadratic terms.

$$\frac{N(x)}{(a_1x^2+b_1x+c_1)\dots(a_Nx^2+b_Nx+c_N)} = \frac{A_1+B_1x}{(a_1x^2+b_1x+c_1)} + \cdots + \frac{A_N+B_Nx}{(a_Nx^2+b_Nx+c_N)}$$

4. $D(x)$ contains repeated irreducible quadratic terms.

$$\frac{N(x)}{(ax^2+bx+c)^M} = \frac{A_1+B_1x}{(ax^2+bx+c)} + \cdots + \frac{A_M+B_Mx}{(ax^2+bx+c)^M}$$

To find the unknown constant we multiply by the denominator on the left-hand side and

1. Consecutively set x value such that one or more of the terms is removed.
2. Expand equation and set the coefficients of the powers of x eaual to get a system of equations, i.e. method of undetermined coefficients.

Improper Integrals

When the region over which the integral is taken is unbounded, we refer to the integral as an *improper integral*. We can identify two types of improper integrals:

1. One or more of the endpoints of the integration interval is infinite.
2. Integrand tends to infinity within the integration interval.

Type 1 Improper Integral Definitions (Infinite Integration Intervals)

$$\int_a^{\infty} f(x)dx \stackrel{\text{def}}{=} \lim_{R\to\infty}\left(\int_a^R f(x)dx\right)$$

$$\int_{-\infty}^{b} f(x)dx \stackrel{\text{def}}{=} \lim_{R\to-\infty}\left(\int_R^b f(x)dx\right)$$

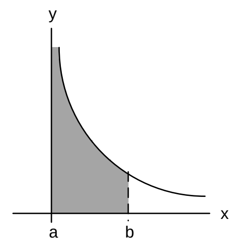
$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^{0} f(x)dx + \int_0^{\infty} f(x)dx \\ &\stackrel{\text{def}}{=} \lim_{R_1\to-\infty}\left(\int_{R_1}^{0} f(x)dx\right) + \lim_{R_2\to\infty}\left(\int_0^{R_2} f(x)dx\right)\end{aligned}$$

In all cases, we say that the improper integral converges if the limit exists (and is finite) and that it diverges if the limit does not exist.

Type 2 Improper Integral Definitions (Unbounded Integrands)

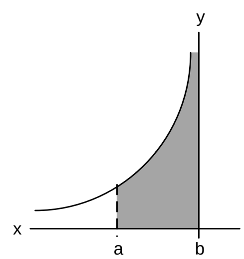
If f is continuous on $(a,b]$ and $\lim_{x\to a^+}\left(f(x)\right) = \pm\infty$, we may define the following

$$\int_a^b f(x)dx \stackrel{\text{def}}{=} \lim_{R\to a^+}\left(\int_R^b f(x)dx\right)$$



If f is continuous on $[a,b)$ and $\lim_{x\to b^-}\left(f(x)\right) = \pm\infty$, we may define the following

$$\int_a^b f(x)dx \stackrel{\text{def}}{=} \lim_{R\to b^-}\left(\int_a^R f(x)dx\right)$$

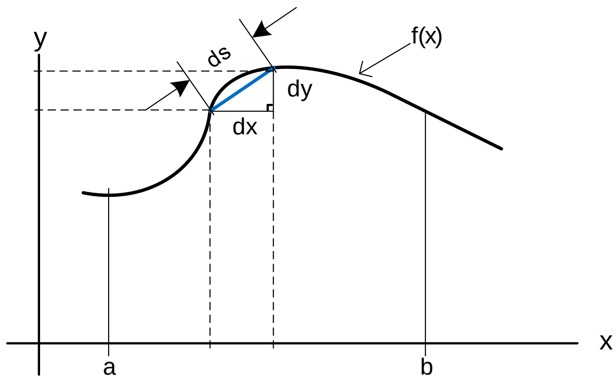
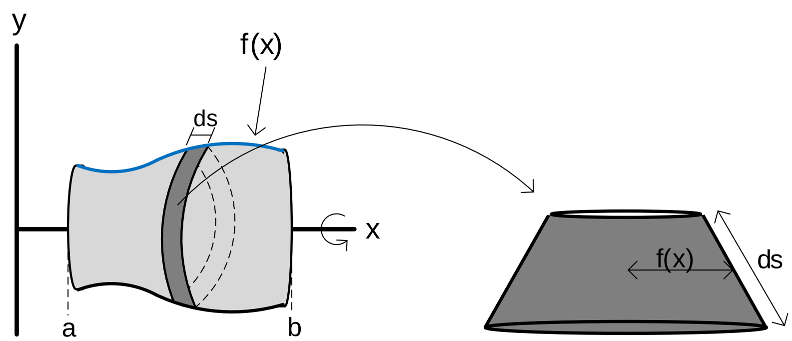


In both cases, we say that the improper integral converges if the limit exists (and is finite) and that it diverges if the limit does not exist.

If there is a single point c in the interval $[a, b]$ such that $\lim_{x\to c^-}\left(f(x)\right) = \pm\infty$ or $\lim_{x\to c^+}\left(f(x)\right) = \pm\infty$ and if both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ converge then we may define the following

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

The *p-integral* over $[a,\infty)$, for $a > 0$
$\int_a^\infty \frac{1}{x^p}dx = \begin{cases} \frac{a^{1-p}}{1-p}, & p > 1 \\ \infty, & p \le 1 \end{cases}$
The *p-integral* over $[0, a]$, for $a > 0$
$\int_0^a \frac{1}{x^p}dx = \begin{cases} \frac{a^{1-p}}{1-p}, & p < 1 \\ \infty, & p \ge 1 \end{cases}$

Formula for Arc Length
If f is continuous and differentiable on $[a,b]$, then the arc length, s, of $y = f(x)$ over $[a,b]$ is equal to $$s = \int_a^b \left(\sqrt{1 + \left(f'(x)\right)^2}\right) dx$$ Similarly, if g is continuous and differentiable on $[a,b]$, then the arc length, s, of $x = g(y)$ over $[a,b]$ is equal to $$s = \int_a^b \left(\sqrt{1 + \left(g'(y)\right)^2}\right) dy$$ 
Formula for Surface Area of a Surface of Revolution
If $f(x) \geq 0$ and if f is continuous and differentiable on $[a,b]$, then the surface area, A_S, of the surface obtained by rotating the graph of f about the x-axis for $a \leq x \leq b$ is equal to $$A_S = \int_a^b 2\pi f(x)\sqrt{1 + \left(f'(x)\right)^2}\, dx$$ 

The next set of lessons focused on the calculus of parametric equations. Parametric equations can be used to describe relationships between two variables that are not strictly functions.

Parametric Equations

Parametric equations are a group of functions that describe various quantities in terms of one or more independent variables, i.e. parameters. In the most general sense, a set of parametric equations is as follows.

$$\begin{cases} f_1(\alpha_1, \alpha_2, \dots, \alpha_M) \\ f_2(\alpha_1, \alpha_2, \dots, \alpha_M) \\ \quad\quad . \\ \quad\quad . \\ f_N(\alpha_1, \alpha_2, \dots, \alpha_M) \end{cases}$$

Where, N is the number of quantities and α_i are the M parameters.

If we let the quantities be 2D space coordinates, i.e. x and y, and the parameter be time, t, then the parametric equations can be said to represent the position of an object over time in a 2D coordinate plane. In this case we say the object moves along the curve, $c(t)$, which can be written as

$$c(t) = [x(t), y(t)]$$

- The curve, $c(t)$, is not unique, i.e. all curves can be parameterized in infinitely many ways, however, the motion of a particle along this path will follow a particular parameterization.
- If the graph of the curve represents an explicit function, we can usually eliminate the parameter by solving for it in one of the equations and substituting it into the other to find $y(x)$ or $x(y)$.

Standard Parameterizations

- Line with slope $m = \frac{\Delta y}{\Delta x}$, that passes through the point, $[x_0, y_0]$.

$$c(t) = [x_0 + \Delta x t, y + \Delta y t]$$

- Circle with radius, R, centered at $[x_c, y_c]$.

$$c(t) = [x_c + R\,cos(t)\,, y_c + \ R\,sin(t)]$$

- Ellipse with major axis $2a$, and minor axis $2b$, centered at $[x_c, y_c]$.

$$c(t) = [x_c + a\,cos(t)\,, y_c + \ b\,sin(t)]$$

- Cycloid generated by a circle of radius, R.

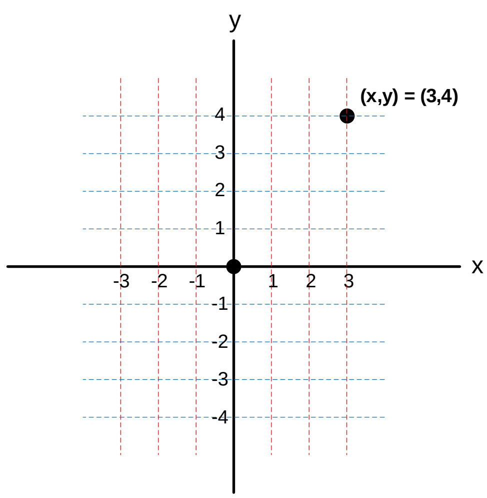
$$c(t) = [Rt - R\,sin(t)\,, R - R\,cos(t)]$$

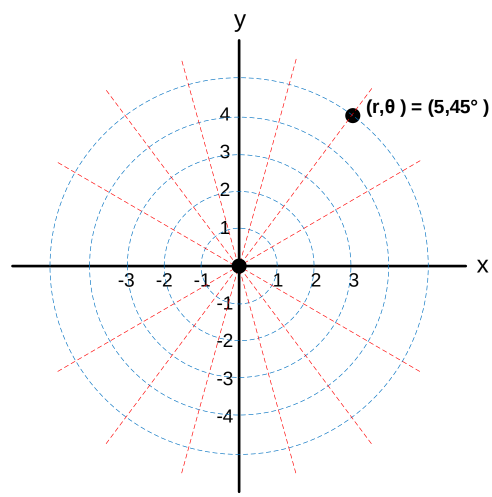
Slope of the Tangent Line for Parametric Equations
If $c(t) = [x(t), y(t)]$, where $x(t)$ and $y(t)$ are both differentiable and $x'(t)$ is continuous and not equal to zero, then $$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$
Area Under the Curve for Parametric Equations
For a parametric curve, $c(t) = [x(t), y(t)]$, that stays above the x-axis for $t_0 \leq t \leq t_1$ and represents a function in the same interval, i.e. passes the vertical line test, the area under this curve is given as $$A = \int_{t_0}^{t_1} y(t)x'(t)dt$$
Formula for Arc Length
If $c(t) = [x(t), y(t)]$, where $x(t)$ and $y(t)$ are differentiable, the arc length for $a \leq t \leq b$ is $$s = \int_a^b \left(\sqrt{x'(t)^2 + y'(t)^2}\right) dt$$
Distance Traveled Along a Parametric Curve
If $r(t) = [x(t), y(t)]$ represents the position of a particle in space over time, then the distance the particle has traveled along the curve at time, t, is $$s(t) = \int_{t_0}^{t} \left(\sqrt{x'(\tau)^2 + y'(\tau)^2}\right) dt$$ Furthermore, the speed of the particle at time, t, is $$\lvert v(t) \rvert = \frac{d}{dt}\big(s(t)\big) = \sqrt{x'(t)^2 + y'(t)^2}$$ The displacement of the particle, Δr, in each direction during a time $t_a \leq t \leq t_b$ is given as $$\Delta r = r(t_b) - r(t_a) = \left[\Delta r_x, \Delta r_y\right]$$ The magnitude of the displacement is then $$\lvert \Delta r \rvert = \sqrt{(\Delta r_x)^2 + \left(\Delta r_y\right)^2}$$
Formula for Surface Area of a Surface of Revolution
If $c(t) = [x(t), y(t)]$ and $y(t) \geq 0$ on $[a, b]$, then the surface area, A_S, of the surface obtained by rotating the curve about the x-axis for $a \leq t \leq b$ is equal to $$A_S = \int_a^b 2\pi y(t) \left(\sqrt{x'(t)^2 + y'(t)^2}\right) dt$$

One alternate to the rectangular coordinate system is called the polar coordinates system. For many problems, the polar coordinates system is much more convenient to use.

Polar Coordinates

A point P can be specified in both the rectangular coordinate system as (x, y) and a polar coordinate system as (r, θ). The two systems are shown below.





2D Coordinate Conversion Formulas

Polar to Rectangular: $(r, \theta) \rightarrow (x, y)$	***Rectangular to Polar:*** $(x, y) \rightarrow (r, \theta)$
$x = r\,cos(\theta)$ $y = r\,sin(\theta)$	$r = \sqrt{x^2 + y^2}$ $\theta = tan^{-1}\left(\frac{y}{x}\right)$ Note: The quadrant where the point lies must be considered when using $tan^{-1}\left(\frac{y}{x}\right)$.

Common Polar Curves

Description of Curve	Polar Equation
Circle of radius r_0 centered at the origin	$r = r_0$
Line through the origin with slope $\tan(\theta_0)$	$\theta = \theta_0$
Line on which the point (r_0, θ_0) is the point closest to the origin	$r(\theta) = r_0\,sec(\theta - \theta_0)$
Circle of radius a centered at $(a, 0)$	$r(\theta) = 2a\,cos(\theta)$
Circle of radius b centered at $(0, b)$	$r(\theta) = 2b\,sin(\theta)$
Circle of radius $\sqrt{a^2 + b^2}$ centered at (a, b)	$r(\theta) = 2a\,cos(\theta) + 2b\,sin(\theta)$

Area in Polar Coordinates
The area bounded by the curve, $r = f(\theta)$, and the rays $\theta = \theta_A$ and $\theta = \theta_B$, with $\theta_A < \theta_B$, is equal to $$A = \frac{1}{2}\int_{\theta_A}^{\theta_B} f^2(\theta)d\theta$$

Arc Length in Polar Coordinates
The arc length of a curve, $r = f(\theta)$, between the rays $\theta = \theta_A$ and $\theta = \theta_B$, with $\theta_A < \theta_B$, is equal to $$s = \int_{\theta_A}^{\theta_B} \left(\sqrt{f'(\theta)^2 + f(\theta)^2}\right) d\theta$$

Conic sections are an important class of functions that are used in a variety of applications. Hence, a lesson on conic sections is included in the study of calculus 2.

Conic Sections
Conic sections are curves which are obtained by intersecting the surface of a cone and a plane. The curves are the hyperbola, the parabola, the ellipse, and the circle (which can also be considered a special case of the ellipse). The algebraic equations that describe the conic sections are special cases of a general equation of degree 2 in x and y shown here. $$ax^2 + bxy + cy^2 + dx + ey + f = 0$$ Apart from certain "degenerate" conditions, the equation above describes a conic section that is not necessarily in standard position. The term bxy is called the cross term. When $b \neq 0$ the conic section is rotated with respect to the coordinate axes.

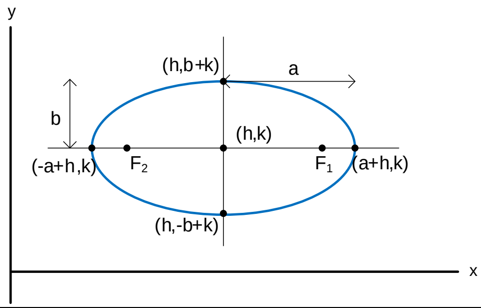
Ellipse

The algebraic equation for an ellipse can be written as

$$\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1$$

Where:

- $a > b > 0$ and $c = \sqrt{a^2 - b^2}$
- The center of the ellipse at located at (h, k)
- The focal points are $F_1 = (c + h, k)$ and $F_2 = (-c + h, k)$
- The semimajor axis is a and the semiminor axis is b
- The focal vertices are $(\pm a + h, k)$ and the minor vertices are $(h, \pm b + k)$.



Hyperbola

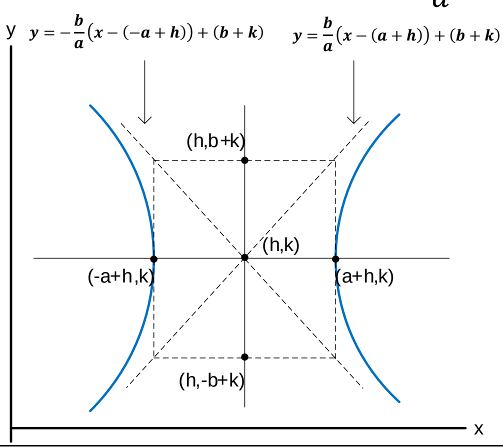
The algebraic equation for a hyperbola can be written as

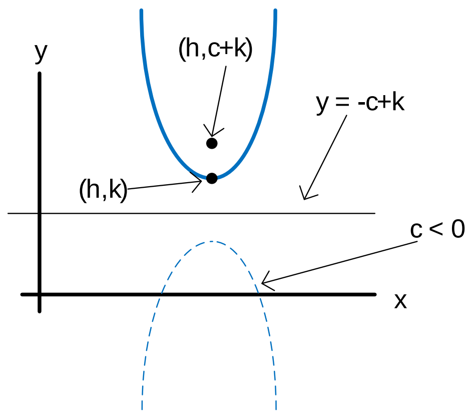
$$\left(\frac{x-h}{a}\right)^2 - \left(\frac{y-k}{b}\right)^2 = 1$$

Where:

- $a > 0, b > 0$ and $c = \sqrt{a^2 + b^2}$
- The center of the hyperbola at located at (h, k)
- The focal points are $F_1 = (c + h, k)$ and $F_2 = (-c + h, k)$
- Horizontal endpoints of the rectangle are $x = (\pm a + h)$
- Vertical endpoints of the rectangle are $y = (\pm b + k)$
- The slant asymptotes are given as:

$$y = \frac{b}{a}\big(x - (a + h)\big) + (b + k) \qquad y = -\frac{b}{a}\big(x - (-a + h)\big) + (b + k)$$



Parabola
The algebraic equation for a parabola can be written as $$y = \frac{1}{4c}(x-h)^2 + k$$ For $c \neq 0$ Where: • The vertex is located at (h, k) • The focus $F = (h, c+k)$ • The directrix is given as $y = -c + k$ • The parabola opens upward if $c > 0$ and downward if $c < 0$ 

Differential equations express how the derivatives of one or more independent variables vary with respect to one or more dependent variables. Differential equations are a fundamental tool used in science and engineering to model all types of physical systems. The next set of lessons provided a preliminary introduction into these types of equations.

Differential Equation Definition
An equation that relates derivatives of one or more dependent variables to one or more independent variables.

Differential Equation Classifications

Type:

1. *Ordinary differential equation,* (ODE): Has only one independent variable.
2. *Partial differential equation,* (PDE): has more than one independent variable.

Ordinary Differential Equation Example	***Partial Differential Equation Example***
$3\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 6y = 0$	$3\frac{\partial u}{\partial x} - 4\frac{\partial u}{\partial y} + 6u = 0$

Order:

Order is determined by the highest order derivative in a differential equation.

First-Order	$x^2\frac{dy}{dx} + 6y = 40$
Second-Order	$3\frac{d^2y}{dx^2} - sin(x)\frac{dy}{dx} = 0$
Third-Order	$3\frac{d^3y}{dx^3} = x\frac{dy}{dx} + cos(x^4)$

Linear vs. Non-Linear:

A differential equation is called linear if it can be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

The above equation is characterized by the following two properties:

1. The dependent variable, y, is only to the first degree.
   - E.g. $5x^2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + \boldsymbol{y^2} = e^x$ is non-linear because of the y^2 term.
2. Each coefficient, a_n, depends only on the independent variable, x.
   - E.g. $5x^2\frac{d^2y}{dx^2} + 3\text{y}\frac{dy}{dx} + y = e^x$ is non-linear because of the coefficient $3y$.

Separation of Variables

A first-order differential equation is called *separable* if the first-order derivative can be expressed as the ratio of two functions: one a function of x and the other a function of y.

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

First-order separable differential equations are solved using the method of the *Separation of Variables* as follows:

1. Move the terms involving y and dy to one side and the terms involving x and dx to the other.

$$\frac{1}{g(y)} dy = f(x)dx$$

2. Integrate both sides of the equation.

$$\int \frac{1}{g(y)} dy = \int f(x)dx$$

Assuming these integrals can be evaluated, we can then try to solve for y as a function of x.

$$y = f(x) + C$$

The set of solutions defined by the above equation is known as the general solution to the differential equations.

Initial Value Problem

A differential equation that is subject to an initial condition is called an *initial value problem*. An example is given as follows:

Solve the following differential equation

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

Subject to the initial condition

$$y(1) = 2$$

After finding the general solution, the initial condition can be used to solve for the unknown constant to obtain the so-called *particular solution*.

Exponential Growth/Decay – Rate ∝ Amount Present

The differential equation that models a system in which the rate of change of a certain quantity, $y(t)$, is proportional to the quantity present is as follows:

$$\frac{dy}{dt} = ky$$

Where k is referred to as the proportionality, or time, constant.

The general solution to this differential equation is given as

$$y(t) = Ce^{kt}$$

Assuming we know the quantity at $t = 0$, the particular solution is given as

$$y(t) = y(0)e^{kt}$$

- If $k < 0$ the solution is an exponential decaying function.
- If $k > 0$ the solution is an exponential growth function.

Exponential Growth/Decay – Rate ∝ (Amount Present – Fixed Value)

The differential equation that models a system in which the rate of change of a certain quantity, $y(t)$, is proportional to the quantity present minus a fixed value, b, is as follows:

$$\frac{dy}{dt} = k(y - b)$$

Where k is referred to as the proportionality constant and b is a fixed value.

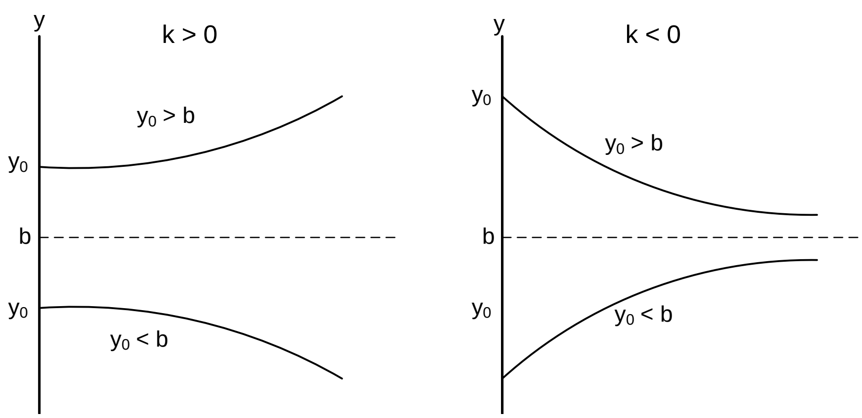
The general solution to this differential equation is given as

$$y(t) = Ce^{kt} + b$$

Assuming we know the quantity at $t = 0$, the particular solution is given as

$$y(t) = (y(0) - b)e^{kt} + b$$

We can identify the behavior of 4 cases based on the sign of k and $(y(0) - b)$.



Note: this behavior would be identical to the first type of differential equation if let $b = 0$.

The Logistic Equation for Population Growth
Around 1840 P.F. Verhulst proposed an alternate model for population growth which is based on a logistic differential equation and is written as $$\frac{dP}{dt} = kP\left(1 - \frac{P}{A}\right)$$ Where, $k > 0$ is the growth constant and $A > 0$ is called the carrying capacity. The solution is given as $$P(t) = \frac{A}{(1 + Be^{-kt})}$$ Where, $$B = \frac{A - P(0)}{P(0)}, \;\; P(0) > 0$$

The final series of lessons focused on the so-called third branch of calculus, the theory of infinite series. There are many important and interesting applications of infinite series. As an example, the Fourier Series has innumerable practical applications in many different fields.

Sequence
A sequence, $\{a_n\}$, is an ordered collection of numbers that may or may not be defined by a function, f, on a set of sequential integers. The values a_n are called the terms of the sequence, and n is called the index. $$\{a_n\} = a_1, a_2, a_3, \ldots$$ If the sequence is defined by a function, we can say that $$a_n = f(n)$$ Note: The sequence does not have to start at $n = 1$. It can start at any other integer.
Limit of a Sequence
We say that the sequence, $\{a_n\}$, converges to a limit L, and we write $$\lim_{n\to\infty} a_n = L$$ if, for every $\varepsilon > 0$, there is a number M such that $\lvert a_n - L\rvert < \epsilon$ for all $n > M$. • If no limit exists, we say that $\{a_n\}$ diverges. If the terms increase without bound, we say that $\{a_n\}$ diverges to infinity.
Sequence Defined by a Function
If $\lim_{x\to\infty} f(x)$ exists, then the sequence $a_n = f(n)$ converges to the same limit: $$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$$
A Function of a Sequence
If f is continuous and $\lim_{n\to\infty}(a_n) = L$, then $$\lim_{n\to\infty}(a_n) = f\left(\lim_{n\to\infty}(a_n)\right) = f(L)$$

Bounded Sequences

A sequence $\{a_n\}$ is:

- **Bounded from above** if there is a number M_u such that $a_n \le M_u$ for all n. The number M_u is called the upper bound.
- **Bounded from below** if there is a number M_d such that $a_n \ge M_d$ for all n. The number M_d is called the lower bound.

The sequence $\{a_n\}$ is called **bounded** if it is bounded from above **and** below. A sequence that is not bounded is called an **unbounded** sequence.

Monotonic Sequences

A sequence $\{a_n\}$ is **monotonic** if:

$a_{n+1} > a_n$, i.e. it is increasing.

or

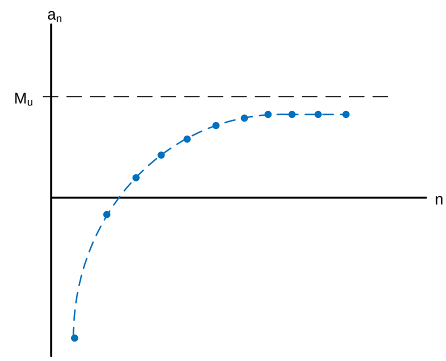
$a_{n+1} < a_n$, , i.e. it is decreasing.

Monotonic Sequences Bounded from Above or Below

A sequence $\{a_n\}$ is:

- **Monotonically increasing,** $a_{n+1} > a_n$, and
- **Bounded from above,** $a_n \le M_u$

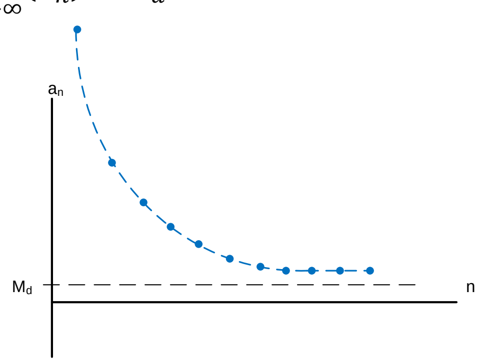
Then $\{a_n\}$ converges and $\lim_{n\to\infty}(a_n) \le M_u$



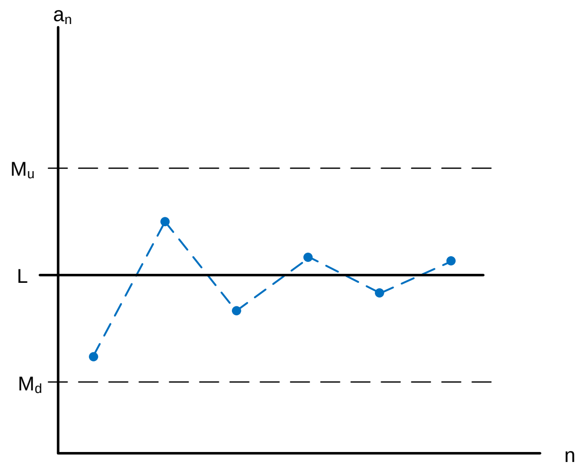
A sequence $\{a_n\}$ is:

- **Monotonically decreasing,** $a_{n+1} < a_n$, and
- **Bounded from below,** $a_n \ge M_d$

Then $\{a_n\}$ converges and $\lim_{n\to\infty}(a_n) \ge M_d$



Convergent Sequences are Bounded
If $\{a_n\}$ converges, then it is bounded from above and below.



Note: This theorem does **not** state that all bounded sequences converge.

Function Growth Rates

In some cases, we can use the following guidelines to help us evaluate limits of rational functions.

$$\lim_{n\to\infty}\left(\frac{higher\ growth\ rate\ function}{lower\ growth\ rate\ function}\right)=\infty$$

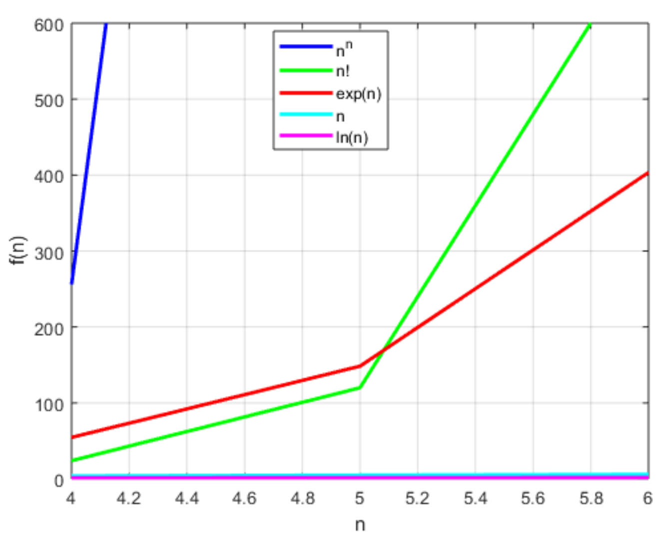
$$\lim_{n\to\infty}\left(\frac{lower\ growth\ rate\ function}{higher\ growth\ rate\ function}\right)=0$$

We can also classify the relative growth rate of some common functions as shown below. As n gets very large the following is true.

$$ln(n) \ll n^a \ll b^n \ll n! \ll n^n$$

For $a > 0$ and $b > 1$.

Illustrative Plot



Infinite Series
An infinite series is a summation of the terms of an infinite sequence, e.g. $\{a_n\}$. $$\sum_{n=k}^{\infty} a_n = \lim_{N\to\infty}\left(\sum_{n=k}^{N} a_n\right)$$

Convergence of an Infinite Series
An infinite series $\sum_{n=k}^{\infty} a_n$ converges to a value, S, if the sequence of its partial sums, $\{S_N\}$, converges to S. $$\lim_{N\to\infty}\left(\sum_{n=k}^{N} a_n\right) = \lim_{N\to\infty}(S_N) = S$$ Where, $$S_N = \sum_{n=k}^{N} a_n$$ • If no limit exists, we say that the infinite series diverges. • If the terms increase without bound, we say that infinite series diverges to infinity.

Telescoping Series
A telescoping has the general from $$S = \sum_{n=1}^{\infty} b(n) - b(n+A)$$ Expanding this series and cancelling terms we have $$S = [b(1) + b(2) + \ldots + b(A)] - \left[\lim_{N\to\infty}\big(b(N+1) + b(N+2) + \ldots + b(N+A)\big)\right]$$ Assuming $\lim_{N\to\infty}\big(b(N+1) + b(N+2) + \ldots + b(N+A)\big) = 0$ $$S = [b(1) + b(2) + \ldots + b(A)]$$

Geometric Series
A geometric series, with $C \neq 0$ has the general form $$S = \sum_{n=M}^{\infty} Cr^n$$ If $\lvert r\rvert < 1$ the geometric series converges and $$S = \sum_{n=M}^{\infty} Cr^n = \frac{Cr^M}{(1-r)}$$ Note: if $M = 0$ we can write $$S = \sum_{n=0}^{\infty} Cr^n = \frac{C}{(1-r)}$$ If $\lvert r\rvert \geq 1$ the geometric series diverges.

Summary of Tests

nth Term Divergence Test

If $\lim_{n\to\infty}(a_n) \neq 0$ then the series

$\sum_{n=1}^{\infty} a_n$, diverges

If $\lim_{n\to\infty}(a_n) = 0$ then the test is inconclusive.

Direct Comparison Test

Assume that there exists $M > 0$ such that $0 \leq a_n \leq b_n$ for all $n \geq M$.

i. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

ii. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.

Limit Comparison Test

Let a_n and b_n be positive sequences and let the following limit exist.

$$L = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)$$

i. If $L > 0$ then $a_n \cong Lb_n$ for large n, and both series either converge or diverge.

ii. If $L = 0$ then $b_n \gg a_n$ for large n, and if $\sum_{n=1}^{\infty} b_n$ converge, so does $\sum_{n=1}^{\infty} a_n$

iii. If $L = \infty$ then $a_n \gg b_n$ for large n, and if $\sum_{n=1}^{\infty} a_n$ converge, so does $\sum_{n=1}^{\infty} b_n$.

Integral Test

Let $a_n = f(n)$, where f is a positive, decreasing, and continuous function of x for $x \geq 1$.

- If $\int_1^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

If $\int_1^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Ratio Test

Given the series $\sum a_n$ we define the following

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

Then,

- If $L < 1$ then $\sum a_n$ converges absolutely.
- If $L > 1$ then $\sum a_n$ diverges.
- If $L = 1$ the test is inconclusive.

Root Test

Given the series $\sum a_n$ we define the following

$$L = \lim_{n\to\infty}\left(\sqrt[n]{|a_n|}\right)$$

Then,

- If $L < 1$ then $\sum a_n$ converges absolutely.
- If $L > 1$ then $\sum a_n$ diverges.
- If $L = 1$ the test is inconclusive.

Alternating Series Test

An alternating series of the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

Converges if

- $b_n \geq 0$, Non-negative
- $b_{n+1} < b_n$, Decreasing

$$\lim_{n \to \infty} (b_n) = 0$$

Absolute Convergence

The series $\sum a_n$ ***converges absolutely*** if $\sum |a_n|$ converges.

Absolute Convergence Implies Convergence

- If $\sum |a_n|$ converges, then $\sum a_n$ also converges.
- If $\sum |a_n|$ diverges, then the behavior of $\sum a_n$ is inconclusive.

Conditional Convergence

A series $\sum a_n$ ***converges conditionally*** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Convergence of p-Series

The infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Converges for $p > 1$, and diverges otherwise.

Geometric Series

A geometric series, with $C \neq 0$ has the general form

$$S = \sum_{n=M}^{\infty} Cr^n$$

If $|r| < 1$ the geometric series converges and

$$S = \sum_{n=M}^{\infty} Cr^n = \frac{Cr^M}{(1-r)}$$

Note: if $M = 0$ we can write

$$S = \sum_{n=0}^{\infty} Cr^n = \frac{C}{(1-r)}$$

If $|r| \geq 1$ the geometric series diverges.

Test Determination Strategies
It's not always obvious which test to apply to a particular series. Below are some general guidelines for determining which test to apply when first looking at an arbitrary series.
1. The n^{th} term Divergence Test
One should always check this test first, as it is usually relatively east to check. If the series diverges, i.e. $\lim_{n\to\infty}(a_n) \neq 0$, no other test are required.
2. Positive Series

a. The Direct Comparison Test

For this test we should consider whether dropping terms in the numerator or denominator of the sequence results in a series that we know either converges for diverges.
If by dropping terms we create a series with a sequence, b_n, that converges, we can prove the original series with the sequence, a_n, also converges if $a_n < b_n$ for all $n > M$.
For example, if given $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2+n}$, we can create a new series, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$.
Then since $\sum_{n=1}^{\infty} b_n$ is a convergent p-series and $a_n < b_n$ we can conclude that $\sum_{n=1}^{\infty} a_n$ converges.

Conversely, if by dropping terms we create a series with a sequence, b_n, that diverges, we can prove the original series with the sequence, a_n, also diverges if $a_n > b_n$ for all $n > M$
For example, if given $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$, we can create a new series, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$.
Then since $\sum_{n=1}^{\infty} b_n$ is a divergent p-series and $a_n > b_n$ we can conclude that $\sum_{n=1}^{\infty} a_n$ diverges.
An example of a series where this test doesn't work is $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2-n}$. In this case we can use the limit comparison test, which we explain next.

a. The Limit Comparison Test

For this test we should consider the dominant term in the numerator and denominator as our comparison sequence, b_n. Take the example from the previous test.
Given $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2-n}$, we would create a new series as $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$.
Performing the limit comparison test we have, $L = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{n^2}{n^2-n}\right) = 1$, and since $L > 0$ the series behave the same, and therefore $\sum_{n=1}^{\infty} a_n$ converges.

b. The Ratio Test

The ratio test can be useful for series containing factorials and constants to the power n. When forming the ratio, the power n cancels and there is also quite a bit of cancellation with the factorials. For example, when we apply the ratio test to the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{5^n}{n!}$, the result is as follows:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{5^{n+1}}{(n+!)!}\cdot\frac{n!}{5^{n+1}}\right) = \lim_{n\to\infty}\left(\frac{5}{n+1}\right) = 0$$

And since $L < 1$ the series converges.

c. The Root Test

The root test can be useful for series containing a term of the form $f(n)^{g(n)}$. For example, applying the root test to the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left(\frac{3n^3+5n}{7n^3+2}\right)^n$ yields the following.

$$L = \lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}\left(\left(\frac{3n^3+5n}{7n^3+2}\right)^n\right)^{1/n} = \lim_{n\to\infty}\left(\frac{3n^3+5n}{7n^3+2}\right) = \lim_{n\to\infty}\left(\frac{3\cancel{n^3}}{7\cancel{n^3}}\right) = \frac{3}{7}$$

And since $L < 1$ the series converges

f. The Integral Test

When other tests fail and the series is positive and decreasing, we can consider the integral test. With the integral test the series converges if $\int_1^{\infty} f(x)dx$, where we map the discrete sequence a_n to the continuous function, $f(x)$. For example, the following series is difficult to evaluate using any of the above tests but can be easily evaluated using the integral test as shown below. Given the series

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty}\frac{1}{n\, ln\, n}$$

We find

$$\int_2^{\infty}\frac{1}{x\, ln\, x}dx = ln(ln\, x)|_2^{\infty} = \infty$$

Therefore, the series diverges.

3. Non-Positive Series

a. The Alternating Series Test

An alternating series is of the form $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$, and converges if:

- $b_n \geq 0$, Non-negative
- $b_{n+1} < b_n$, Decreasing
- $\lim_{n\to\infty}(b_n) = 0$

For example, the alternating harmonic series shown below.

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

Converges since

- $\frac{1}{n} \geq 0$, Non-negative
- $\frac{1}{n+1} < \frac{1}{n}$, Decreasing
- $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$

b. Absolute Convergence

If the series is not alternating, but nonetheless non-positive we may compute the absolute value of the sequence and check if this series converges. If so, the original series converges as well.

General Form of a Power Series

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

Where, x is a variable and c is a constant. We refer to this as a power series with center c.

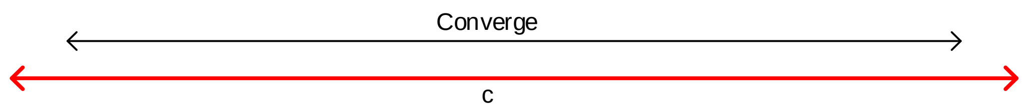
The series can also be expressed in expanded form as shown below.

$$F(x) = a_0 + a_1(x-c)^1 + a_2(x-c)^2 + a_3(x-c)^3 + \ldots$$

Radius of Convergence for Power Series

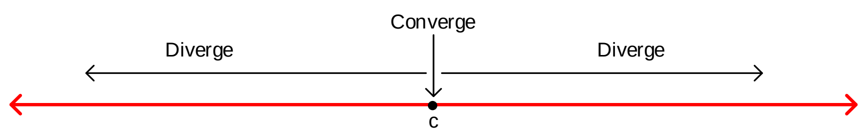
Case 1:

The series converges for all values of x, i.e. $R = \infty$



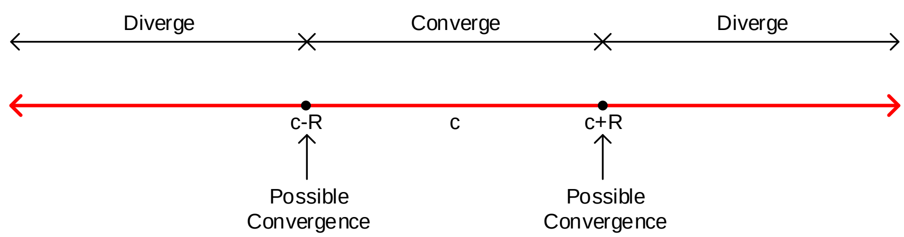
Case 2:

The series diverges for all values of x except $x = c$, i.e. $R = 0$



Case 3:

The series converges for $|x - c| < R$, i.e. $R > 0$



Geometric Series as a Power Series

$$F(x) = \sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$$

For $|x| < 1$

Note: Power series for other functions can be found with substitution using this series as a abase.

Power Series Differentiation and Integration
Assume that the power series $$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$ Has a radius of convergence, $R > 0$. Then $F(x)$ can be differentiated and integrated for $c - R < x < c + R$. The differentiation and integration is done term by term and can be expressed as follows:
Differentiation
$$\frac{d}{dx}\big(F(x)\big) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n (x-c)^n\right) = \sum_{n=0}^{\infty} a_n \frac{d}{dx}(x-c)^n$$ $$\boldsymbol{\frac{d}{dx}\big(F(x)\big) = \sum_{n=1}^{\infty} a_n n (x-c)^{n-1}}$$
Integration
$$\int F(x)dx = \int\left(\sum_{n=0}^{\infty} a_n (x-c)^n\right)dx = \sum_{n=0}^{\infty}\left(\int a_n (x-c)^n\right)dx$$ $$\boldsymbol{\int F(x)dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1} + C}$$
The resulting series have the same radius of convergence, R.

Taylor Series Expansion
If $f(x)$ is represented as a power series centered at c in an interval $\|x-c\|<R$ with $R>0$, then the power series is the **Taylor series** $$T(x)=\sum_{n=0}^{\infty}\frac{f^n(c)}{n!}(x-c)^n$$ In the special case where $c=0$, $T(x)$ is also called the **Maclaurin series** $$T(x)=\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n$$
Taylor Series Convergence
Suppose there exists a $K>0$ such that all derivatives of $f(x)$ are bounded by K on an interval, $I=(c-R,c+R)$, i.e. $$\|f^k(x)\|\leq K\text{ , for all } k>0 \text{ and } x\in I$$ Then $f(x)$ can be represented by the Taylor series, $T(x)$ on I $$T(x)=\sum_{n=0}^{\infty}\frac{f^n(c)}{n!}(x-c)^n\text{ , for all } x\in I$$
Binomial Theorem
If a is any positive integer then, $$(x+y)^a=\sum_{k=0}^{a}\binom{a}{k}x^{a-k}y^k$$ $$(x+y)^a=\binom{a}{0}x^a y^0+\binom{a}{1}x^{a-1}y^1+\binom{a}{2}x^{a-2}y^2+\cdots+\binom{a}{a-1}x^1y^{n-1}+\binom{a}{a}x^0y^a$$ Where, the binomial coefficient, $\binom{a}{k}$, is defined as $$\binom{a}{k}=\frac{a!}{k!\,(a-k)!}$$ or equivalently as $$\binom{a}{k}=\frac{a(a-1)(a-2)\cdots\big(a-(k-1)\big)}{k!} \qquad \binom{a}{0}\overset{\text{def}}{=}1$$

Table of Common Maclaurin Series		
$f(x)$	**Maclaurin Series**	**Converges to $f(x)$**
e^x	$\sum_{n=0}^{\infty} \frac{(x)^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$All\ x$
$sin(x)$	$\sum_{n=0}^{\infty} (-1)^n \frac{(x)^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$All\ x$
$cos(x)$	$\sum_{n=0}^{\infty} (-1)^n \frac{(x)^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$All\ x$
$(1+x)^a$	$\sum_{n=0}^{\infty} \binom{a}{n} x^n = \binom{a}{0} x^0 + \binom{a}{1} x^1 + \binom{a}{2} x^2 + \binom{a}{3} x^3 + \cdots$ $= 1 + ax + \frac{a(a-1)}{2!} x^2 + \frac{a(a-1)(a-2)}{3!} x^3 + \cdots$	$\|x\| < 1$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	$\|x\| < 1$
$\frac{1}{1+x}$	$\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$	$\|x\| < 1$
$ln(1+x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \frac{x^1}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$-1 < x \leq 1$

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