Calculus 1 Summary and Formulas

At the heart of Calculus is the concept of an infinitesimal. In order to understand infinitesimals, we first need to understand the concept of the limit. The first series of lessons studied this concept.

Definition of a Limit

If $f(x)$ is defined for all x in an open interval containing c, but not necessarily defined at c, we can say that:

The limit of $f(x)$ as x approaches c is equal to the number L

If $|f(x) - L|$ can be made arbitrarily small by taking x sufficiently close (but not equal) to c. In this case we can write:

$$\lim_{\substack{x \to c \\ x \to c}} \{f(x)\} = L$$

And we can say: "The limit of $f(x)$ as x approaches c is equal to L"

One-Sided Limits

Left-Sided Limit:

The limit of $f(x)$ as x approaches c from values less than c is equal to L.

$$\lim_{x \to c^-} \{f(x)\} = L$$

Right-Sided Limit:

The limit of $f(x)$ as x approaches c from values greater than c is equal to L.

$$\lim_{x \to c^+} \{f(x)\} = L$$

- The limit itself exists only when both one-sided limits exists and are equal.

$$\lim_{x \to c} \{f(x)\} = L \text{ if and only if } \lim_{x \to c^-} \{f(x)\} = \lim_{x \to c^+} \{f(x)\} = L$$

Infinite Limits

- $\lim_{x \to c} \{f(x)\ \} = \infty$ if $f(x)$ increases without bound as x approaches c.
- $\lim_{x \to c} \{f(x)\ \} = -\infty$ if $f(x)$ decreases without bound as x approaches c.

Note: The same one-sided limit rules apply to infinite limits.

Basic Limit Laws

If $\lim_{x \to c}\{f(x)\}$ and $\lim_{x \to c}\{g(x)\}$ exist, then the following laws apply

Sum and Difference Law

$$\lim_{x \to c}\{f(x) \mp g(x)\} = \lim_{x \to c}\{f(x)\} \mp \lim_{x \to c}\{g(x)\}$$

Constant Multiple Law

$$\lim_{x \to c}\{Kf(x)\} = K \lim_{x \to c}\{f(x)\}$$

Product Law

$$\lim_{x \to c}\{f(x) \cdot g(x)\} = \left(\lim_{x \to c}\{f(x)\}\right) \cdot \left(\lim_{x \to c}\{g(x)\}\right)$$

Quotient Law

$$\lim_{x \to c}\left\{\frac{f(x)}{g(x)}\right\} = \frac{\lim_{x \to c}\{f(x)\}}{\lim_{x \to c}\{g(x)\}}$$

Provided, $\lim_{x \to c}\{g(x)\} \neq 0$

Power Law

$$\lim_{x \to c}\left\{\left(f(x)\right)^n\right\} = \left(\lim_{x \to c}\{f(x)\}\right)^n$$

Where, n is a positive integer

Root Law

$$\lim_{x \to c}\left\{\sqrt[n]{f(x)}\right\} = \sqrt[n]{\lim_{x \to c}\{f(x)\}}$$

Where, if n is even we assume $\lim_{x \to c}\{f(x)\} \geq 0$

Continuity at a Point

A function, $f(x)$, is continuous at a point, c, if

$$\lim_{x\to c}\{f(x)\} = f(c)$$

Otherwise, the function is discontinuous at $x = c$.

Conditions for Continuity of a Function

1. $f(c)$ is defined
2. $\lim_{x\to c}\{f(x)\}$ exists
3. The values of 1 and 2 are equal.

Types of Discontinuities

1. ***Removable Discontinuity:*** Occurs if $\lim_{x\to c}\{f(x)\}$ exists but is not equal to $f(c)$.

2. ***Jump Discontinuity:*** Occurs when $\lim_{x\to c^-}\{f(x)\}$ and $\lim_{x\to c^+}\{f(x)\}$ both exists but are not equal, and therefore $\lim_{x\to c}\{f(x)\}$ does not exist.

3. ***Infinite Discontinuity:*** Occurs when $\lim_{x\to c^-}\{f(x)\}$, $\lim_{x\to c^+}\{f(x)\}$, or both are infinite.

Common Continuous Functions

- Polynomials
- $\sin(x)$ and $\cos(x)$
- Exponential functions
- Logarithmic functions on its domain
- $x^{\frac{1}{n}}$ on its domain where n is any natural number.

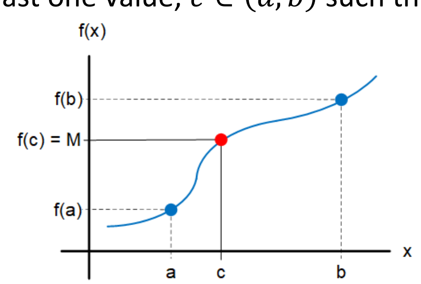
Laws of Continuity

If $f(x)$ and $g(x)$ are continuous at $x = c$, then the following functions are also continuous at $x = c$:

1. $f(x) \mp g(x)$
2. $f(x) \cdot g(x)$
3. $kf(x)$, for any k
4. $f(x)/g(x)$, if $g(c) \neq 0$

Intermediate Value Theorem

If $f(x)$ is continuous on a closed interval, $[a, b]$, then for every value M between $f(a)$ and $f(b)$, there exists at least one value, $c \in (a, b)$ such that $f(c) = M$.



Direct Substitution

If a function is continuous at $x = c$, we can evaluate the limit using direct substitution.

$$\lim_{x \to c} \{f(x)\} = f(c)$$

Indeterminate Forms

We say that $f(x)$ has an indeterminate form at $x = c$ when $f(c)$ evaluates to one of the following forms:

$$\frac{0}{0}, \qquad \frac{\infty}{\infty}, \qquad \infty \cdot 0, \qquad \infty - \infty$$

Algebraic Evaluation Techniques

To evaluate $\lim_{x \to c} \{f(x)\}$ when $f(c)$ is an indeterminate we can try to algebraically rewrite $f(x)$ so that $f(c)$ no longer evaluates to an indeterminate. **If possible**, the limit can then be evaluated using direct substitution.

Some common algebraic techniques to rewrite $f(x)$ are as follows:

1. Factor all factorable polynomials
2. Combine terms using a common denominator where possible
3. Expand factored terms where possible
4. Multiply numerator and denominator by the conjugate of the binomial with a square root term.

Note: Given an indeterminate form, it is NOT always possible to algebraically manipulate the expression to allow for evaluation of the limit using direct substitution. In these cases, we need to rely on other techniques.

Squeeze Theorem

If for all points in an open interval (a, b), excluding the point $x = c$, where $a < c < b$, the following is true:

$$l(x) \leq f(x) \leq u(x) \quad \text{and} \quad \lim_{x \to c} \{l(x)\} = \lim_{x \to c} \{u(x)\} = L$$

Then:

$$\lim_{x \to c} \{f(x)\} = L$$

Important Trigonometric Limits

$$\lim_{x \to 0} \left\{ \frac{\sin(x)}{x} \right\} = \lim_{x \to 0} \left\{ \frac{x}{\sin(x)} \right\} = 1 \qquad\qquad \lim_{x \to 0} \left\{ \frac{1 - \cos(x)}{x} \right\} = 0$$

Limits at Infinity of Power Functions

For all positive integers, n

$$\lim_{x\to\infty}\{x^n\} = \infty \qquad \lim_{x\to\infty}\{x^{-n}\} = 0$$

$$\lim_{x\to-\infty}\{x^n\} = \begin{cases} \infty \ if\ n\ is\ even \\ -\infty\ if\ n\ is\ odd \end{cases} \qquad \lim_{x\to-\infty}\{x^{-n}\} = 0$$

Limits at Infinity of Radical Functions

For all positive integers, p and q

q is even, e.g. $x^{1/2}, x^{1/4}$	p is even, q is odd, e.g. $x^{2/3}, x^{2/5}$	p is odd, q is odd, e.g. $x^{1/3}, x^{1/5}$
$\lim_{x\to\infty}\left\{x^{p/q}\right\} = \infty$	$\lim_{x\to\infty}\left\{x^{p/q}\right\} = \infty$	$\lim_{x\to\infty}\left\{x^{p/q}\right\} = \infty$
$\lim_{x\to-\infty}\left\{x^{p/q}\right\} = DNE$	$\lim_{x\to-\infty}\left\{x^{p/q}\right\} = \infty$	$\lim_{x\to-\infty}\left\{x^{p/q}\right\} = -\infty$

Limits at Infinity of Rational Functions

The limits at infinity of a rational function depend ONLY on the leading terms of the numerator and denominator.

If $a_n, b_m \neq 0$, then

$$\lim_{x\to\pm\infty}\left\{\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}\right\} = \left(\frac{a_n}{b_m}\right)\lim_{x\to\pm\infty}\left\{\frac{x^n}{x^m}\right\}$$

With the following cases that depend on the relative values of n and m

1. $n = m$:

$$\lim_{x\to\pm\infty}\left\{\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}\right\} = \frac{a_n}{b_m}$$

2. $n < m$:

$$\lim_{x\to\pm\infty}\left\{\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}\right\} = 0$$

3. $n > m$,

$n - m$: Even

$$\lim_{x\to\pm\infty}\left\{\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}\right\} = \infty$$

$n - m$: Odd

$$\lim_{x\to\pm\infty}\left\{\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}\right\} = \pm\infty$$

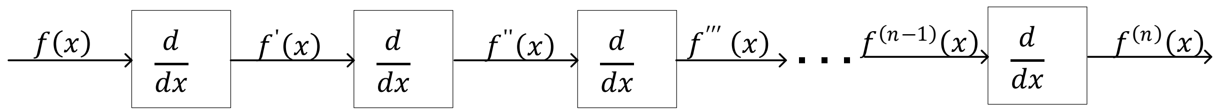
The second set of lessons introduced the first major branch of calculus, differentiation.

The Definitional of the Derivative
The derivative of a function, $f(x)$, with respect to x at a point $x = a$, which we refer to as $f'(a)$, is the limit of the difference quotient (if it exists): $$f'(a) = \lim_{h \to 0} \left\{ \frac{f(a+h) - f(a)}{h} \right\}$$ If the limit exists, then we say the function is differentiable at the point $x = a$.   Note: The derivative at a point $x = a$ can also be interpreted as the slope of the tangent line to the graph of $f(x)$ at the point, $P = (a, f(a))$. Using the point-slope formula, the tangent line can be expressed as: $$y = f'(a)(x - a) + f(a)$$
The Derivative as a Function
The derivative of a function, $f(x)$, with respect to x is another function, $f'(x)$, defined as: $$f'(x) = \lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} \right\}$$ The domain of $f'(x)$ consists of all values of x in the domain of $f(x)$ for which the limit above exists. We say that $f(x)$ is differentiable for wherever $f'(x)$ exists.
Various Notations for the Derivative
Assuming $y = f(x)$, notations for the derivative may be indicated as follows: **Lagrange Notation** $f'(x)$, f', $y'(x)$, f' **Leibniz Notation** $\frac{df(x)}{dx}$, $\frac{df}{dx}$, $\frac{dy(x)}{dx}$, $\frac{dy}{dx}$ And to specify the value of the derivative for a fixed value of x, e.g. a: **Lagrange Notation** $f'(a)$, $y'(a)$ **Leibniz Notation** $\left.\frac{df(x)}{dx}\right\|_{x=a}$ $\left.\frac{dy(x)}{dx}\right\|_{x=a}$

Derivative of a Constant Rule
$$\frac{d}{dx}(C) = 0$$
Linearity Rules of the Derivative
Sum and Difference Rule: $$\frac{d}{dx}\big(f(x) \pm g(x)\big) = \frac{d}{dx}\big(f(x)\big) \pm \frac{d}{dx}\big(g(x)\big)$$ *Constant Multiple Rule:* $$\frac{d}{dx}\big(Cf(x)\big) = C\frac{d}{dx}\big(f(x)\big)$$
The Power Rule for the Derivative
For all exponents, n: $$\frac{d}{dx}(x^n) = nx^{n-1}$$
The Product Rule for the Derivative
If $f(x)$ and $g(x)$ are differentiable functions, then $f(x)g(x)$ is also differentiable and: $$\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x)$$ Dropping the independent variable in the notation we can write this more compactly as: $$(fg)' = f'g + fg'$$
The Quotient Rule for the Derivative
If $f(x)$ and $g(x)$ are differentiable functions, then $f(x)/g(x)$ is also differentiable and: $$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$ Dropping the independent variable in the notation we can write this more compactly as: $$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Higher Order Derivatives

Higher derivatives are obtained by repeatedly differentiating a function according to the diagram below.



Where we refer to $f'(x)$ as the first derivative, $f''(x)$ as the second derivative, and so on. After three prime marks we usually switch to using integer values, but in parenthesis to distinguish from raising the function to a power. For example, the fourth derivative is notated as $f^{(4)}(x)$.

Using Leibniz notation, higher order differentiation is notated as follows:

$$\frac{df}{dx}, \qquad \frac{d^2f}{dx}, \qquad \frac{d^3f}{dx}, \qquad \frac{d^4f}{dx}, \ldots$$

The Chain Rule of Differentiation

If $f(x)$ and $g(x)$ are differentiable functions, then the composite function, $f\big(g(x)\big)$, is differentiable and

Using Leibniz notation:

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

Using Lagrange notation:

$$\Big(f\big(g(x)\big)\Big)' = f'\big(g(x)\big)g'(x)$$

Using this notation, we usually refer to g as the inside function and f as the outside function.

With this the above form can be stated in words as follows:
"The derivative of $f\big(g(x)\big)$ is equal to the derivative of the outside function evaluated at the inside function, multiplied by the derivative of the inside function."

Implicit Differentiation

Implicit differentiation is used to compute dy/dx when x and y are related through an implicit relation only. The steps to compute dy/dx can be summarized as follows:

1. Compute the derivative of both sides of the equation with respect to x.
2. Solve the resulting equation for dy/dx.

Remember, when differentiating any y terms, we need to use the chain rule since we are differentiating with respect to x and y is a function of x. Example shown below:

$$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$$

Derivative of Inverse Trigonometric Functions

$$\frac{d}{dx}(sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(tan^{-1}(x)) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(cot^{-1}(x)) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(csc^{-1}(x)) = -\frac{1}{|x|\sqrt{1-x^2}}$$

$$\frac{d}{dx}(sec^{-1}(x)) = \frac{1}{|x|\sqrt{1-x^2}}$$

Derivative of General Inverse Functions

Suppose f has an inverse function, f^{-1}. If f is differentiable at $f^{-1}(x)$ and $f'(f^{-1}(x))$ is not zero, then $f^{-1}$is differentiable at x and we may apply the following formula to evaluate.

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Derivative of Exponential Functions

If $f(x) = b^x$ for any $b > 0$,

$$\frac{d}{dx}(b^x) = ln(b)\, b^x$$

For $b = e$, $ln(e) = 1$, which results in a function whose derivative is the function itself.

$$\frac{d}{dx}(e^x) = e^x$$

Derivative of Logarithmic Functions

If $f(x) = log_b(x)$ for any $b > 0$, and for $x > 0$

$$\frac{d}{dx}(log_b(x)) = \frac{1}{xln(b)}$$

For $b = e$, $ln(e) = 1$, therefore the derivative of the natural logarithm is given as

$$\frac{d}{dx}(ln(x)) = \frac{1}{x}$$

Logarithmic Differentiation

Logarithmic differentiation can be used to save time when differentiating functions containing a product and/or quotient with *several* factors or functions of the type, $f(x)^{g(x)}$.

The procedure calls for first taking the logarithm of the function and then using various logarithm properties to simplify the differentiation process. The derivative of the original function is then computed as

$$\frac{d}{dx}\big(f(x)\big) = \big(f(x)\big)\frac{d}{dx}\Big(ln\big(f(x)\big)\Big)$$

The next set if lessons introduced various applications of derivatives.

Related Rate Problem Summary
We are generally presented with two variables, e.g. x, and y, that are both functions of time and that can be related through either an explicit or implicit function, e.g. explicit: $y = f(x)$. We are then given the time rate of change of one of the variables, e.g. dx/dt, and we are asked to find the time rate of change of the other variable, e.g. dy/dt. Solving this problem generally involves the following steps: 1. Determine the relationship, if not given, between the two variables and write as either an explicit or implicit equation. 2. Take the time derivative of the entire equation from step 1. 3. Solve the equation from step 2 for the unknown rate. 4. Evaluate for the unknown rate using the values given in the problem.

Linear Approximation
The linear approximation, $L(x)$, to a differentiable function, $f(x)$, near the point $x = a$ is given by the tangent line to $f(x)$ at the given point $$L(x) = f'(a)(x - a) + f(a)$$
Linear Approximation of Δf
If $f(x)$ is differentiable at $x = a$ and Δx is small, then $$\Delta f \cong \widetilde{\Delta f} = f'(a)\Delta x$$ Where, $$\Delta f = f(a + \Delta x) - f(a)$$ and $$\widetilde{\Delta f} = L(a + \Delta x) - L(a)$$
Approximating Values
Using the value $\frac{1}{\sqrt{17}}$ as an example, the procedure is described as follows. 1. Define a function that matches the value to be evaluated, e.g. $\frac{1}{\sqrt{x}}$ for this example. 2. Choose the point for linearization close to the value to be evaluated, but that can be easily computed without a calculator, e.g. use $a = 16$ for the function $\frac{1}{\sqrt{x}}$ above. 3. Define the value to be evaluated as $X = a + \Delta x$, e.g. in the example above $\Delta x = 1$ 4. Write the linear approximation and evaluate as follows: $$L(X) = f'(a)(X - a) + f(a)$$ $$L(a + \Delta x) = f'(a)\Delta x + f(a)$$ Using the above example: $$L(17) = f'(16)(1) + f(16)$$ $$L(17) = \left(-\frac{1}{2\left(\sqrt{x}\right)^3}\right)\Bigg\|_{x=16}(1) + \frac{1}{\sqrt{16}}$$ $$L(17) = \left(-\frac{1}{128}\right) + \frac{32}{128}$$ $$L(17) = \frac{31}{128} \cong \frac{1}{\sqrt{17}}$$

L'Hôpital's Rule for Quotient Indeterminate Forms

If $\lim_{x \to a}\{f(x)\} = \lim_{x \to a}\{g(x)\} = 0$ or $\pm\infty$, and the following hold:

1. $f(x)$ and $g(x)$ are both differentiable functions in an open interval containing a, except possibly at $x = a$.
2. $g'(x) \neq 0$, except possibly at $x = a$.

Then,

$$\lim_{x \to a}\left\{\frac{f(x)}{g(x)}\right\} = \lim_{x \to a}\left\{\frac{f'(x)}{g'(x)}\right\}$$

So long as the limit exists or is $\pm\infty$.

Note 1: The goal in differentiating the numerator and denominator is to make the quotient simpler to evaluate.
Note 2: This procedure can be applied repeatedly until the limit is able to be evaluated, assuming it exists.
Note 3: The rule is also valid for one-sided limits.
Note 4: The rule also applies for limits as $x \to \infty$ or $x \to -\infty$.

L'Hôpital's Rule for Other Indeterminate Forms

- Indeterminate forms of the type, $0 \cdot \infty$ or $\infty - \infty$, may also be evaluate using L'Hôpital's Rule once that are algebraically rearranged to be in the quotient form.

- Indeterminate forms of the type, $1^{\infty}, 0^0, or\ \infty^0$, arising from functions of the form $f(x)^{g(x)}$ may also be evaluated with L'Hôpital's Rule according to

$$\lim_{x \to a}\left\{f(x)^{g(x)}\right\} = e^{\lim_{x \to a}\left\{\left(\frac{ln(f(x))}{1/g(x)}\right)\right\}} = e^{L}$$

Where, L, if exists is equal to $\lim_{x \to a}\left\{\left(\frac{ln(f(x))}{1/g(x)}\right)\right\}$

L'Hôpital's Rule to Compare Growth of Functions

If $f(x)$ and $g(x)$ are both differentiable functions, we can say that $f(x)$ grows faster that $g(x)$ if

$$\lim_{x \to \infty}\left\{\frac{f(x)}{g(x)}\right\} = \infty \qquad \text{Or, equivalently} \qquad \lim_{x \to \infty}\left\{\frac{g(x)}{f(x)}\right\} = 0$$

Absolute Extreme Values Definition
Consider the function, $f(x)$, over an interval, I, and let $a \in I$. We say that $f(a)$ is the: • **Absolute minimum** of f on I if $f(a) \leq f(x)$ for all $x \in I$. • **Absolute maximum** of f on I if $f(a) \geq f(x)$ for all $x \in I$.
Absolute Extreme Value Theorem
If a function, $f(x)$, is continuous over a closed interval, $[a, b]$, then $f(x)$ has both a minimum and maximum on $[a, b]$.
Local Extreme Values Definition
Consider the function, $f(x)$. We say that $f(a)$ is a: • **Local minimum** that occurs at $x = a$ if $f(a)$ is the minimum value of f on some small interval containing a. • **Local maximum** that occurs at $x = a$ if $f(a)$ is the maximum value of f on some small interval containing a.
Critical Points Definition
A number c in the domain of $f(x)$ is called a **critical point** if either of the following are true: • $f'(c) = 0$ • $f'(c)$ does not exist.
Fermat's Theorem of Local Extrema
If $f(c)$ is a local minimum or maximum, then c is a critical point of f. Note: This theorem does not claim that all critical points are local extreme values, but rather that all local extreme values are critical points.
Finding Absolute Extreme Values
To find the absolute extreme values of a function, $f(x)$, over a closed interval, $[a, b]$ we: 1. Find critical points, $x = \{c_1, c_2, \dots c_N\}$, of $f(x)$ in $[a, b]$. 2. Evaluate $f(x)$ at the critical points, $\{f(c_1), f(c_2), \dots f(c_N)\}$, and at the endpoints, $\{f(a), f(b)\}$. 3. The absolute minimum and maximum values are the smallest and largest among the evaluated values from step 2.

Increasing and Decreasing Function Behavior

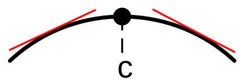
Consider a differentiable function, f, over an open interval, (a, b).

- If $f'(x) > 0$ for $x \in (a, b)$, then f is increasing on (a, b).
- If $f'(x) < 0$ for $x \in (a, b)$, then f is decreasing on (a, b).

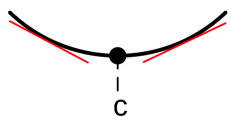
First Derivative Test for Critical Points

Let c be a critical point of the function, $f(x)$. Then:

- If $f'(x)$ changes from $+$ to – at c, $\Longrightarrow$ $f(c)$ is a local maximum.



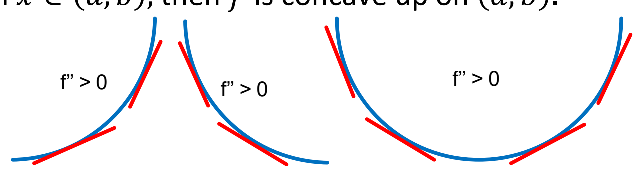
- If $f'(x)$ changes from – to $+$at c, $\Longrightarrow$ $f(c)$ is a local minimum.



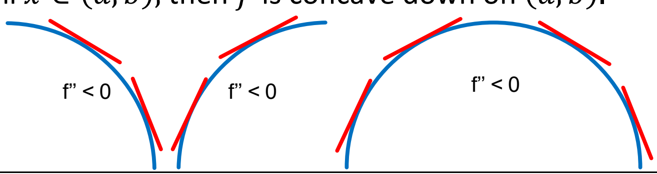
Concavity of a Function

Consider a differentiable function, f, over an open interval, (a, b).

- If $f''(x) > 0$ for all $x \in (a, b)$, then f is concave up on (a, b).



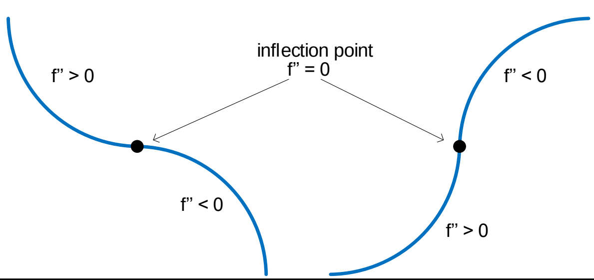
- If $f''(x) < 0$ for all $x \in (a, b)$, then f is concave down on (a, b).



Test for Inflection Points

A number c in the domain of $f(x)$ is called an **inflection point** if either of the following are true:

- $f''(c) = 0$ and $f''(x)$ changes sign at $x = c$
- $f''(c)$ does not exist and $f''(x)$ changes sign at $x = c$



Second Derivative Test for Critical Points

Let c be a critical point of the function, $f(x)$. Then:

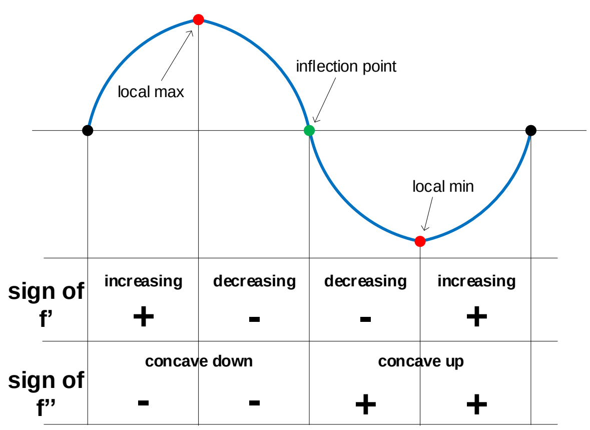
- If $f''(c) < 0 \implies f(c)$ is a local maximum.
- If $f''(c) > 0 \implies f(c)$ is a local minimum.
- If $f''(c) = 0 \implies$ The test is inconclusive.
    - $f(c)$ may be a local maximum, a local minimum, an inflection point, or none of the above.
    - Use the first derivative test and/or the test for inflection points.

Graph Sketching Procedure

To obtain the sketch of a function, $f(x)$, can be broken into the following steps.

1. Determine the Domain of $f(x)$.
2. Compute $f'(x)$ and $f''(x)$ and identify all potential transition points, i.e. points where either of these functions is zero or undefined.
3. Divide the x-axis into intervals based on the transition points in step 2.
4. Choose a "test point" in each interval and evaluate $f'(x)$ and $f''(x)$ to determine the sign.
5. Determine any local minimums, local maximums, inflection points, and the asymptotic behavior of $f(x)$.
6. Compute $f(x)$ at all local minimums, local maximums, and inflection points.
7. Draw arcs of the appropriate shape based on the sign combination found in step 4 in each interval and "connect" at the computed values from step 6, noting the asymptotic behavior.

The basic function shapes, determined by the sign of $f'(x)$ and $f''(x)$, are shown below.



Applied Optimization Problem Solving

We can identify 5 main steps to help with solving applied optimization problems.

1. Understand what the problem is asking, often by drawing a diagram.
2. Write an expression for the objective function, i.e. the function that requires optimization.
3. If the objective function is not of a single variable, look for additional function(s) that relates the variables, i.e. constraint function(s).
4. Rewrite the objective function as a function of a single variable if required.
5. Identify the interval of optimization if required and optimize the objective function.

The second major branch of calculus is integration, which we focused on for the next set of lessons.

Area Approximation Using Right Endpoints
$$R_N = \Delta x \sum_{j=0}^{N-1} f\left(x_{j+1}\right)$$
Area Approximation Using Left Endpoints
$$L_N = \Delta x \sum_{j=0}^{N-1} f\left(x_j\right)$$
Area Approximation Using Midpoints
$$M_N = \Delta x \sum_{j=0}^{N-1} \frac{f\left(x_{j+1}\right) + f\left(x_j\right)}{2} = \frac{1}{2}\left(\Delta x \sum_{j=0}^{N-1} f\left(x_{j+1}\right) + \Delta x \sum_{j=0}^{N-1} f\left(x_j\right)\right)$$ $$M_N = \left(\frac{R_N + L_N}{2}\right)$$
Where the following definitions apply to all three approximations. If the interval over which the area is computed is defined as $[a, b]$, the subinterval width is $$\Delta x = \frac{b-a}{N}$$ and the evaluation points are $$x_i = a + i\Delta x$$
Summation Formulas
$$\sum_{j=0}^{N-1} j = 0 + 1 + 2 \cdots (N-1) = \frac{N(N-1)}{2} = \frac{N^2}{2} - \frac{N}{2}$$ $$\sum_{j=0}^{N-1} j^2 = 0 + 1^2 + 2^2 \cdots (N-1)^2 = \frac{N(N-1)(2N-1)}{6} = \frac{N^3}{3} - \frac{N^2}{2} + \frac{N}{6}$$
Area Under the Graph
If f is continuous on $[a, b]$, then the endpoint and midpoint approximations approach the same value, A, in the limit as $N \to \infty$. In other words: $$\lim_{N\to\infty} R_N = \lim_{N\to\infty} L_N = \lim_{N\to\infty} M_N = A$$ If $f(x) > 0$ on $[a, b]$, then A represents the area under the graph of $f(x)$ on $[a, b]$ Note: If $f(x)$ takes on negative values the above theorem still holds, but we interpret A as the *signed area,* which is discussed in the next section.

Riemann Sum

The Riemann Sum is a general method for computing the *signed area* of a graph over a given interval, and is defined as

$$R(f, P, C) = \sum_{i=1}^{N} f(c_i)\Delta x_i$$

Where,

- f represents the function whose graph we are computing the area for.
- P defines the partition of the interval, $[a, b]$, for which the area is computed over. The points, x_i, divide the interval into N subintervals.
$$P: a = x_0 < x_1 < x_2 \cdots < x_N = b$$
- $C = \{c_1, c_2, \cdots c_N\}$ are the sample points within each subinterval where the function is evaluated.
- Δx_i is the width of each subinterval.
$$\Delta x_i = x_i - x_{i-1}$$

Definite Integral Definition

The definite integral of f over $[a, b]$, denoted by the integral sign, is the limit of the Riemann Sum.

$$\int_a^b f(x)dx = \lim_{\|P\| \to 0} \{R(f, P, C)\} = \lim_{\|P\| \to 0} \left\{\sum_{i=1}^{N} f(c_i)\Delta x_i\right\}$$

When this limit exists, we say f is integrable over $[a, b]$.

$\|P\|$ is defined as the largest of the lengths, Δx_i.

Integrable Theorem

If f is continuous on $[a, b]$, or if f is continuous with at most finitely many jump discontinuities, then f is *integrable* on $[a, b]$.

Definite Integral Formulas

$$\int_a^b C\,dx = C(b - a) \qquad \int_0^b x\,dx = \frac{1}{2}b^2 \qquad \int_0^b x^2\,dx = \frac{1}{3}b^3$$

Definite Integral Properties
The Zero Rule: $$\int_a^a f(x)dx = 0$$
Order of Integration: $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$
Linearity Properties: **Sum and Difference Property** $$\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$ **Constant Multiplier Property** $$\int_a^b Cf(x)\,dx = C\int_a^b f(x)\,dx$$
Additivity of Adjacent Regions: $$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

Antiderivative Definition
A function, F, is **an** antiderivative of f on an open interval (a, b) if: $F'(x) = f(x)$ for all x in (a, b).
General Antiderivative Theorem
If $F(x)$, is an antiderivative of $f(x)$ on (a, b), then: Every other antiderivative on (a, b) is of the form $F(x) + C$. Where C is any constant.
Indefinite Integrable Notation
$\int f(x)dx = F(x) + C$ means that $F'(x) = f(x)$ We say that $F(x) + C$ is the **indefinite integral,** (the general antiderivative), of $f(x)$. Note: The function, $f(x)$, appearing in the integral is called the *integrand*, and dx is a differential.
Power Rule for Integration
$\int x^n dx = \left(\frac{1}{n+1}\right) x^{n+1} + C$, For $n \neq -1$
Integral Formula for x^{-1}
$\int \frac{1}{x} dx = ln\|x\| + C$, For $\{x: x \neq 0\}$
Integral Formula for e^x
$\int e^x dx = e^x + C$
Basic Trigonometric Integrals
$\int sin(x)\, dx = -cos(x) + C$ $\int cos(x)\, dx = sin(x) + C$ $\int sec^2(x)\, dx = tan(x) + C$ $\int csc^2(x)\, dx = -cot(x) + C$ $\int csc(x)\, cot(x)\, dx = -csc(x) + C$ $\int sec(x)\, tan(x)\, dx = sec(x) + C$
Sine and Cosine Integral Formulas
$\int sin(kx)\, dx = -\frac{1}{k} cos(kx) + C$ $\int cos(kx)\, dx = \frac{1}{k} sin(kx) + C$
Integral Formula for e^{kx}
$\int e^{kx} dx = \frac{1}{k} e^x + C$

The Fundamental Theorems of Calculus

The fundamental theorems of calculus reveal the deep connection between the two main branches of calculus, i.e. differentiation and integration. It is one of the most important theorems in mathematics. The theorem is split into two parts. Part 1 allows for the computation of definite integrals without having to use the Riemann sum. Part 2 formally defines the inverse relationship between integration and differentiation.

The Fundamental Theorem of Calculus Part 1

Assume that f is continuous on $[a, b]$. If F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Note: We generally use the notation where $F(b) - F(a)$ is denoted as: $F(x)|_a^b$

The Fundamental Theorem of Calculus Part 2

Assume that f is continuous on an open interval, I, and let a be a point in I. The area function

$$A(x) = \int_a^x f(t)dt$$

is an antiderivative of f on I. In other words

$$\frac{d}{dx}\big(A(x)\big) = f(x)$$

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

Integration and Differentiation Inverse Relationship

If we integrate first and then differentiate, we get the original function back.

$$f(x) \underset{\longrightarrow}{Integrate} \int_a^x f(t)dt \underset{\longrightarrow}{Differntiate} \frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

If we differentiate first and then integrate, we get the original function back, BUT only up to a constant $f(a)$.

$$f(x) \underset{\longrightarrow}{Differntiate} f'(x) \underset{\longrightarrow}{Integrate} \int_a^x f'(t)dt = f(x) - \boldsymbol{f(a)}$$

Generalized Derivative of an Integral

$$\frac{d}{dx}\left(\int_{l(x)}^{u(x)} f(t)dt\right) = f\big(u(x)\big)u'(x) - f\big(l(x)\big)l'(x)$$

Change of Variables Formula, (Substitution Method of Integration)

$$\int \underbrace{f\big(g(x)\big)}_{f(u)} \underbrace{g'(x)dx}_{du} = \int f(u)du$$

The substitution method of integration is a technique that may sometimes be used to evaluate integrals that we otherwise cannot integrate. Unlike the various differentiation rules, it is not always straightforward, (or even possible), to apply the substitution method for integration. In general, we need to

- Identify $g(x)$ contained within the integrand such that when we let $u = g(x)$
    - There is another part of the integrand that is some scalar version of $g'(x) = \frac{du}{dx}$.
    - After substituting u and du the new integrand $f(u)$ is simpler to evaluate.

Substitution Method for Definite Integrals

The same change of variables formula is used for definite integrals. The limits if integration need to be handled appropriately. Two different methods can be employed.

1. Evaluate the u integral, resubstitute x in the solution, then apply the x limits.

2. Evaluate the u integral, replace the limits using $u(x)$, then apply the u limits.

$$\int_{x=a}^{x=b} f\big(g(x)\big)g'(x)dx = \int_{u=x(a)}^{u=x(b)} f(u)du$$

The final set of lessons covered applications of integrals.

Net Change as the Integral of the Rate of Change
The net change of $f(x)$ over an interval, $[x_1, x_2]$, is given by the following definite integral. $$\int_{x_1}^{x_2} f'(x)dx = f(x_2) - f(x_1)$$ Note: Since $\frac{d}{dx}\left(f(x)\right) = f'(x)$ the above results from FTC I.
The Integral of Velocity
If an object is in linear motion, i.e. one-dimensional motion, with a velocity, $v(t)$, then The **displacement**, Δx , i.e. , net change in position, during $[t_1, t_2]$ is given as: $$\Delta x = \int_{t_i}^{t_f} v(t)dt$$ The **distance traveled**, D, during $[t_1, t_2]$ is given as: $$D = \int_{t_i}^{t_f} \lvert v(t) \rvert dt$$
Production Costs
The cost, C, to produce x units of a certain product is represented as $C(x)$. The rate of change of cost with respect to the number of units produced is referred to as the *marginal cost* and is given $C'(x)$. The net change in cost, ΔC, to go from producing x_a units to x_b units is given as: $$\Delta C = \int_{x_a}^{x_b} C'(x)dx$$

Area Between Curves

Given two curves in the x-y plane, the area between these curves, either over a given interval or where they naturally intersect, may be of interest. The region formed may be "simple", as defined below, in which case the area can be computed with a single integral.

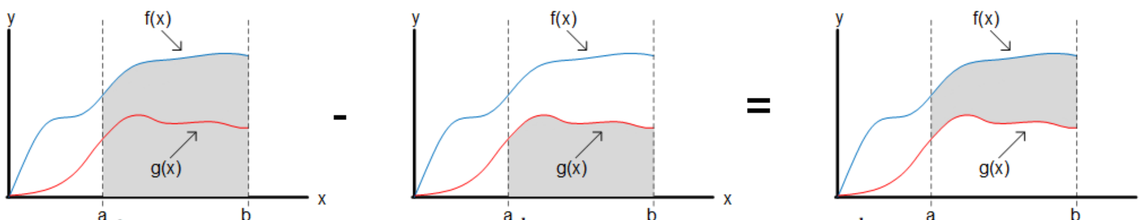
Non-simple regions can be subdivided into multiple "simple" regions and integrated separately to find the total area.

In either case, it's always best to sketch or graph the functions to visualize the region in order to properly set up the integral(s).

Area Between Curves that form a Vertically Simple Region

If $f(x) \geq g(x)$ for all x in the interval $[a, b]$, i.e. they form a vertically simple region, then the area between the two curves is given as

$$\int_a^b (f(x) - g(x))dx$$

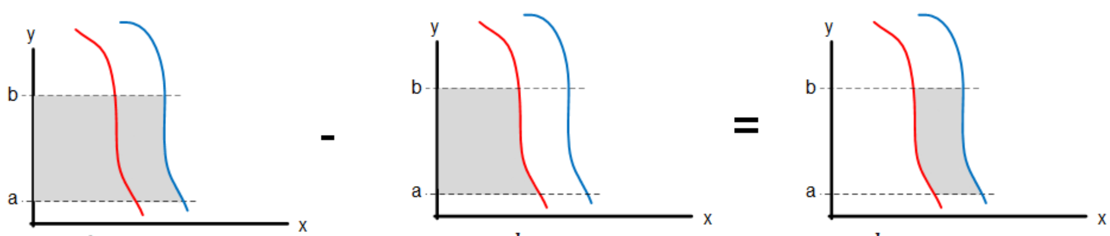


$$\int_a^b f(x)dx \quad - \quad \int_a^b g(x)dx \quad = \quad \int_a^b (f(x) - g(x))dx$$

Area Between Curves that form a Horizontally Simple Region

If $f(y) \geq g(y)$ for all y in the interval $[a, b]$, i.e. they form a horizontally simple region, then the area between the two curves is given as

$$\int_a^b (f(y) - g(y))dy$$



$$\int_a^b f(y)dy \quad - \quad \int_a^b g(y)dy \quad = \quad \int_a^b (f(y) - g(y))dx$$

Setting up Integrals

- In general terms the integral symbol, $\int$, is used to represent a summation symbol, $\sum$, when we are summing infinitely many infinitesimal objects, e.g. $dx's, dA's$.

- The limits of integration represent the interval over which these infinitesimal objects exist, e.g. $0 \le x \le 3$.

In some application where integration is required the following high-level procedure can provide guidance for setting up the integral.

1. Identify the infinitesimal "segment" from which the "total amount" can be computed from. e.g. If the "object" is area, the infinitesimal segment is a rectangle with area dA.

2. Identify how this segment changes as we compute the total amount. e.g. For the area under a curve the rectangle height, $f(x)$, changes as a function of x.

3. Write an expression for the infinitesimal segment in terms of the variable you will change to compute the total amount. e.g. For the area under a curve the infinitesimal rectangle has an area of $dA = f(x)dx$.

4. Integrate the relationship from step 3 over an interval, e.g. $[a, b]$, to find the total amount, e.g. $A = \int_a^b f(x)dx$

Volume as the Integral of the Cross-Sectional Area

Let $A(y)$ be the area of a x-z plane cross section at height y of a solid body that extends from $y = a$ to $y = b$. The volume of the object, V, can then be computed as

$$V = \int_a^b A(y)dy$$

Total Mass

The total mass of an object with a linear mass density, $\rho(x)$, in $mass/length$, and a length of L is given by

$$M = \int_0^L \rho(x)dx$$

Average Value

The average value of a continuous function, $f(x)$, over $[a, b]$ is

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$$

Volumes of Revolution: Disk Method
Rotating a region in the x-y plane around an axis results in a solid with cross sections that are either disks or washers. The general procedure to find the volume of the solid can be summarized as: 1. Find the radius of the disk or the outer and inner radius of the washer in terms of the functions that specify the region in the x-y plane. 2. Give the disk or washer an infinitesimal width, dx or dy, and write an expression for the infinitesimal volume. 3. Integrate over the appropriate interval to find the total volume of the solid. There are many variations for these types of solids and some general expressions were shown, however, it is always best to follow the guideline above and derive the expressions for each specific case. With that we show the expressions for some of the specific cases we reviewed.
Volume of Revolution: Single Function Rotated About x-axis (Basic Disk Method)
Consider a continuous function $f(x) \geq 0$ on $[a, b]$. Rotating this function about the x-axis creates a solid object with a circular cross section and a radius, $r = f(x)$. The volume of such objects is found by evaluating the following definite integral. $$V = \pi \int_a^b f^2(x) dx$$
Volume of Revolution: Region Between Two Curves: (Basic Washer Method)
Consider two continuous functions, $f(x)$ and $g(x)$, where $f(x) > g(x) \geq 0$ on $[a, b]$. Rotating these functions about the x-axis while considering only the region between the functions creates a solid object with a cross section in the shape of a circular washer. The outer radius of the washer is $f(x)$ and the inner radius is $g(x)$. The volume of such objects is found by evaluating the following definite integral. $$V = \pi \int_a^b \left(f^2(x) - g^2(x)\right) dx$$
Volume of Revolution: Rotation Around a Horizontal Line
Consider two continuous functions, $f(x)$ and $g(x)$, where $f(x) > g(x) \geq 0$ on $[a, b]$. Rotating these functions about the horizontal line $y = C$, where C is a positive constant such that $C < g(x) < f(x)$ on $[a, b]$ creates a solid object with a cross section in the shape of a circular washer. The volume of such objects is found by evaluating the following definite integral. $$V = \pi \int_a^b ((f(x) - C)^2 - (g(x) - C)^2) dx$$

Volumes of Revolution: Shell Method
Rotating a region in the x-y plane around an axis results in a solid which can be divided into infinitely many cylindrical shells. The general procedure to find the volume of the solid can be summarized as: 4. Find the radius and height of the shell in terms of the functions that specify the region in the x-y plane. 5. Give the shell an infinitesimal width, dx or dy, and write an expression for the infinitesimal volume. $dV = 2\pi \cdot (radius)(height)$ 6. Integrate over the appropriate interval to find the total volume of the solid. There are many variations for these types of solids and some general expressions were shown, however, it is always best to follow the guideline above and derive the expressions for each specific case. With that we show the expressions for some of the specific cases we reviewed.
Volume of Revolution: Single Function Rotated About y-axis (Basic Shell Method)
Consider a continuous function $f(x) \geq 0$ on $[a, b]$. Rotating this function about the y-axis creates a solid object which can be divided into infinitely many cylindrical shells with radius x, height $f(x)$, and width dx. The volume of such objects may be found by evaluating the following definite integral. $V = \int_a^b (2\pi \cdot radius)(height)dx = 2\pi \int_a^b xf(x)dx$
Volume of Revolution: Region Between Two Curves: (Shell Method)
Consider two continuous functions, $f(x)$ and $g(x)$, where $f(x) > g(x)$ on $[a, b]$. Rotating these functions about the y-axis while considering only the region between the functions creates a solid object which can be divided into infinitely many cylindrical shells with radius x, height $f(x) - g(x)$, and width dx. The volume of such objects is found by evaluating the following definite integral. $V = 2\pi \int_a^b x\big(f(x) - g(x)\big)dx$
Volume of Revolution: Rotation Around a Vertical Line
Consider two continuous functions, $f(x)$ and $g(x)$, where $f(x) > g(x)$ on $[a, b]$. Rotating these functions about the horizontal line $x = C$, where $C \leq a$, while considering only the region between the functions creates a solid object which can be divided into infinitely many cylindrical shells with radius $x - C$, height $f(x) - g(x)$, and width dx. The volume of such objects is found by evaluating the following definite integral. $V = 2\pi \int_a^b (x - C)\big(f(x) - g(x)\big)dx$

Disk vs. Shell Method General Guidelines
Observations: ***Disk Method:*** For horizontal axis rotation the disks are oriented such that the radius is directly related to the function(s) given. For vertical axis rotation the disks are oriented such that the radius is related to the inverse function(s) given. ***Shell Method:*** For horizontal axis rotation the shells are oriented such that the height is related to the inverse function(s) given. For vertical axis rotation the shells are oriented such that the height is directly related to the function(s) given. **General Guideline** Since finding the inverse function is not always simple, these observations suggest that we should choose the method which **does not** require the inverse to compute. To avoid inverse functions, we should choose the disk method when the rotation is about a horizontal axis and the shell method when the rotation is about a vertical axis. Note: If the function(s) is easily inverted than either method should suffice.

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