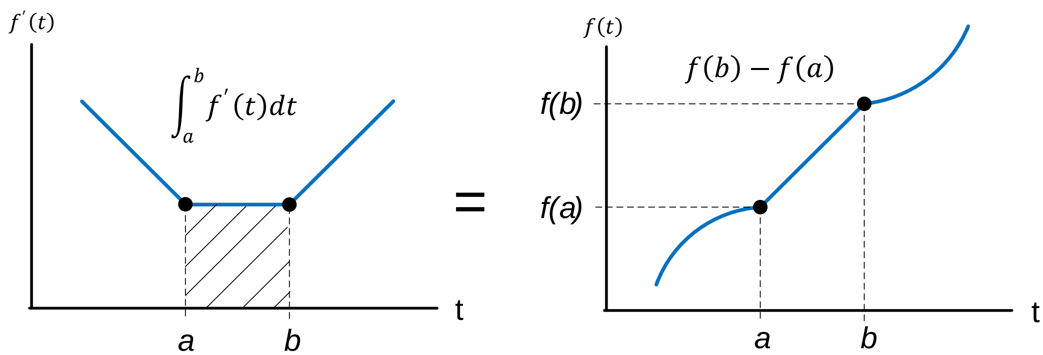
Fundamental Theorems of Vector Calculus – Green's Theorem

In this final series of lessons, we introduce what are known as *The Fundamental Theorems of Vector Calculus*. The four theorems commonly associated with this group are: 1. Gradient Theorem, 2. Green's Theorem, 3. Stokes Theorem, and 4. Divergence Theorem. The theorems can be seen as generalizations of The Fundamental Theorem of Single Variable Calculus.

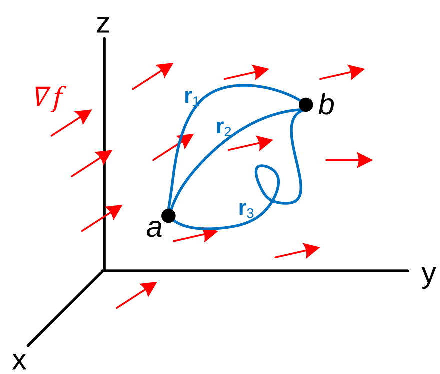
$$\int_a^b f'(t)dt = f(b) - f(a)$$

In a general sense this theorem *relates the integral of some type of derivative of a certain object to the values of that function along the boundary of that object*.



The first theorem, *the gradient theorem,* most closely resembles the theorem above. We have seen this theorem when discussing conservative vector fields and path independence. The path independence theorem relates to the fact that the path integral of a conservative vector field along a path depends only on the endpoints of the path. Additionally, we learned that the gradient vector fields is a conservative vector field. The gradient theorem, in a very similar way to the fundamental theorem of calculus, *relates the integral of some type of derivative, i.e. the gradient, of a certain object to the values of that function along the boundary of that object*.

The Gradient Theorem
$\int_C \nabla f \cdot d\mathbf{s} = f(b) - f(a)$



In this section we introduce Green's theorem, which we will show *relates the double integral of some type of derivative of a certain object to the values of that function along the boundary of that object.*

Green's Theorem Introduction

Similar to the Fundamental Theorem of Calculus and the Gradient Theorem, Green's Theorem can also be said to *relate the integral of some type of derivative of a certain object to the values of that function along the boundary of that object.* Specifically, Green's theorem relates the double integral of a two dimensional curl of a vector field over a given domain to a vector line integral around a curve that encloses the given domain.

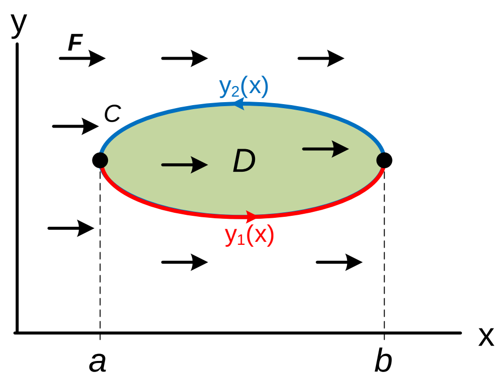
$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D curl_z(\boldsymbol{F}) dA$$

Where, C is a simple closed curve that bounds D.

We'll start by deriving the above relationship. Following that we will provide a slightly more intuitive understanding before doing examples to illustrate how the theorem is used.

Derivation

Consider a closed curve that is oriented counterclockwise and can be described as the union of two functions, $y_1(x)$ and $y_2(x)$, that is placed in a vector field that has an x-component only, $\boldsymbol{F}(x, y) = \langle F_1(x, y), 0\rangle$.



Using the component-wise form for $d\mathbf{r}$**,** i.e. $d\mathbf{r} = \langle dx, dy\rangle$, the restriction on the vector field, i.e. $\boldsymbol{F}_x(x, y) = \langle F_1(x, y), 0\rangle$, and the fact that we can traverse the entire curve, C, by first going from a to b along $y_1(x)$ and then from b to a along $y_2(x)$, we can write the following.

$$\oint_C \boldsymbol{F}_x \cdot d\boldsymbol{r} = \oint_C \langle F_1(x, y), 0\rangle \cdot \langle dx, dy\rangle$$

$$= \oint_C F_1(x, y) dx$$

$$= \int_a^b F_1\big(x, y_1(x)\big) dx + \int_b^a F_1\big(x, y_2(x)\big) dx$$

Switching the order of integration on the second integral and pulling out a negative sign we have

$$= \int_a^b F_1(x, y_1(x))dx - \int_a^b F_1(x, y_2(x))dx$$

$$= -\int_a^b \Big(F_1(x, y_2(x)) - F_1(x, y_1(x))\Big)\, dx$$

Next, we note the integrand can be rewritten using the Fundamental Theorem of Calculus, $\int_a^b \left(\frac{df(t)}{dt}\right) dt \;=\; f(b) - f(a)$. Therefore, $F_1(x, y_2(x)) - F_1(x, y_1(x)) = \int_{y_1(x)}^{y_2(x)} \left(\frac{\partial F_1(x,y)}{\partial y}\right) dy$

Applying this we have

$$= -\int_a^b \Big(F_1(x, y_2(x)) - F_1(x, y_1(x))\Big)\, dx$$

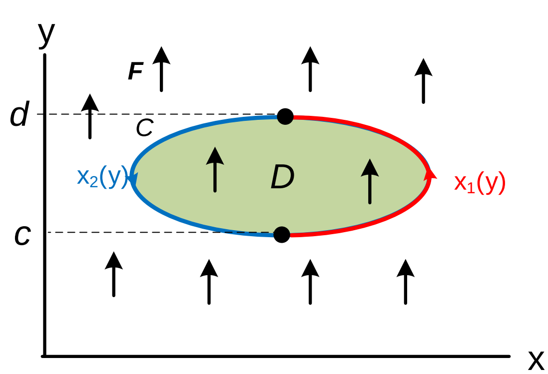
$$= -\int_a^b \left(\int_{y_1(x)}^{y_2(x)} \left(\frac{\partial F_1(x,y)}{\partial y}\right) dy\right) dx$$

$$\oint_C \boldsymbol{F}_x \cdot d\boldsymbol{r} = -\iint_D \frac{\partial F_1(x,y)}{\partial y}\, dA$$

We have completed our derivation of Green's theorem in that sense that we have related the double integral over a region, D, of the derivative of a vector field, $\frac{\partial F_1(x,y)}{\partial y}$, to the line integral of the vector field along the boundary of the region, C, i.e. $\oint_C \boldsymbol{F}_x \cdot d\boldsymbol{r}$. However, the formula does not match the given form of Green's theorem from above.

One issue is that we used a special vector field that only had an x-component. In an attempt to deal with this issue, we will repeat the exercise, this time using a vector field with only a y-component, and then see if we can combine the results in the end.

We start in a similar fashion, i.e. by creating a y-component only vector field, $\boldsymbol{F}_y(x, y) = \langle 0, F_2(x, y)\rangle$, and splitting the curve into $x_1(y)$ and $x_2(y)$ as shown below.



Using the same procedure from above, we have

$$\oint_C \boldsymbol{F}_y \cdot d\boldsymbol{r} = \oint_C \langle 0, F_2(x,y)\rangle \cdot \langle dx, dy\rangle$$

$$= \oint_C F_2(x,y)dy$$

$$= \int_c^d F_2(x_1(y), y)dy - \int_c^d F_2(x_2(y), y)dy$$

$$= \int_c^d \big(F_2(x_1(y), y) - F_2(x_2(y), y)\big)dy$$

$$= \int_c^d \left(\int_{x_2(y)}^{x_1(y)} \frac{\partial F_2(x,y)}{\partial x} dx\right) dy$$

$$= \int_c^d \int_{x_2(y)}^{x_1(y)} \frac{\partial F_2(x,y)}{\partial x} dA$$

$$\oint_C \boldsymbol{F}_y \cdot d\boldsymbol{r} = \iint_D \frac{\partial F_2(x,y)}{\partial x} dA$$

To complete the derivation, we use the results from above to compute the line integral of the combined vector field, $\boldsymbol{F} = \left(\boldsymbol{F}_x + \boldsymbol{F}_y\right)$.

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_C \left(\boldsymbol{F}_x + \boldsymbol{F}_y\right) \cdot d\boldsymbol{r}$$

$$= \oint_C \boldsymbol{F}_x \cdot d\boldsymbol{r} + \oint_C \boldsymbol{F}_y \cdot d\boldsymbol{r}$$

$$= -\iint_D \frac{\partial F_1(x,y)}{\partial y} dA + \iint_D \frac{\partial F_2(x,y)}{\partial x} dA$$

$$\boxed{\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D \left(\frac{\partial F_2(x,y)}{\partial x} - \frac{\partial F_1(x,y)}{\partial y}\right) dA}$$

The final step is to recognize the fact that we define $curl_z(\boldsymbol{F})$ as the z-component of the curl of a two dimensional vector field, $\boldsymbol{F}(x,y) = \langle F_1(x,y), F_2(x,y), 0\rangle$. We illustrate this below.

$$
\begin{aligned}
curl(\boldsymbol{F}) &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1(x,y) & F_2(x,y) & 0 \end{vmatrix} \\
&= \langle -\frac{\partial F_2(x,y)}{\partial z}, \frac{\partial F_1(x,y)}{\partial z}, \frac{\partial F_2(x,y)}{\partial x} - \frac{\partial F_1(x,y)}{\partial y} \rangle \\
&= \langle 0,0, \frac{\partial F_2(x,y)}{\partial x} - \frac{\partial F_1(x,y)}{\partial y} \rangle
\end{aligned}
$$

Therefore,

$$
curl_z(\boldsymbol{F}) = \left(\frac{\partial F_2(x,y)}{\partial x} - \frac{\partial F_1(x,y)}{\partial y}\right)
$$

Green's Theorem
Let D be a domain in R^2 whose boundary, C, is a simple closed curve, oriented counterclockwise. Then, $$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D \left(\frac{\partial F_2(x,y)}{\partial x} - \frac{\partial F_1(x,y)}{\partial y}\right) dA$$ Where, $\left(\frac{\partial F_2(x,y)}{\partial x} - \frac{\partial F_1(x,y)}{\partial y}\right) = curl_z(\boldsymbol{F})$

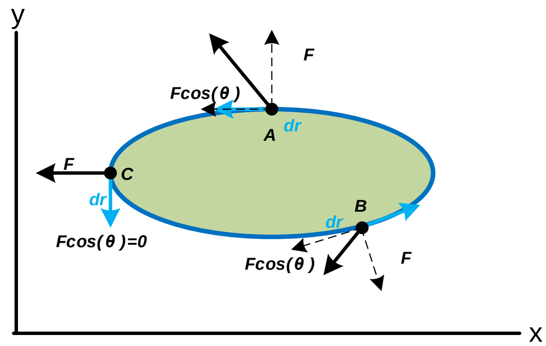
Green's Theorem Intuition

In this section we give a picture of Green's Theorem that is designed to supplement the more formal derivation above. The goal is to provide you with an intuitive understanding of the theorem. Green's Theorem is stated as follows.

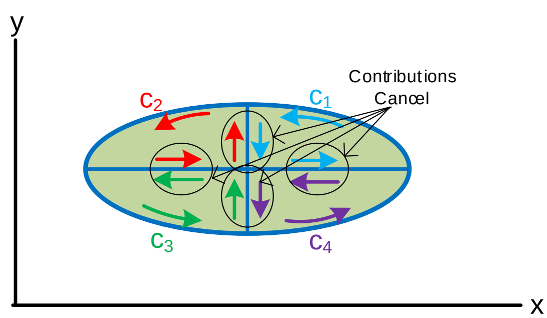
$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D curl_z(\boldsymbol{F})dA$$

For sake of discussion we assume the vector field, $\boldsymbol{F}$, represents the velocity of a fluid on a $2D$ plane. We start with a heuristic look at the meaning of the line integral on the left-hand side of Green's Theorem.

As the figure illustrates, the dot product in the integrand, $\boldsymbol{F} \cdot d\boldsymbol{r} = \|\boldsymbol{F}\| \, cos(\theta) \, \|d\boldsymbol{r}\|$, attempts to capture the portion of the fluid flow that is in a direction that is tangent to the curve. For example, at point A the dot product is positive value, at B it would be negative, and at C there is zero fluid flowing along curve. As we move along the curve, we continue to sum these quantities, resulting in a value that represents the amount of overall fluid flow that is in the direction of the curve.



Next, we imagine chopping the region into different sections as illustrated below with four regions.

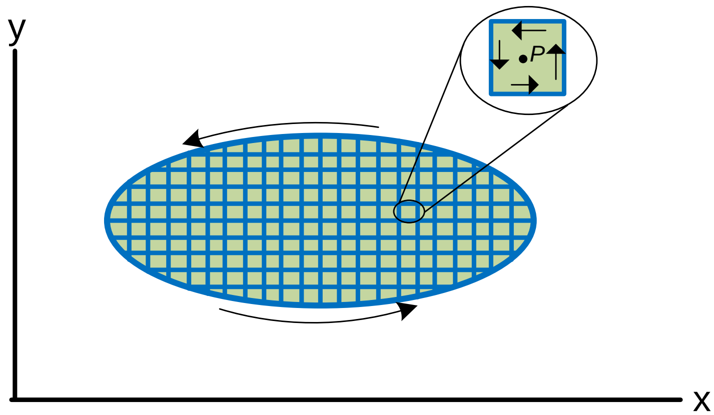


Notice that if we were to compute the line integral for each path, e.g. C_1, C_2, C_3, C_4, and sum the results, we would obtain the same result as if we computed the integral using the single path around the exterior boundary only. This is due to the fact that each portion of a path that lies within the region would cancel with a portion of the path from a different path.

$$\oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} + \oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} + \oint_{C_3} \boldsymbol{F} \cdot d\boldsymbol{r} + \oint_{C_4} \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

Breaking the larger region into many small regions we can write,

$$\sum_{k=1}^{N}\left(\oint_{C_k} \boldsymbol{F}\cdot d\boldsymbol{r}\right) = \oint_{C} \boldsymbol{F}\cdot d\boldsymbol{r}$$



Another way we can measure the fluid rotation of one of these small regions is by way of the curl. If we assume the region is small enough then the curl is constant and the integral on the right-hand side of Green's Theorem for this small region can be written as

$$\iint_{D_k} curl_z(\boldsymbol{F}(P))dA = curl_z(\boldsymbol{F}(P))\iint_{D_k} dA = curl_z(\boldsymbol{F}(P_k))dA_k$$

Which, according to Green's Theorem, is approximately equal to the line integral for the same small region.

$$\oint_{C_k} \boldsymbol{F}\cdot d\boldsymbol{r} \cong curl_z(\boldsymbol{F}(P_k))dA_k$$

Summing over all k regions we have

$$\sum_{k=1}^{N}\left(\oint_{C_k} \boldsymbol{F}\cdot d\boldsymbol{r}\right) \cong \sum_{k=1}^{N}\left(curl_z(\boldsymbol{F}(P_k))dA_k\right)$$

$$\oint_{C} \boldsymbol{F}\cdot d\boldsymbol{r} \cong \sum_{k=1}^{N}\left(curl_z(\boldsymbol{F}(P_k))dA_k\right)$$

We recognize the right-hand side as an approximation of a double integral of $curl_z(\boldsymbol{F}(P_k))$. With small enough regions we can make the approximation an equality for Green's Theorem.

$$\oint_{C} \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_{D} curl_z(\boldsymbol{F})dA$$

Finally, let's look at some examples to see Green's Theorem in action.

Example 1: Verify Green's Theorem for the region of a unit circle placed in the vector field

$$\boldsymbol{F} = \langle xy^2, x\rangle$$

Solution: The line integral around a unit circle is best computed using polar coordinates.

$$\boldsymbol{r}(\theta) = \langle cos(\theta), sin(\theta)\rangle \quad \rightarrow \quad \boldsymbol{r}'(\theta) = \langle -sin(\theta), cos(\theta)\rangle \quad \rightarrow \quad \boldsymbol{F}(\boldsymbol{r}(\theta)) = \langle cos(\theta)\, sin^2(\theta), cos(\theta)\rangle$$

$$\begin{aligned}
\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} &= \int_0^{2\pi} \boldsymbol{F}(\boldsymbol{r}(\theta))\cdot \boldsymbol{r}'(\theta)d\theta \\
&= \int_0^{2\pi} (-cos(\theta)\, sin^3(\theta) + cos^2(\theta))d\theta \\
&= -\int_0^0 (u^3)du + \frac{1}{2}\int_0^{2\pi} (1 + cos(2\theta))d\theta \\
&= -0 + \frac{1}{2}\left(\left(2\pi + \frac{1}{2}sin(4\pi)\right) - \left(0 + \frac{1}{2}sin(0)\right)\right) \\
&= \pi
\end{aligned}$$

Next, using Green's Theorem, we evaluate the following double integral.

$$\iint_D \left(\frac{\partial F_2(x,y)}{\partial x} - \frac{\partial F_1(x,y)}{\partial y}\right) dA$$

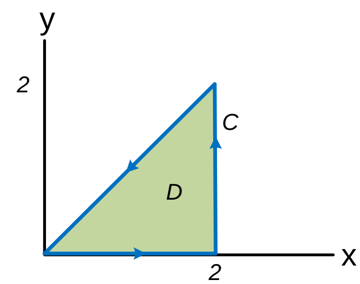
The partials are

$$\frac{\partial F_2(x,y)}{\partial x} = \frac{\partial}{\partial x}(x) = 1 \qquad\qquad \frac{\partial F_1(x,y)}{\partial y} = \frac{\partial}{\partial y}(\, xy^2) = 2yx$$

Using polar coordinates again we have

$$\begin{aligned}
\iint_D (1 - 2yx)dA = \iint_D dA - \int_0^{2\pi}\int_0^1 2r^2\, sin(\theta)\, cos(\theta)\, drd\theta \\
&= \pi - \frac{2}{3}\int_0^{2\pi} sin(\theta)\, cos(\theta)\, d\theta \\
&= \pi - \frac{2}{3}\int_0^0 udu \\
&= \pi - 0 \\
&= \pi
\end{aligned}$$

Example 2: Compute the work done by the vector force field, $\boldsymbol{F} = \langle sin(x)\,, x^2y^3 \rangle$, to move a particle counterclockwise around the triangular path shown below.



Solution: The work done by the force is the vector line integral around the closed path.

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

The triangular path would make this line integral rather time consuming. Using Green's Theorem, we have

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D curl_z(\boldsymbol{F})dA$$

And,

$$\begin{aligned} curl_z(\boldsymbol{F}) &= \frac{\partial F_2(x,y)}{\partial x} - \frac{\partial F_1(x,y)}{\partial y} \\ &= \frac{\partial}{\partial x}(x^2y^3) - \frac{\partial}{\partial y}(sin(x)) \\ &= 2xy^3 - 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \iint_D curl_z(\boldsymbol{F})dA &= \iint_D 2xy^3 dA \\ &= 2\left(\int_0^2 x\left(\int_0^x y^3 dy\right)dx\right) \\ &= \frac{1}{2}\left(\int_0^2 x^5 dx\right) \\ &= \frac{1}{2}\left(\frac{2^6}{6}\right) \\ &= \frac{16}{3} \end{aligned}$$

Note that if the $curl_z(\boldsymbol{F}) = 0$, the force would be conservative, and the work would be zero.

Area Via Green's Theorem

We can use the left-hand side of Green's theorem to find the area of a domain, D, in the xy-plane, enclosed by a simple curve, C.

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D curl_z(\boldsymbol{F}) dA$$

Since the area of D can be computed as $\iint_D dA$, we need to create a vector field with $curl_z(\boldsymbol{F}) = 1$.

$$curl_z(\boldsymbol{F}) = \frac{\partial F_2(x,y)}{\partial x} - \frac{\partial F_1(x,y)}{\partial y} = 1$$

We can do so in the following three ways:

1. Let $\boldsymbol{F} = \langle 0, x \rangle$, then

$$\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(0) = 1 - 0 = 1$$

2. Let $\boldsymbol{F} = \langle -y, 0 \rangle$, then

$$\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(-y) = 0 - (-1) = 1$$

3. Let $\boldsymbol{F} = \langle -y/2, x/2 \rangle$, then

$$\frac{\partial}{\partial x}(x/2) - \frac{\partial}{\partial y}(-y/2) = (1/2) - (-1/2) = 1$$

In each of these cases the left-hand side of Green's Theorem can be used to find the area enclosed by C. Using the following form of the line integral,

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_C \langle F_1, F_2 \rangle \cdot \langle dx, dy \rangle = \oint_C F_1 dx, + F_2 dy$$

We can write the three cases as follows:

$$Area\ Enclosed\ by\ C = \left(\oint_C xdy\right) = \left(\oint_C -ydx\right) = \left(\frac{1}{2}\oint_C xdy - ydx\right)$$

Example 3: Use Green's Theorem to find the area of the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Solution: The formula for the area of an ellipse is $A = \pi ab$. We will attempt to derive this formula using all three methods from above.

We start by parameterizing the ellipse using

$$x = a\,cos(\theta) \rightarrow dx = -a\,sin(\theta)\,d\theta \qquad\qquad y = b\,sin(\theta) \rightarrow\; dy = b\,cos(\theta)\,d\theta$$

1.

$$\begin{aligned} A &= \oint_C \; xdy \\ &= \int_0^{2\pi} (a\,cos(\theta))(b\,cos(\theta)\,d\theta) \\ &= \frac{ab}{2}\int_0^{2\pi} (1 + cos(2\theta))d\theta \\ &= \frac{ab}{2}(2\pi + 0) \\ &= \pi ab \end{aligned}$$

2.

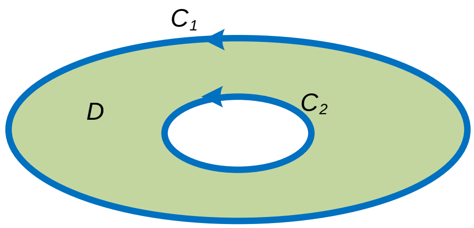
$$\begin{aligned} A &= \oint_C \; -ydx \\ &= -\int_0^{2\pi} (b\,sin(\theta))(-a\,sin(\theta)\,d\theta) \\ &= \frac{ab}{2}\int_0^{2\pi} (1 - cos(2\theta))d\theta \\ &= \frac{ab}{2}(2\pi + 0) \\ &= \pi ab \end{aligned}$$

3.

$$A = \frac{1}{2}\oint_C \; xdy - ydx = \frac{1}{2}\left(\oint_C \; xdy + \oint_C \; -ydx\right) = \frac{1}{2}(\pi ab + \pi ab) = \pi ab$$

More General Form of Green's Theorem

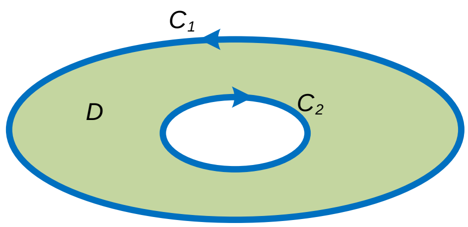
Up to this point we assumed that Green's Theorem applies to simple regions, i.e. no holes, only. The theorem can actually be applied to more general regions as long as we keep in mind the fact that the region to be considered always lies to the left of the curve according to its orientation.



The desired region is shown above as D. Traversing C_1 counterclockwise represents the entire region including the hole, since it lies to the left. Whereas, traversing C_2 counterclockwise represents the region occupied by the hole. Therefore, the region D can be represented as

$$D = C_1 - C_2$$

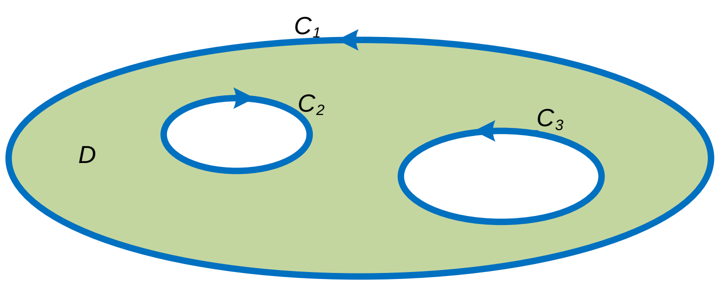
What if the inner curve, C_2, was traversed clockwise as shown below?



In this case C_2 represents the region outside of the whole. i.e. to the left as you traverse the path. The region is then represented as

$$D = C_1 + C_2$$

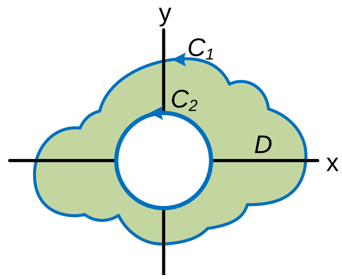
We show one more example to illustrate.



In this example the region, D, is represented as

$$D = C_1 + C_2 - C_3$$

Example 4: Consider the vector field $\boldsymbol{F} = \langle x - y,\ x + y^3 \rangle$ and the non-simple region, D, shown below. Compute the line integral with C_1, an unknown curve, as the boundary of the region. The curve, C_2, is a unit circle and it is known that the area of the region, D, is 8.



Solution: The region can be defined using the two curves as $D = C_1 - C_2$, Therefore,

$$\oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} - \oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D curl_z(\boldsymbol{F}) dA$$

$$\oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D curl_z(\boldsymbol{F}) dA + \oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}$$

The first integral can be evaluated as follows.

$$\iint_D curl_z(\boldsymbol{F}) dA = \iint_D \left(\frac{\partial F_2(x,y)}{\partial x} - \frac{\partial F_1(x,y)}{\partial y} \right) dA$$

$$= \iint_D \left(\frac{\partial}{\partial x}(\ x + y^3) - \frac{\partial}{\partial y}(x - y) \right) dA$$

$$= \iint_D (2) dA = 2A(D) = 2 \cdot 8 = 16$$

For the line integral we use polar coordinates

$$\oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_{C_2} \langle cos(\theta) - sin(\theta), cos(\theta) + sin^2(\theta) \rangle \cdot \langle - sin(\theta), cos(\theta) \rangle d\theta$$

$$= \int_0^{2\pi} (cos(\theta) sin(\theta) + sin^2(\theta) + cos^2(\theta) + cos(\theta) sin^2(\theta)) d\theta$$

$$= \int_0^{2\pi} cos(\theta) sin(\theta)\, d\theta + \int_0^{2\pi} 1 d\theta + \int_0^{2\pi} cos(\theta) sin^2(\theta)\, d\theta$$

$$= 0 + 2\pi + 0$$

$$= 2\pi$$

Finally, we have

$$\oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D curl_z(\boldsymbol{F}) dA + \oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} = (16 + 2\pi)$$

Green's Theorem using the Normal Vector to the Curve - Flux

The vector line integral from Green's Theorem uses the component of the vector field that is *tangential* to the path. This type of line integral was used to compute the work done by a force field when an object moves along the path. Recall, however, that in our lesson on vector line integrals we also introduced line integrals that used the component of the vector field that is *normal* to the path. In this case, the quantity computed was referred to as the flux across the curve, Φ. Referring back to that lesson we showed that the flux across a curve is given as

$$\Phi = \int_a^b \left(\boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{N}(t)\right) dt$$

Where, $\boldsymbol{N}(t) = \langle y'(t), -x'(t)\rangle$.

Substituting we can write the following

$$\oint_C \left(\boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{N}(t)\right) dt = \int_a^b (\langle F_1, F_2\rangle \cdot \langle y'(t), -x'(t)\rangle) dt$$

$$= \int_a^b \left(F_1 \frac{dy}{dt} - F_2 \frac{dx}{dt}\right) dt$$

$$= \int_a^b F_1 dy - F_2 dx$$

Whereas, the line integral using the tangential component can be written as

$$\oint_C (\boldsymbol{F} \cdot d\boldsymbol{r}) = \int_a^b (\langle F_1, F_2\rangle \cdot \langle dx, dy\rangle) = \int_a^b F_1 dx + F_2 dy$$

Using this notation for the line integral we can write Green's Theorem as follows

$$\int_a^b F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

This line integral can be mapped to the one using the normal component as follows.

$Tangential \rightarrow Normal$	
$F_1 \rightarrow -F_2$	$F_2 \rightarrow F_1$

Using these mappings on the right-hand side of Green's theorem we have

$$\int_a^b F_1 dy - F_2 dx = \iint_D \left(\frac{\partial F_1}{\partial x} - \frac{\partial(-F_2)}{\partial y}\right) dA = \iint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dA = \iint_D div(\boldsymbol{F}) dA$$

Finally, we can write the Flux version of Green's Theorem.

$$\boxed{\oint_C (\boldsymbol{F} \cdot \boldsymbol{N}) dt = \iint_D div(\boldsymbol{F}) dA}$$

Example 5: Calculate the flux out for a unit circle that is placed in a vector field, $\boldsymbol{F} = \langle x^3, y^3 + y \rangle$.

Solution: The flux version of Green's Theorem was given above as

$$\oint_C (\boldsymbol{F} \cdot \boldsymbol{N})dt = \iint_D div(\boldsymbol{F})dA$$

Using the right-hand side to compute the flux we have

$$\begin{aligned} div(\boldsymbol{F}) &= \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) \\ &= \left(\frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3 + y)\right) \\ &= 3x^2 + 3y^2 + 1 \end{aligned}$$

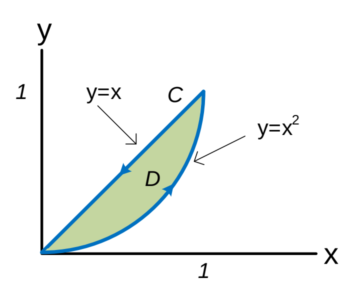
Then using polar coordinates, the integral is evaluated as follows

$$\begin{aligned} \iint_D div(\boldsymbol{F})dA &= \int_0^{2\pi}\int_0^1 (3r^2 \cos^2(\theta) + 3r^2 \sin^2(\theta) + 1)r\,dr\,d\theta \\ &= \int_0^{2\pi}\left(\int_0^1 (3r^3 + r)dr\right)d\theta \\ &= \int_0^{2\pi}\left(\frac{5}{4}\right)d\theta \\ &= \frac{5\pi}{2} \end{aligned}$$

Before ending this lesson let's do additional examples to get more comfortable with the ideas presented.

Example 6: Calculate the line integral for a closed curve that consists of arcs $y = x^2$ and $y = x$ for $0 \le x \le 1$ placed in a vector field, $\boldsymbol{F} = \langle x^2, x^2 \rangle$.

Solution: The curve, oriented counterclockwise, and the resulting region is shown below.



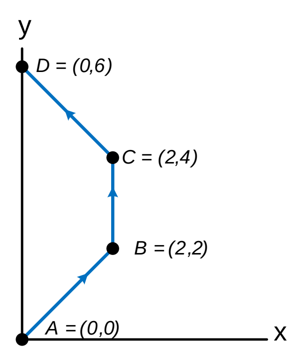
Since the closed curve is made two different functions, directly evaluating the line integral could be time consuming. Green's Theorem allows for an alternate method.

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D curl_z(\boldsymbol{F}) dA$$
$$= \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$
$$= \iint_D (2x) dA$$

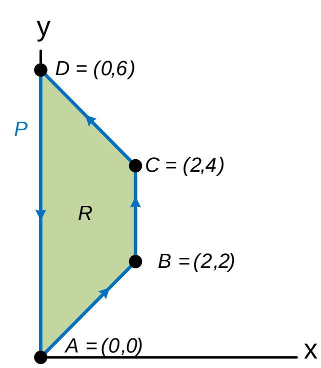
Treating the region as vertically simple we have

$$= \int_0^1 \left(\int_{x^2}^{x} (2x) dy\right) dx$$
$$= \int_0^1 (2x^2 - 2x^3)\, dx$$
$$= \frac{2}{3} - \frac{2}{4}$$
$$= \frac{1}{6}$$

Example 7: Calculate the line integral for the non-closed path shown below using the vector field, $\boldsymbol{F} = \langle sin(x) + y, 3x + y\rangle$.



Solution: We can utilize Green's Theorem by first adding a path so that the curve is closed.



Therefore, we have

$$\oint_P \boldsymbol{F}\cdot d\boldsymbol{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

$$= \iint_R \left(\frac{\partial}{\partial x}(3x+y) - \frac{\partial}{\partial y}(sin(x)+y)\right) dA$$

$$= \iint_R (3-1) dA$$

$$= 2\iint_R dA$$

$$= 2\cdot 8 = 16$$

Next, we can write the line integral of the non-closed path as follows

$$\int_{AD} \boldsymbol{F}\cdot d\boldsymbol{r} = \oint_P \boldsymbol{F}\cdot d\boldsymbol{r} - \int_{DA} \boldsymbol{F}\cdot d\boldsymbol{r}$$

$$= 16 - \int_{DA} \boldsymbol{F}\cdot d\boldsymbol{r}$$

The path DA can be parameterized as $\boldsymbol{r} = \langle 0, t\rangle$ for t: $6 \to 0$. Therefore,

$$\int_{DA} \boldsymbol{F}\cdot d\boldsymbol{r} = \int_6^0 \boldsymbol{F}(\boldsymbol{r}(t))\cdot\langle\frac{dx}{dt},\frac{dy}{dt}\rangle\, dt$$

$$= \int_6^0 \langle t, t\rangle\cdot\langle 0, 1\rangle\, dt$$

$$= \int_6^0 t\, dt$$

$$= -18$$

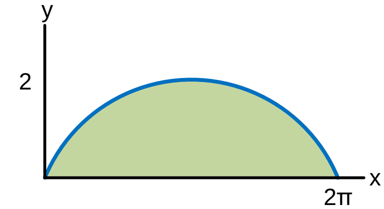
Finally, we have

$$\int_{AD} \boldsymbol{F}\cdot d\boldsymbol{r} = \oint_P \boldsymbol{F}\cdot d\boldsymbol{r} - \int_{DA} \boldsymbol{F}\cdot d\boldsymbol{r}$$

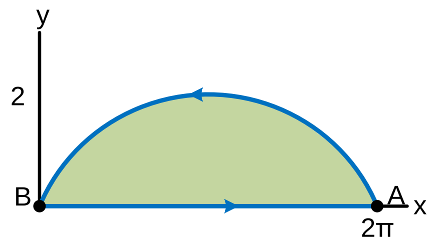
$$= 16 - (-18)$$

$$= 34$$

Example 8: Use one of the area formulas derived from Green's Theorem to find the area between the x-axis and the cycloid, $\boldsymbol{r}(t) = \langle t - sin(t)\, , 1 - cos(t)\rangle$, $0 \le t \le 2\pi$.



Solution: Since Green's Theorem applies to closed curves, we start by adding a path along the x-axis as shown below.



Using the second formula for area we have

$$\oint_C -ydx = \int_A^B -ydx + \oint_B^A -ydx$$

For the first integral we have

$$y = 1 - cos(t) \qquad\qquad dx = (1 - cos(t))dt$$

Therefore,

$$\int_A^B -ydx = \int_{2\pi}^0 -(1 - cos(t))^2 dt$$
$$= \int_0^{2\pi} (1 - 2\,cos(t) + cos^2(t))dt$$
$$= 2\pi - 0 + \frac{1}{2}\int_0^{2\pi} (1 + cos(2t))dt$$
$$= 2\pi + \frac{1}{2}(2\pi + 1)$$
$$= 3\pi$$

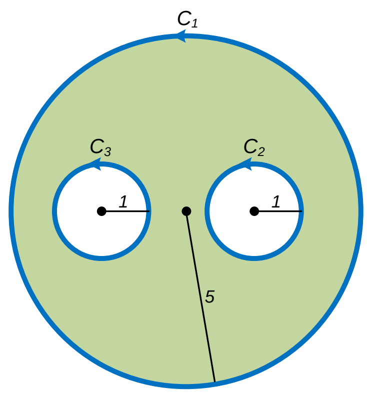
For the second integral, since $y = 0$, the integral is zero and

$$\oint_C -ydx = 3\pi$$

Example 9: Referring to the figure below, suppose

$$\oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} = 3\pi \qquad\qquad \oint_{C_3} \boldsymbol{F} \cdot d\boldsymbol{r} = 4\pi$$

Use Green's Theorem to determine the circulation of $\boldsymbol{F}$ around C_1, assuming that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 9$, on the shaded region.



Solution: The shaded region, D, is defined as $D = C_1 - C_2 - C_3$. Therefore, we can write

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} - \oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} - \oint_{C_3} \boldsymbol{F} \cdot d\boldsymbol{r}$$

Solving for $\oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r}$ we have

$$\oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA + \oint_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r} + \oint_{C_3} \boldsymbol{F} \cdot d\boldsymbol{r}$$

$$\oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} = 9 \iint_D dA + 3\pi + 4\pi$$

$$\oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} = 9(\pi 5^2 - \pi 1^2 - \pi 1^2) + 7\pi$$

$$\oint_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} = 214\,\pi$$

Example 10: A buffalo stampede is described by a velocity vector field $\boldsymbol{F} = \langle xy - y^3, x^2 + y\rangle\ km/h$ in the region $2 \leq x, y \leq 3$ in km. Assuming a density of buffalo, $\rho = 500$ buffalo per square kilometers. Find the net number of buffalo leaving the region per minute.

Solution: The number of buffalo crossing the boundary can be related to the flux across the curve, given as

$$\oint_C (\boldsymbol{F} \cdot \boldsymbol{N})dt$$

Note the units of flux in this case is km^2/hr. Multiplying the flux by the density of buffalo we get the desired measure.

$$\frac{km^2}{hr} \cdot \frac{buffalo}{km^2} = \frac{buffalo}{hr}$$

To evaluate the line integral, we can use Green's Theorem

$$\oint_C (\boldsymbol{F} \cdot \boldsymbol{N})dt = \iint_D div(\boldsymbol{F})dA$$

$$= \iint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dA$$

$$= \int_2^3 \left(\int_2^3 (y + 1)dy\right) dx$$

$$= \int_2^3 \left(\frac{9}{2} + 3\right) - \left(\frac{4}{2} + 2\right) dx$$

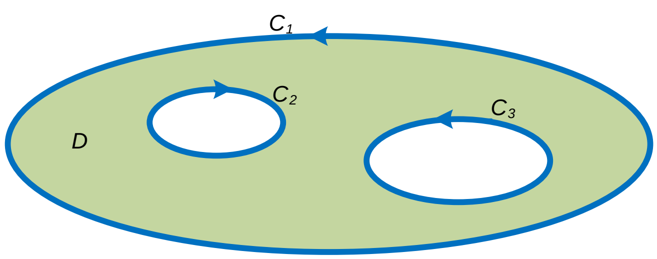
$$= \int_2^3 \left(\frac{7}{2}\right) dx$$

$$= \frac{7}{2}\ km^2/hr$$

The net number of buffalo leaving the region per minute is then computed as

$$\left(\frac{7}{2}\ km^2/hr\right) \cdot \left(\frac{1\ hr}{60\ min}\right) \cdot \left(500 \frac{buffalo}{km^2}\right) \cong 29.2\ buffalo/hr$$

Final Summary for Theorem of Vector Calculus – Green's Theorem

Green's Theorem
Let D be a domain in R^2 whose boundary is a simple closed curve, C, oriented counterclockwise. Then $$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D \left(curl_z(\mathbf{F})\right) dA$$ Where, $curl_z(\boldsymbol{F}) = \left(\frac{\partial F_2(x,y)}{\partial x} - \frac{\partial F_1(x,y)}{\partial y}\right)$ With $\boldsymbol{F} = \langle F_1(x,y), F_2(x,y)\rangle$ and $d\boldsymbol{r} = \langle dx, dy\rangle$, we can also express the line integral as $$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_C F_1(x,y)dx + F_2(x,y)dy$$
Area of Region Using Green's Theorem
There are three equivalent formulas we can use for the area of a region, D, enclosed by a simple curve, C. $$Area\ Enclosed\ by\ C = \left(\oint_C xdy\right) = \left(\oint_C -ydx\right) = \left(\frac{1}{2}\oint_C xdy - ydx\right)$$
Green's Theorem Using Normal Vector – Flux
Using the *normal* vector to the curve, Green's Theorem can be used to express the flux across the curve as follows $$\oint_C (\boldsymbol{F} \cdot \boldsymbol{N})dt = \iint_D div(\boldsymbol{F})dA$$ Where, $\boldsymbol{N}(t) = \langle y'(t), -x'(t)\rangle$ and $div(\boldsymbol{F}) = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right)$
General Form of Green's Theorem
Green's theorem can also be applied to non-simple regions as long as we keep in mind the fact that the region to be considered always lies to the left of the curve according to its orientation.      In this example the region, D, is represented as $$D = C_1 + C_2 - C_3$$

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