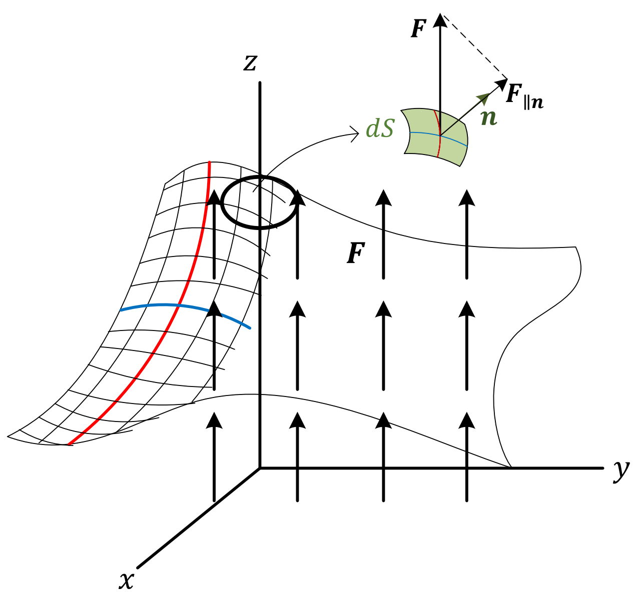
Line and Surface Integrals – Vector Field Surface Integrals

The final lesson in this series will cover integrals of vector fields over surfaces. Recall that we introduced two types of vector line integrals. The first used the component of the vector field that is *tangential* to the curve. In this case, if the vector field is interpreted as a force field the integral represents the amount of work done on a particle traveling along the curve. The second used the component of the vector field that is *normal* to the curve. In this case, the integral represents the flow of a substance across the curve, i.e. *flux*. The vector surface integral is analogous to the second type of the vector line integral. Therefore, the vector surface integral is used to measure the flux *through a surface*. One example is the flux of molecules through a cell membrane, which would be measured in number of molecules per unit of time.

Surface Integral of Vector Field Introduction

We can derive an expression of the vector surface integral by considering an infinitesimal section of a surface, dS, and a vector field, $\boldsymbol{F}$, as shown below.



The flow rate of a substance, i.e. flux, through a section of a surface is equal to the amount of flow that is directed perpendicular to the surface multiplied by the area of the surface section. In the figure above we have

$$d\Phi \;=\; F_{\parallel \boldsymbol{n}} dS$$

Where, $\boldsymbol{n}$ is the unit normal vector for the surface. Furthermore, since $F_{\parallel \boldsymbol{n}} = \boldsymbol{F} \cdot \boldsymbol{n}$, we can write the following

$$d\Phi \;=\; (\boldsymbol{F} \cdot \boldsymbol{n}) dS$$

Next, recall from the scalar surface integral lesson we have the following relationships.

$$\boldsymbol{n} = \frac{\boldsymbol{N}(u,v)}{\|\boldsymbol{N}(u,v)\|} \qquad\qquad dS = \|\boldsymbol{N}(u,v)\| du dv$$

Therefore, we can write

$$d\Phi = (\boldsymbol{F} \cdot \boldsymbol{n})dS = \left(\boldsymbol{F}\big(\boldsymbol{G}(u,v)\big) \cdot \frac{\boldsymbol{N}(u,v)}{\|\boldsymbol{N}(u,v)\|}\right) \|\boldsymbol{N}(u,v)\| dudv = \boldsymbol{F}\big(\boldsymbol{G}(u,v)\big) \cdot \boldsymbol{N}(u,v) dudv$$

With this we can define the vector surface integral, which measures the flux, Φ, as given below.

Vector Surface Integral
Let $\boldsymbol{G}(u,v)$ be a parameterization of a surface, $\mathcal{S}$, on the domain, D. The vector surface integral of the vector field $\boldsymbol{F}(x,y,z)$ over the surface on the given domain is $$\iint_{\mathcal{S}} (\boldsymbol{F} \cdot \boldsymbol{n})dS = \iint_D \left(\boldsymbol{F}\big(\boldsymbol{G}(u,v)\big) \cdot \boldsymbol{N}(u,v)\right) dudv$$

Example 1: Calculate the vector surface integral of the parameterized surface $\boldsymbol{G}(u,v) = \langle u^2, v, u^3 - v^2 \rangle, 0 \le u \le 1, \quad 0 \le v \le 1$ placed in a vector field $\boldsymbol{F} = \langle 0,0,x \rangle$.

Solution: From above the surface integral is given as

$$\iint_D \left(\boldsymbol{F}\big(\boldsymbol{G}(u,v)\big) \cdot \boldsymbol{N}(u,v)\right) dudv = \int_0^1 \int_0^1 \left(\boldsymbol{F}\big(\boldsymbol{G}(u,v)\big) \cdot \boldsymbol{N}(u,v)\right) dudv$$

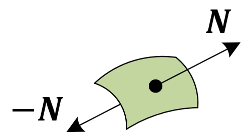
We can start by finding $\boldsymbol{N}(u,v)$, which is equal to the cross product of the tangent vectors.

$$\begin{aligned} \boldsymbol{T}_u &= \frac{\partial \boldsymbol{G}(u,v)}{\partial u} \\ &= \langle \frac{\partial}{\partial u}(u^2), \frac{\partial}{\partial u}(v), \frac{\partial}{\partial u}(u^3 - v^2) \rangle \\ &= \langle 2u, 0, 3u^2 \rangle \end{aligned}$$

$$\begin{aligned} \boldsymbol{T}_v &= \frac{\partial \boldsymbol{G}(u,v)}{\partial v} \\ &= \langle \frac{\partial}{\partial v}(u^2), \frac{\partial}{\partial v}(v), \frac{\partial}{\partial v}(u^3 - v^2) \rangle \\ &= \langle 0,1,-2v \rangle \end{aligned}$$

$$\begin{aligned} \boldsymbol{N}(u,v) &= \boldsymbol{T}_u \times \boldsymbol{T}_v \\ &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ 2u & 0 & 3u^2 \\ 0 & 1 & -2v \end{vmatrix} \\ &= \langle -3u^2, 4uv, 2u \rangle \end{aligned}$$

Before we go any further it's important to note that we can define two normal vectors for every surface, depending on which side we choose the normal to point out from.



Therefore, we must specify our choice for each vector surface integral. In this case we define the unit normal as upward-pointing. Since the z-component of the normal vector we computed above is positive for $0 \leq u \leq 1$ the normal is upward-pointing. If we instead chose to use the downward-pointing normal we would instead use

$$-\boldsymbol{N}(u,v) = -\langle -3u^2, 4uv, 2u\rangle = \langle 3u^2, -4uv, -2u\rangle$$

Next, we compute the integrand dot product

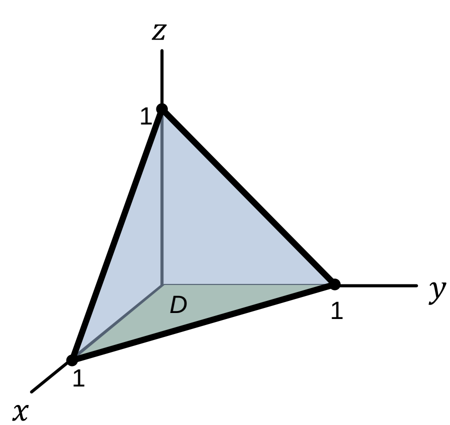
$$\boldsymbol{F}\big(\boldsymbol{G}(u,v)\big)\cdot\boldsymbol{N}(u,v) = \langle 0,0,u^2\rangle\cdot\langle -3u^2, 4uv, 2u\rangle = 2u^3$$

The integral is then

$$\int_0^1\int_0^1\Big(\boldsymbol{F}\big(\boldsymbol{G}(u,v)\big)\cdot\boldsymbol{N}(u,v)\Big)\,dudv = \int_0^1\left(\int_0^1 2u^3du\right)dv = \int_0^1\left(\frac{1}{2}\right)dv = \frac{1}{2}$$

Example 2: Calculate the vector surface integral for the portion of the plane $x + y + z = 1$ in the octant $x, y, z, \geq 0$ placed in a vector field $\boldsymbol{F} = \langle y^2, 2, -x\rangle$. Use an upward-pointing surface normal.

Solution: The surface is shown below



Just as with scalar surface integrals when the surface can be written as $z = g(x, y)$, we can use the simple parameterization

$$\boldsymbol{G}(x, y) = \langle x, y, g(x, y)\rangle$$

The normal vector is then

$$\boldsymbol{N} = \boldsymbol{T}_x \times T_y = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ \dfrac{\partial G_1}{\partial x} & \dfrac{\partial G_2}{\partial x} & \dfrac{\partial G_3}{\partial x} \\ \dfrac{\partial G_1}{\partial y} & \dfrac{\partial G_2}{\partial y} & \dfrac{\partial G_3}{\partial y} \end{vmatrix} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{vmatrix} = \langle -g_x, -g_y, 1\rangle$$

In this case since $z = g(x, y) = 1 - x - y$, we have

$$\boldsymbol{N} = \langle 1, 1, 1\rangle$$

Furthermore

$$\begin{aligned} \boldsymbol{F}\big(\boldsymbol{G}(x, y)\big) &= \langle (1 - x - z)^2, 2, (z + y - 1)\ \rangle \\ &= \langle \big(1 - x - (1 - x - y)\big)^2, 2, (1 - x - y + y - 1)\ \rangle \\ &= \langle y^2, 2, -x\ \rangle \end{aligned}$$

The surface integral is then

$$\iint_D \boldsymbol{F}\big(\boldsymbol{G}(x, y)\big) \cdot \boldsymbol{N}(x, y)dxdy = \iint_D (\langle y^2, 2, -x\ \rangle \cdot \langle 1, 1, 1\rangle)dxdy = \iint_D (y^2 + 2 - x)dxdy$$

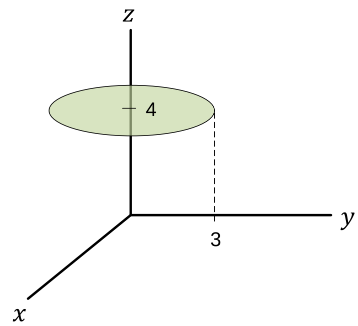
The domain in the xy plane can be deduced from the figure above

$$0 \le x \le 1, \qquad 0 \le y \le 1 - x$$

Therefore,

$$\begin{aligned} \iint_D (y^2 + 2 - x)dxdy &= \int_0^1 \left(\int_0^{1-x} (y^2 + 2 - x)\, dy\right) dx \\ &= \int_0^1 \left(\frac{1}{3}(1 - x)^3 + 2(1 - x) - x(1 - x)\right) dx \\ &= \int_0^1 \left(-\frac{1}{3}x^3 + 2x^2 - 4x + \frac{7}{3}\right) dx \\ &= -\frac{1}{12} + \frac{2}{3} - 2 + \frac{7}{3} \\ &= \frac{11}{12} \end{aligned}$$

Example 3: Calculate the vector surface integral for a disk of radius 3 at height 4 parallel to the xy plane placed in a vector field $\boldsymbol{F} = \langle xz, yz, z^{-1}\rangle$. Use an upward-pointing surface normal.



Solution: The surface can be looked at as the entire horizontal plane at height 4. i.e. $z = 4$. The domain for the surface integral is, however, given by

$$D: x^2 + y^2 \leq 9$$

Since the surface can be described as $z = g(x, y) = 4$, we can parameterize is as follows

$$\boldsymbol{G}(x, y) = \langle x, y, g(x, y)\rangle = \langle x, y, 4\rangle$$

With the normal being computed as

$$\boldsymbol{N} = \langle -g_x, -g_y, 1\rangle = \langle 0, 0, 1\rangle$$

Therefore,

$$\boldsymbol{F}\big(\boldsymbol{G}(x, y)\big) = \langle 4x, 4y, 1/4\rangle$$

And the surface integral is then

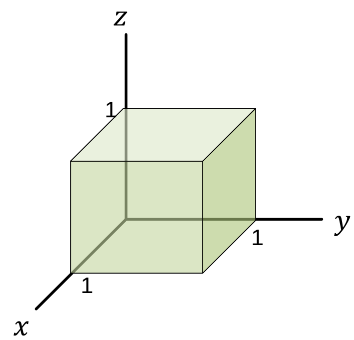
$$\iint_D \boldsymbol{F}\big(\boldsymbol{G}(x, y)\big) \cdot \boldsymbol{N}(x, y) dxdy = \iint_D \langle 4x, 4y, 1/4\rangle \cdot \langle 0,0,1\rangle dxdy$$

$$= \frac{1}{4} \iint_D dxdy$$

Since the domain is a circle of radius 3 in the xy plane we have

$$= \frac{1}{4} \iint_D dxdy = \frac{1}{4} A = \frac{1}{4} \pi 3^2 = \frac{9\pi}{4}$$

Example 4: Calculate the vector surface integral for unit cube shown below placed in a vector field $\boldsymbol{F} = \langle 0,0,e^{y+z}\rangle$. Use an outward-pointing surface normal.



Solution: To find the surface integral for the cube we need to compute the integral for each of the six surfaces. However, since the field has a component only in the z direction the integral is non-zero for the top and bottom surfaces only.

Top surface:

The surface can be written as $z = g(x,y) = 1$. Therefore

$$\boldsymbol{N} = \langle -g_x, -g_y, 1\rangle = \langle 0,0,1\rangle$$

And with $\boldsymbol{G}(x,y) = \langle x,y,1\rangle$, we have

$$\boldsymbol{F}\big(\boldsymbol{G}(x,y)\big) = \langle 0,0,e^{y+1}\rangle$$

Therefore,

$$\iint_D \boldsymbol{F}\big(\boldsymbol{G}(x,y)\big)\cdot\boldsymbol{N}(x,y)dxdy = \iint_D \langle 0,0,e^{y+1}\rangle\cdot\langle 0,0,1\rangle dxdy$$

$$= e^1\int_0^1\left(\int_0^1 e^y dy\right)dx$$

$$= e^1\cdot(e^1-1)\int_0^1 dx$$

$$= e^2 - e^1$$

Bottom surface:

The normal vector is said to be outward-pointing, therefore in this case

$$\boldsymbol{N} = \langle 0,0,-1\rangle$$

And with $\boldsymbol{G}(x,y) = \langle x,y,0\rangle$, we have

$$\boldsymbol{F}\big(\boldsymbol{G}(x,y)\big) = \langle 0,0,e^{y+0}\rangle$$

Therefore,

$$\iint_D \boldsymbol{F}(\boldsymbol{G}(x,y)) \cdot \boldsymbol{N}(x,y)dydx = \iint_D \langle 0,0,e^y\rangle \cdot \langle 0,0,-1\rangle dydx$$

$$= -\int_0^1 \left(\int_0^1 e^y dy\right) dx$$

$$= -(e^1 - 1)\int_0^1 dx$$

$$= 1 - e^1$$

Summing the two we have

$$e^2 - e^1 + 1 - e^1 = (e^2 - 2e^2 + 1) \ = (e^1 - 1)^2$$

Example 4: Calculate the vector surface integral for the paraboloid $z = 9 - x^2 - y^2$ for $x, y, z, \geq 0$ placed in a vector field $\boldsymbol{F} = \langle z, z, x\rangle$. Use an upward-pointing surface normal.

Solution: We can start with the following parameterization

$$\boldsymbol{G}(x,y) = \langle x, y, g(x,y)\rangle = \langle x, y, 9 - x^2 - y^2\rangle$$

The normal vector is then given by

$$\boldsymbol{N} = \langle -g_x, -g_y, 1\rangle = \langle 2x, 2y, 1\rangle$$

Next, we map the vector field with the parameterized surface

$$\boldsymbol{F}(\boldsymbol{G}(x,y)) = \langle 9 - x^2 - y^2, 9 - x^2 - y^2, x\rangle$$

Therefore, we can integrate as follows

$$\iint_D \boldsymbol{F}(\boldsymbol{G}(x,y)) \cdot \boldsymbol{N}(x,y)dxdy = \iint_D \langle 9 - x^2 - y^2, 9 - x^2 - y^2, x\rangle \cdot \langle 2x, 2y, 1\rangle dxdy$$

$$= \iint_D (2x(9 - x^2 - y^2) + \ 2y(9 - x^2 - y^2) + x)dxdy$$

$$= \iint_D \left(2x\left(9 - (x^2 + y^2)\right) + \ 2y\left(9 - (x^2 + y^2)\right) + x\right)dxdy$$

The surface is defined for $x, y, z, \geq 0$. Therefore, the domain is a quarter circle in the xy plane located in the first quadrant. With this domain it's easier to change the variables of integration to polar coordinates where $r = x^2 + y^2, x = r\,cos(\theta)\,, y = r\,sin(\theta)$ and $dxdy = rdrd\theta$. With that we have

$$= \int_0^{\pi/2} \int_0^3 (2r\,cos(\theta)\,(9 - r^2) + 2r\,sin(\theta)\,(9 - r^2) + r\,cos(\theta))rdrd\theta$$

$$= \int_0^{\pi/2} \int_0^3 (18r^2\,cos(\theta) - 2r^4\,cos(\theta) + 18r^2\,sin(\theta) - 2r^4\,sin(\theta) + r^2\,cos(\theta))drd\theta$$

$$= \int_0^{\pi/2} \left(cos(\theta) \int_0^3 (19r^2 - 2r^4)dr \right) d\theta + \int_0^{\pi/2} \left(sin(\theta) \int_0^3 (18r^2 - 2r^4)dr \right)$$

$$= 73.8 \int_0^{\pi/2} cos(\theta)\,d\theta + 64.8 \int_0^{\pi/2} sin(\theta)\,d\theta$$

$$= 73.8(sin(\pi/2) - sin(0)) + 64.8(cos(0) - cos(\pi/2))$$

$$= 73.8(1 - 0) + 64.8(1 - 0)$$

$$= 138.6$$

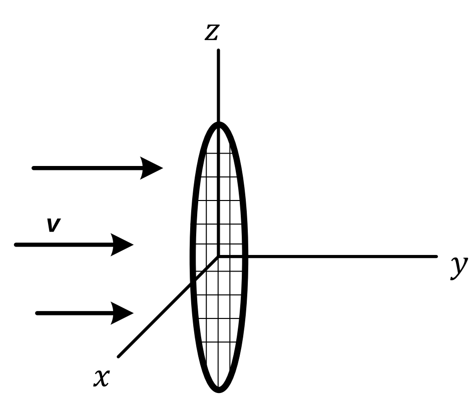
Vector Surface Integral Applications - Flux

As mentioned, an important application of the vector surface integral is for computing the flow of a substance through a surface. We generally refer to this as flux, Φ. The substance 'flowing' could be a fluid, e.g., water, air, or a more esoteric substance like the electric or magnetic field. The remaining examples in this lesson will illustrate some of the various applications.

Example 5 - Water flowing in a River: The flow rate in meters per second of water in a river can be described by the following vector field.

$$\boldsymbol{v} = \langle x - y, z + y + 4, z^2 \rangle$$

If a net a dipped into the river find the flow rate, in meters cubed per second, through the net. The surface of the net can be described by $x^2 + z^2 \leq 1, y = 0$, with the normal pointing in the positive y-direction.



Solution: In this case, the vector field describes the velocity of the river water.

$$\boldsymbol{v} = \langle x - y, z + y + 4, z^2 \rangle$$

The boundary of the net is the surface through which we would like to measure the flow rate, i.e. flux. The surface can be parameterized using modified polar coordinates as

$$\boldsymbol{G}(r, \theta) = \langle r\, cos(\theta)\,, 0, r\, sin(\theta) \rangle, \qquad 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$$

The normal is then found as shown.

$$\begin{aligned} \boldsymbol{T}_r &= \frac{\partial \boldsymbol{G}(r,\theta)}{\partial r} \\ &= \langle \frac{\partial}{\partial r}(r\, cos(\theta)), \frac{\partial}{\partial r}(0), \frac{\partial}{\partial r}(r\, sin(\theta)) \rangle \\ &= \langle cos(\theta)\,, 0, sin(\theta) \rangle \end{aligned}$$

$$\begin{aligned} \boldsymbol{T}_\theta &= \frac{\partial \boldsymbol{G}(r,\theta)}{\partial \theta} \\ &= \langle \frac{\partial}{\partial \theta}(r\, cos(\theta)), \frac{\partial}{\partial \theta}(0), \frac{\partial}{\partial \theta}(r\, sin(\theta)) \rangle \\ &= \langle -r\, sin(\theta)\,, 0, r\, cos(\theta) \rangle \end{aligned}$$

$$\begin{aligned} \boldsymbol{N}(r,\theta) &= \boldsymbol{T}_r \times \boldsymbol{T}_\theta \\ &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ cos(\theta) & 0 & sin(\theta) \\ -r\, sin(\theta) & 0 & r\, cos(\theta) \end{vmatrix} \\ &= \langle 0, -(\, r\, cos^2(\theta) + r\, sin^2(\theta)), 0 \rangle \\ &= \langle 0, -r, 0 \rangle \end{aligned}$$

However, we defined the normal as pointing to the right, therefore we use $\boldsymbol{N}(r,\theta) = \langle 0, r, 0 \rangle$.

Next, we map the vector field with the parameterized surface

$$\boldsymbol{v}\big(\boldsymbol{G}(r,\theta)\big) = \langle r\, cos(\theta)\,, r\, sin(\theta) + 4, r^2\, sin^2(\theta) \rangle$$

Finally, we can evaluate the integral as shown

$$\begin{aligned} \iint_D \Big(\boldsymbol{v}\big(\boldsymbol{G}(r,\theta)\big) \cdot \boldsymbol{N}(r,\theta)\Big)\, dr d\theta &= \iint_D (\langle r\, cos(\theta)\,, r\, sin(\theta) + 4, r^2\, sin^2(\theta) \rangle \cdot \langle 0, r, 0 \rangle) dr d\theta \\ &= \int_0^{2\pi} \left(\int_0^1 (r^2\, sin(\theta) + 4r)\, dr \right) d\theta \\ &= \int_0^{2\pi} \left(\frac{1}{3} sin(\theta) + 2 \right) d\theta \\ &= \int_0^{2\pi} \frac{1}{3} sin(\theta)\, d\theta + \int_0^{2\pi} 2\, d\theta \\ &= 0 + 4\pi \\ \Phi &= 4\pi\ m^3/s \end{aligned}$$

Example 6 - Wind: The velocity of the wind in meters per second can be described

$$\boldsymbol{w} = \langle z, x, 1 \rangle$$

If we model the surface of an umbrella as the upper hemisphere of a sphere, $x^2 + y^2 + z^2 = 1$, find the flux through the undersurface of the umbrella.

Solution: The surface is most conveniently represented in spherical coordinates. Therefore, the parameterized representation is given as

$$\boldsymbol{G}(\theta, \phi) = \langle sin(\phi)\, cos(\theta)\, , sin(\phi)\, sin(\theta)\, , cos(\phi) \rangle, \qquad 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2}$$

The normal is then found as shown.

$$\begin{aligned} \boldsymbol{T}_\theta &= \frac{\partial \boldsymbol{G}(\theta, \phi)}{\partial \theta} \\ &= \langle \frac{\partial}{\partial \theta}(sin(\phi)\, cos(\theta)), \frac{\partial}{\partial \theta}(sin(\phi)\, sin(\theta)), \frac{\partial}{\partial \theta}(cos(\phi)) \rangle \\ &= \langle - sin(\theta)\, sin(\phi)\, , sin(\phi)\, cos(\theta)\, , 0 \rangle \end{aligned}$$

$$\begin{aligned} \boldsymbol{T}_\phi &= \frac{\partial \boldsymbol{G}(\theta, \phi)}{\partial \phi} \\ &= \langle \frac{\partial}{\partial \phi}(sin(\phi)\, cos(\theta)), \frac{\partial}{\partial \phi}(sin(\phi)\, sin(\theta)), \frac{\partial}{\partial \phi}(cos(\phi)) \rangle \\ &= \langle cos(\phi)\, cos(\theta)\, , cos(\phi)\, sin(\theta)\, , - sin(\phi) \rangle \end{aligned}$$

$$\begin{aligned} \boldsymbol{N}(\theta, \phi) &= \boldsymbol{T}_\theta \times \boldsymbol{T}_\phi \\ &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ - sin(\theta)\, sin(\phi) & sin(\phi)\, cos(\theta) & 0 \\ cos(\phi)\, cos(\theta) & cos(\phi)\, sin(\theta) & - sin(\phi) \end{vmatrix} \\ &= \langle - sin^2(\phi)\, cos(\theta)\, , - sin(\theta)\, sin^2(\phi)\, , -(cos(\phi)\, sin^2(\theta)\, sin(\phi) \\ &\qquad + sin(\phi)\, cos^2(\theta)\, cos(\phi)) \rangle \\ &= \langle - sin^2(\phi)\, cos(\theta)\, , -sin(\theta)\, sin^2(\phi)\, , - cos(\phi)\, sin(\phi) \rangle \end{aligned}$$

Note that for , $0 \le \phi \le \frac{\pi}{2}$ the z component of the normal is pointing down. Since we are interested in the flow of the wind through the underside of the umbrella, we switch the sign so the normal points upward.

$$\boldsymbol{N}(\theta, \phi) = \langle sin^2(\phi)\, cos(\theta)\, , sin(\theta)\, sin^2(\phi)\, , cos(\phi)\, sin(\phi) \rangle$$

Next, we map the field via the surface as follows

$$\boldsymbol{w}\big(\boldsymbol{G}(\theta,\phi)\big) = \langle cos(\phi)\,, sin(\phi)\,cos(\theta)\,, 1\rangle$$

The integrand, $\boldsymbol{w}\big(\boldsymbol{G}(\theta,\phi)\big)\cdot\boldsymbol{N}(\theta,\phi)$ is then computed as follows

$$= \langle cos(\phi)\,, sin(\phi)\,cos(\theta)\,, 1\rangle \cdot \langle sin^2(\phi)\,cos(\theta)\,, sin(\theta)\,sin^2(\phi)\,, cos(\phi)\,sin(\phi)\rangle$$
$$= \; cos(\theta)\,sin^2(\phi)\,cos(\phi) + sin(\theta)\,cos(\theta)\,sin^3(\phi) + cos(\phi)\,sin(\phi)$$

Therefore,

$$\iint_D \Big(\boldsymbol{w}\big(\boldsymbol{G}(\theta,\phi)\big)\cdot\boldsymbol{N}(\theta,\phi)\Big)\,d\theta d\phi$$
$$= \iint_D cos(\theta)\,sin^2(\phi)\,cos(\phi) + sin(\theta)\,cos(\theta)\,sin^3(\phi) + cos(\phi)\,sin(\phi)\,d\theta d\phi$$
$$= \int_0^{\pi/2}\left(\int_0^{2\pi} cos(\theta)\,sin^2(\phi)\,cos(\phi) + sin(\theta)\,cos(\theta)\,sin^3(\phi) + cos(\phi)\,sin(\phi)\,d\theta\right)d\phi$$

The inner integral is evaluated as follows

$$= \left(sin^2(\phi)\,cos(\phi)\int_0^{2\pi} cos(\theta)\,d\theta\right) + \left(sin^3(\phi)\int_0^{2\pi} sin(\theta)\,cos(\theta)\,d\theta\right)$$
$$+ \left(cos(\phi)\,sin(\phi)\int_0^{2\pi} d\theta\right)$$
$$= (0) + \left(sin^3(\phi)\int_0^{0} udu\right) + (cos(\phi)\,sin(\phi)\,2\pi)$$
$$= \; cos(\phi)\,sin(\phi)\,2\pi$$

The outer integral is then

$$2\pi\int_0^{\pi/2}(cos(\phi)\,sin(\phi)\,)d\phi = 2\pi\int_0^1 udu$$
$$= 2\pi\int_0^1 udu$$
$$\Phi = \; \pi\, m^3/s$$

Where, we use the following substitution

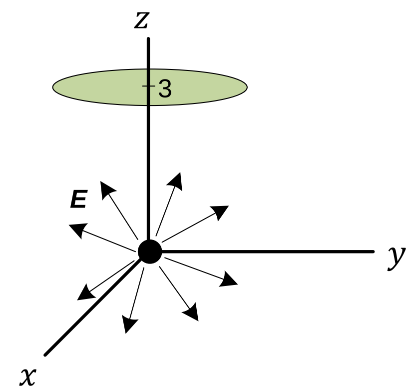
$$u = sin(\phi) \quad \rightarrow \quad du = cos(\phi)\,d\phi$$

Example 7 - Electric Flux: It is readily observable that electricity and magnetism exert forces on certain objects without making physical contact. Michael Faraday introduced the idea of a "force field" to describe these phenomena. Faraday's 'lines of force' translate directly into the vector fields being discussed in this section. If a positive point charge is placed at the origin of a rectangular coordinate system an electric vector field fills all of space and is described as

$$\boldsymbol{E} = kq\left\langle \frac{x}{d^3}, \frac{y}{d^3}, \frac{z}{d^3} \right\rangle$$

Where, $d = \sqrt{x^2 + y^2 + z^2}$, $k = 9E^9$ and $q = 1.6E^{-19}$

Find the electric flux through a disk of radius 2 parallel to the xy plane with its center at $(0,0,3)$.



Solution: The electric flux through the surface is given by the vector surface integral

$$\Phi_E = \iint_D \Big(\boldsymbol{E}\big(\boldsymbol{G}(x,y)\big) \cdot \boldsymbol{N}(x,y)\Big)\, dxdy$$

The surface can be described using an equation of an entire plane, $z = g(x,y) = 3$, with domain restricted by the circle of radius 2. The normal to the surface is than

$$\boldsymbol{N}(x,y) = \langle -g_x, -g_y, 1\rangle = \langle 0,0,1\rangle$$

We can parameterize the surface as $\boldsymbol{G}(x,y) = \langle x, y, 3\rangle$. Therefore,

$$\boldsymbol{E}\big(\boldsymbol{G}(x,y)\big) = kq\left\langle \frac{x}{(x^2+y^2+9)^{3/2}}, \frac{y}{(x^2+y^2+9)^{3/2}}, \frac{3}{(x^2+y^2+9)^{3/2}} \right\rangle$$

And the integrand is given as

$$\boldsymbol{E}\big(\boldsymbol{G}(x,y)\big) \cdot \boldsymbol{N}(x,y) = \frac{3kq}{(x^2+y^2+9)^{3/2}}$$

The region in the xy-plane can be described much easier using polar coordinates. Converting the integral to polar coordinates we have

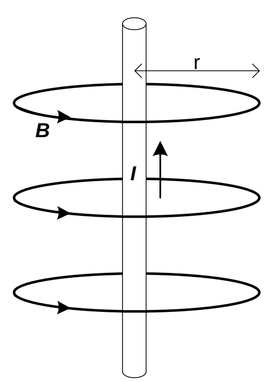
$$\Phi_E = \int_0^{2\pi}\int_0^2 \frac{3kq}{(r^2+9)^{3/2}} r dr d\theta$$
$$= 3kq \int_0^{2\pi}\left(\int_0^2 \frac{r}{(r^2+9)^{3/2}} dr\right) d\theta$$

We can evaluate the inner integral using substation as shown.

$$u = r^2 + 9 \qquad\qquad du = 2rdr$$

$$\Phi_E = \frac{3kq}{2}\int_0^{2\pi}\left(\int_9^{13} u^{-3/2}\, du\right) d\theta$$
$$= \frac{3kq}{2}\int_0^{2\pi} 2\left(\frac{1}{\sqrt{9}} - \frac{1}{\sqrt{13}}\right) d\theta$$
$$= 3kq\left(\frac{1}{3} - \frac{1}{\sqrt{13}}\right)\int_0^{2\pi} d\,\theta$$
$$= 6\pi kq\left(\frac{1}{3} - \frac{1}{\sqrt{13}}\right)$$
$$\Phi_E \cong 1.52E^{-9}\ \frac{N}{C}/m^2$$

Example 8 - Magnetic Flux: In 1820 Hans Christian Oersted noticed that a magnetic field fills the space around a current carrying wire as shown below.



Further experiments revealed a relationship describing the magnitude, measured in Teslas, of the magnetic field as follows

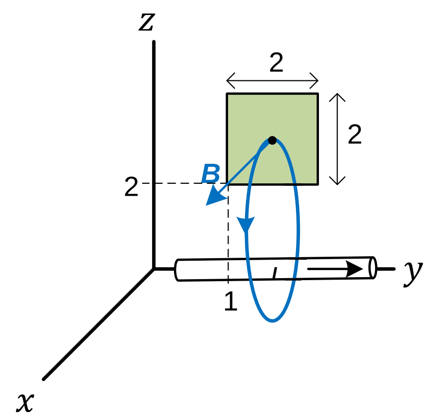
$$\|\boldsymbol{B}\| = \frac{\mu_0 I}{2\pi r}$$

Where, I is the current in amps, r is the distance from the wire in meters, and μ_0 is called the permeability of free space and is equal to $4\pi E^{-7}\ T\cdot m/A$.

Similar to the electric flux the magnetic flux through a surface can be computed as

$$\Phi_B = \iint_D \Big(\boldsymbol{B}\big(\boldsymbol{G}(x,y)\big)\cdot \boldsymbol{N}(x,y)\Big)\,dxdy$$

Compute the magnetic flux through a square with side length 2 due to a current carrying wire placed on the y-axis. The current in the wire is 5 amps and the disk is placed in the yz plane with center at $(0,0,3)$ as shown below.



Solution: Although the magnetic field wraps around the wire we see that at each point on the surface the magnetic field vector points in the x direction only. Furthermore, its magnitude depends on its distance along the z axis only. Therefore, we can express the magnetic field through the surface as

$$\boldsymbol{B}(x,y,z) = \langle \frac{\mu_0 I}{2\pi z}, 0, 0\rangle$$

The surface can be described using an equation of the entire yz plane, i.e. $x = g(y,z) = 0$, with domain restricted by the circle of radius 2. With this the surface is parameterized as

$$\boldsymbol{G}(y,z) = \langle 0, y, z\rangle$$

Therefore,

$$\boldsymbol{B}\big(\boldsymbol{G}(y,z)\big) = \langle \frac{\mu_0 I}{2\pi z}, 0, 0\rangle$$

The normal to the surface is given as

$$\boldsymbol{N}(y,z) = \langle 1, -g_y, -g_z\rangle = \langle 1,0,0\rangle$$

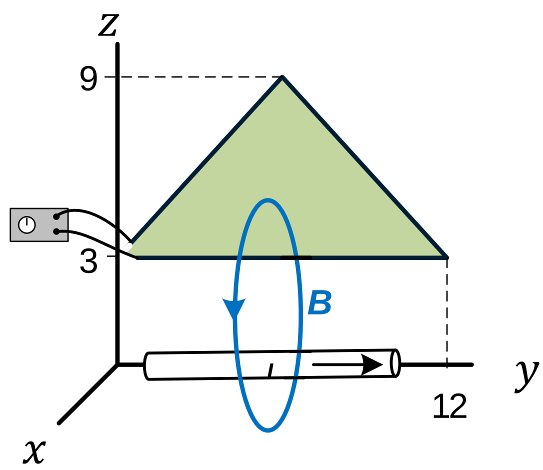
With this information we can compute the magnetic flux as

$$\begin{aligned}
\Phi_B &= \iint_D \Big(\boldsymbol{B}\big(\boldsymbol{G}(y,z)\big)\cdot \boldsymbol{N}(y,z)\Big)\, dydz \\
&= \iint_D \Big(\langle\frac{\mu_0 I}{2\pi z}, 0, 0\rangle \cdot \langle 1,0,0\rangle\Big)\, dydz \\
&= \frac{\mu_0 I}{2\pi}\int_2^4 \frac{1}{z}\left(\int_1^3 dy\right) dz \\
&= \frac{\mu_0 I}{\pi}\int_2^4 \frac{1}{z} dz \\
&= \frac{\mu_0 I}{\pi}\int_2^4 \frac{1}{z} dz \\
&= \frac{4\pi E^{-7}\cdot 5}{\pi}(ln(4) - ln(2)) \\
&\cong 1.4E^{-6}\, T\cdot m^2
\end{aligned}$$

Example 9 - Faraday's Law: In example 7 we mentioned how Michael Faraday created the idea of the field to help describe various electrical and magnetic phenomena. In the previous example we saw that Oersted showed that a magnetic field is created around a current carrying wire. With this in mind Michael Faraday showed how the opposite is also true. The phenomena is described by what is called *Faraday's Law of Induction.* It states that a changing magnetic field will induce a voltage, and therefore a current, around a closed path.

Faraday's Law of Induction
$$\varepsilon = -\frac{d\Phi_B}{dt}$$ Where, ε is the voltage, (or EMF), measured in volts, and the magnetic flux is given by the following surface integral $$\Phi_B = \iint_D \Big(\boldsymbol{B}\big(\boldsymbol{G}(y,z)\big)\cdot \boldsymbol{N}(y,z)\Big)\, dydz$$

A varying current of magnitude $I(t) = 28\,cos(400t)\ \ Amps$ flows through a wire shown below. A triangle wire loop is located directly above the wire and connected to a device that can measure voltage. Using Faraday's Law of Induction, compute the voltage induced in the loop. Note the distances shown in the figure are in centimeters.



We start by finding the flux through the triangular wire loop. From the previous problem we know the magnetic field and the surface normal vector are given as follows

$$\boldsymbol{B}(\boldsymbol{G}(y,z),t) = \langle \frac{\mu_0 I(t)}{2\pi z}, 0, 0\rangle \qquad\qquad \boldsymbol{N}(y,z) = \langle 1,0,0\rangle$$

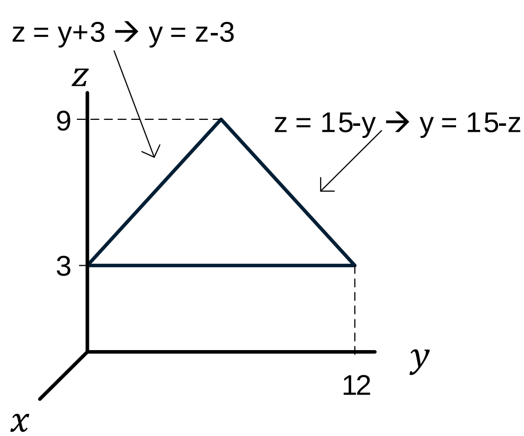
Note for this case the current is a function of time.

The magnetic flux is then computed as follows

$$\Phi_B(t) = \iint_D \left(\boldsymbol{B}(\boldsymbol{G}(y,z)) \cdot \boldsymbol{N}(y,z)\right) dydz$$

$$= \frac{\mu_0 I(t)}{2\pi z} \iint_D \left(\frac{1}{z}\right) dydz$$

We redraw the triangular region below to help determine the integration limits



The integral can then be evaluated, (using meters for the lengths), as follows

$$\Phi_B(t) = \frac{\mu_0 I(t)}{2\pi} \int_{0.03}^{0.09} \frac{1}{z} \left(\int_{z-0.03}^{0.15-z} dy \right) dz$$

$$= \frac{\mu_0 I(t)}{2\pi} \int_{0.03}^{0.09} \frac{1}{z} \big((0.15 - z) - (z - 0.03) \big) \, dz$$

$$= \frac{\mu_0 I(t)}{2\pi} \int_{0.03}^{0.09} \left(\frac{0.18}{z} - 2 \right) dz$$

$$= \frac{\mu_0 I(t)}{2\pi} (0.18 ln(z) - 2x|_{0.03}^{0.09})$$

$$= \frac{\mu_0 I(t)}{2\pi} (0.18 \, ln(3) - 0.12)$$

$$\cong 1.6E^{-8} \cdot I(t)$$

$$\Phi_B(t) \cong 3.1E^{-6} \, cos(400t)$$

Finally, the voltage induced is the negative of the rate of change of the magnetic flux.

$$\varepsilon = -\frac{d\Phi_B(t)}{dt}$$

$$\varepsilon = -\frac{d}{dt}(3.1E^{-6} \, cos(400t))$$

$$\varepsilon = 1.24E^{-3} \, sin(400t)$$

Below we summarize the integrals we have introduced in this series of lessons. Note the similarities as you review.

Final Summary for Line and Surface Integrals

Scalar Line Integral

Let $\boldsymbol{r}(t)$ be a parameterization of a curve, C, for $a \leq t \leq b$, then the scalar line integral is also given as

$$\int_C f(x,y,z)ds = \int_a^b f\big(\boldsymbol{r}(t)\big)\|\boldsymbol{r}'(t)\|dt$$

Vector Line Integral

Let $\boldsymbol{r}(t)$ be a parameterization of a curve, C, for $a \leq t \leq b$, then the vector line integral is also given as by the two equivalent expressions

$$\int_C \boldsymbol{F}(x,y,z) \cdot d\boldsymbol{s} \int_a^b \Big(\boldsymbol{F}\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{r}'(t)\Big)\, dt$$

Work Along a Curve

$$W = \int_a^b \Big(\boldsymbol{F}\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{r}'(t)\Big)\, dt$$

Flux Across a Curve

$$\Phi = \int_a^b \Big(\boldsymbol{v}\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{N}(t)\Big)\, dt$$

Scalar Surface Integral

Let $\boldsymbol{G}(u,v)$ be a parameterization of a surface, $\mathcal{S}$, on the domain. The scalar surface integral of the function $f(x,y,z)$ over the surface on the given domain is

$$\iint_{\mathcal{S}} f(x,y,z)dS = \iint_D f(\boldsymbol{G}(u,v))\,\|\boldsymbol{N}(u,v)\|dudv$$

Vector Surface Integral

Let $\boldsymbol{G}(u,v)$ be a parameterization of a surface, $\mathcal{S}$, on the domain, D. The vector surface integral, also called the flux, of the vector field $\boldsymbol{F}(x,y,z)$ over the surface on the given domain is

$$\iint_{\mathcal{S}} (\boldsymbol{F} \cdot \boldsymbol{n})dS = \iint_D \Big(\boldsymbol{F}\big(\boldsymbol{G}(u,v)\big) \cdot \boldsymbol{N}(u,v)\Big)\, dudv$$

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