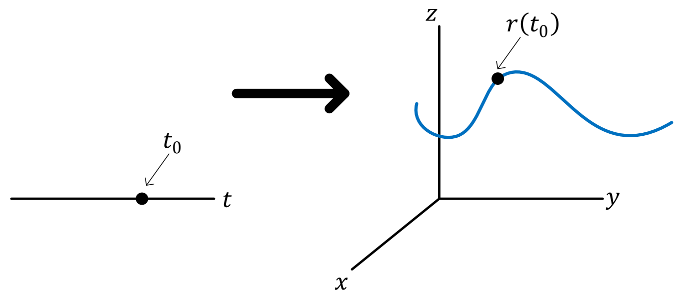
# Line and Surface Integrals – Scalar Surface Integrals

Integration is used in a variety of contexts. One of the stated goals for this series of lessons was to extend our notion of integration. Our first venture into this was evaluating integrals of scalar functions and vector fields over curves, i.e. *line integrals*. A natural progression would be evaluating integrals of these same types of objects over surfaces, i.e. *surface integrals*. That is indeed what we intend to do over the next two lessons, beginning with surface integrals of scalar functions.

*Parameterized Surfaces*

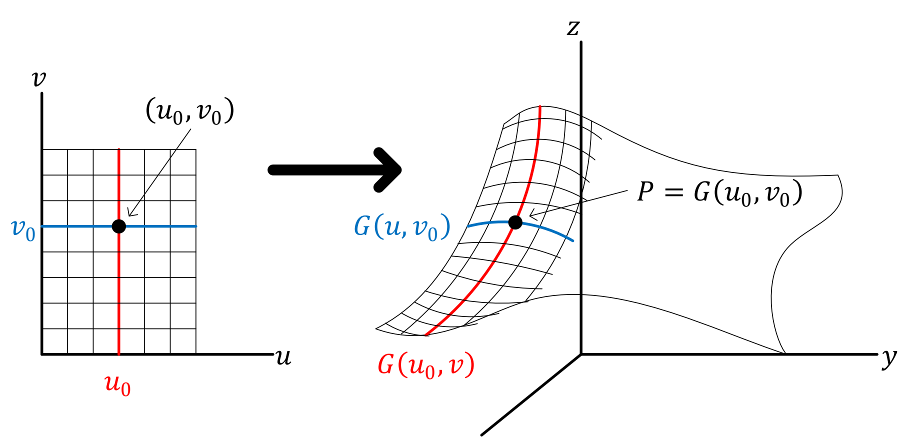
When considering line integrals over curves we used a parameterized description of the curves to facilitate the integration process. One can imagine this parameterization as a mapping of a single parameter, $t$, onto a curve in three dimensions. Each value of the parameter, e.g. $t_0$, corresponds to a point on a curve in 3-space, i.e. $r(t_0) = \big(x(t_0), y(t_0), z(t_0)\big)$. The continued mapping of the points results in the entire curve.

$$C \to \boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$$




For surface integrals we will similarly consider a parameterized description of the surfaces. Of course, two parameters are required to parameterize a surface, $\mathcal{S}$. In this case we can imagine mapping the $uv$-plane onto a surface in three dimensions. The grid lines in the $uv$-plane, e.g. $u = u_0$ and $v = v_0$ map to a curve, $\boldsymbol{G}(u_0, v) = \langle x(u_0, v), y(u_0, v), z(u_0, v)\rangle$ and , $\boldsymbol{G}(u, v_0) = \langle x(u, v_0), y(u, v_0), z(u, v_0)\rangle$ respectively, on the surface. The continued mapping of the $uv$ grid lines result in the entire surface as shown below.

$$\mathcal{S} \to \boldsymbol{G}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$$



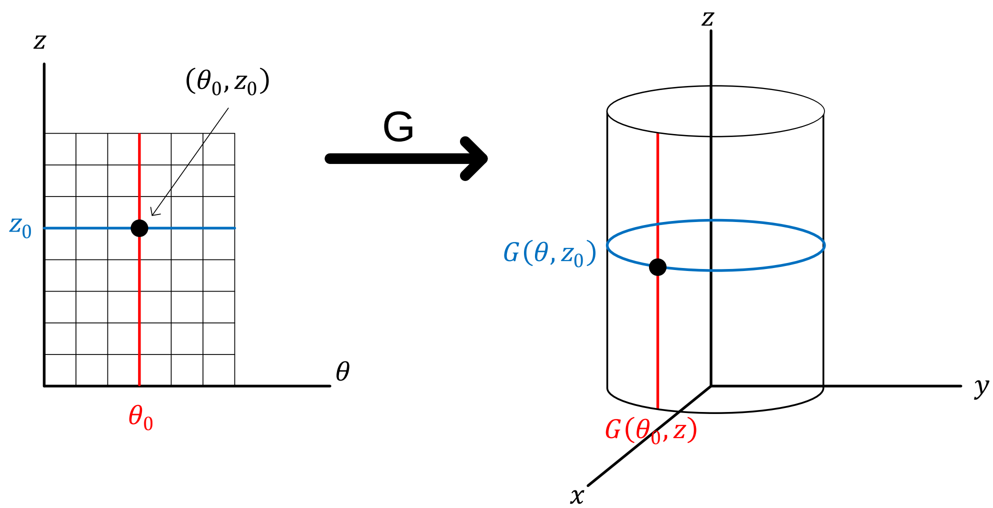
Let's look at some examples before we move onto deriving the surface integral formula.

**Example 1:** Find the parameterization of a cylinder with a radius of 2, i.e. $x^2 + y^2 = 4$ .

Solution: A cylinder is conveniently parameterized using cylindrical coordinates, $(r, \theta, z)$. Since the radius is fixed, we can eliminate $r$ as a parameter. Therefore, we have

$$\boldsymbol{G}(\theta, z) = \langle x(\theta, z), y(\theta, z), z(\theta, z)\rangle = \langle 2\,cos(\theta)\,, 2\,sin(\theta)\,, z\rangle$$
$$0 \leq \theta \leq 2\pi, \quad -\infty < z < \infty$$

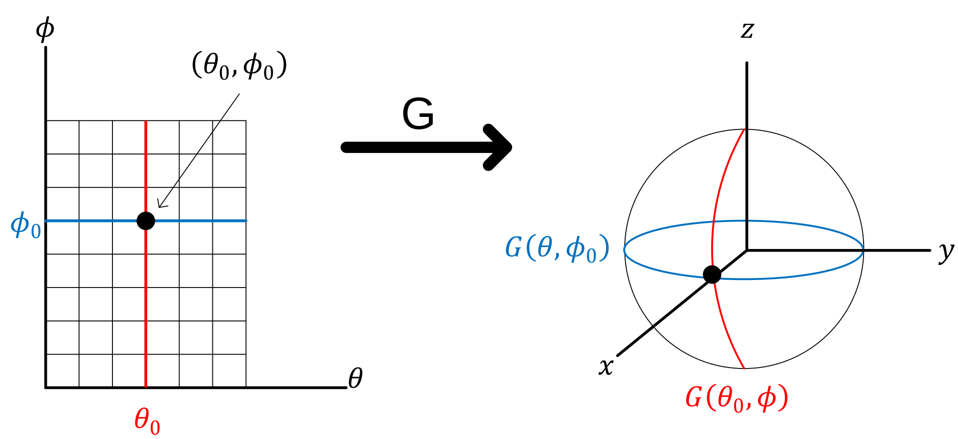
The mapping is shown for illustration.




**Example 2:** Find the parameterization of a sphere with a radius of 2, i.e. $x^2 + y^2 + z^2 = 4$.

Solution: A sphere is conveniently parameterized using spherical coordinates, $(\rho, \theta, \phi)$. Since the radius is fixed, we can eliminate $\rho$ as a parameter. Therefore, we have

$$\boldsymbol{G}(\theta, \phi) = \langle x(\theta, \phi), y(\theta, \phi), z(\theta, \phi)\rangle = \langle 2\,sin(\phi)\,cos(\theta)\,, 2\,sin(\phi)\,sin(\theta)\,, 2\,cos(\phi)\rangle$$
$$0 \leq \theta \leq 2\pi, \quad 0 < \phi < \pi$$



**Example 3:** Find the parameterization of the surface given by $f(x, y) = x^2 + y^2$.

Solution: In this case, since we can explicitly write the third variable as a function of the other two, the parameterization is simple, i.e. $u = x, v = y$

Therefore,

$$\boldsymbol{G}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle = \langle u, v, u^2 + v^2\rangle$$
$$-\infty < u < \infty, \quad -\infty < v < \infty$$

For these cases we generally keep $x$ and $y$ as the variables

$$\boldsymbol{G}(x, y) = \langle x, y, z(x, y)\rangle = \langle x, y, x^2 + y^2\rangle$$
$$-\infty < x < \infty, \quad -\infty < y < \infty$$

*Scalar Surface Area Derivation*

Recall, the scalar line integral of a function, $f(x, y, z)$, over a curve, $C$, was given as

$$\int_C f(x, y, z)ds$$

Where, $ds$ is an infinitesimal length of the curve.

Analogously, the scalar surface integral of a function, $f(x, y, z)$, over a surface, $\mathcal{S}$, can be written as

$$\iint_{\mathcal{S}} f(x, y, z)dS$$

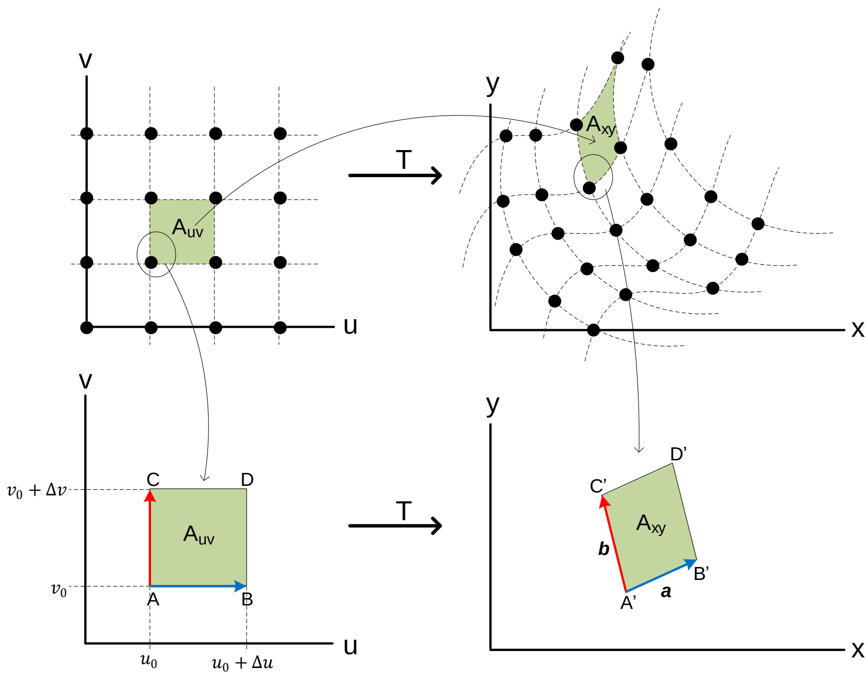
Where, $dS$ is an infinitesimal two dimensional portion of the surface.

In order to evaluate the scalar line integral, we parameterized the curve and showed that the infinitesimal length can be written as $ds = \|\boldsymbol{r}'(t)\|dt$, giving us the following relationship.

$$\int_C f(x, y, z)ds = \int_a^b f(\boldsymbol{r}(t))\|\boldsymbol{r}'(t)\|dt$$

Therefore, it would seem that in order to evaluate the surface integral we need to find a representation of $dS$ in terms of the parameterized surface, $\boldsymbol{G}(u, v)$.

The process is very similar to what was done in the *Change of Variables* lesson. In that lesson we started by imagining a linear mapping of a small rectangular region in $uv$-space to a small parallelogram in $xy$-space as shown below.




The area of this small parallelogram is given by the magnitude of the cross product of the two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\Delta A_{xy} = \|\boldsymbol{a} \times \boldsymbol{b}\| = \left\|\overrightarrow{A'B'} \times \overrightarrow{A'C'}\right\|$$

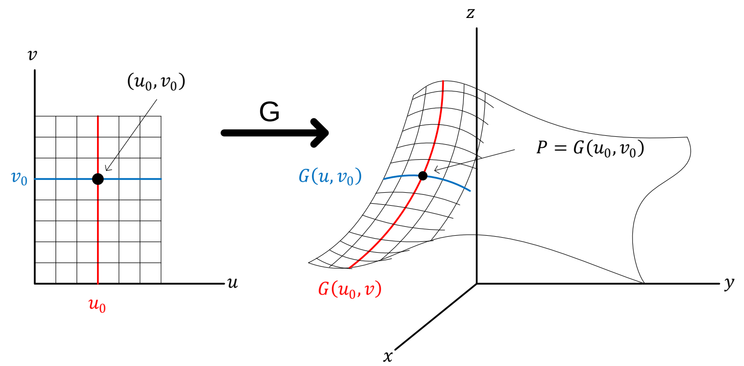
Where $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written as

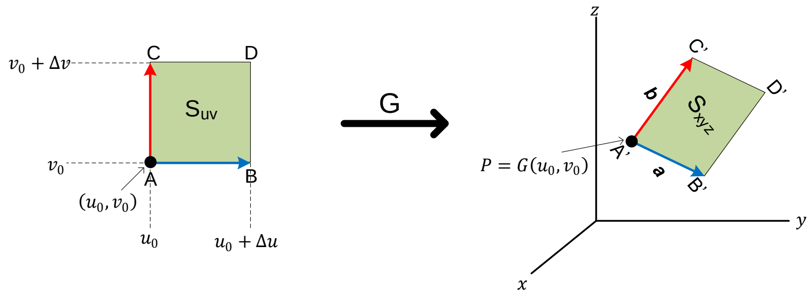
$$\boldsymbol{a} = \langle \boldsymbol{T}(u_0 + \Delta u, v_0) - \boldsymbol{T}(u_0, v_0) \rangle$$

$$\boldsymbol{b} = \langle \boldsymbol{T}(u_0, v_0 + \Delta v) - \boldsymbol{T}(u_0, v_0) \rangle$$

With $\boldsymbol{T}(u, v) = \langle x(u, v), y(u, v) \rangle$

The case here is completely analogous except that we are mapping to a three dimensional space, using $\boldsymbol{G}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$




Therefore, the vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, are given as

$$\boldsymbol{a} = \boldsymbol{G}(u_0 + \Delta u, v_0) - \boldsymbol{G}(u_0, v_0)$$

$$\boldsymbol{b} = \boldsymbol{G}(u_0, v_0 + \Delta v) - \boldsymbol{G}(u_0, v_0)$$

Then, as we did in the previous lesson, we call to mind the definition of the partial derivative, which is shown below as an approximation without the limit.

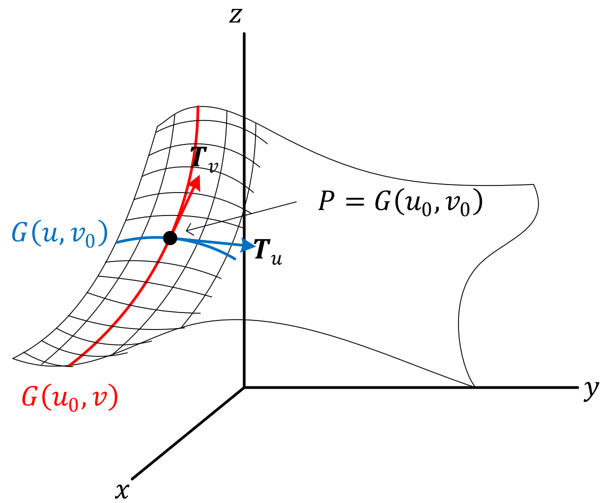
$$\boxed{\frac{\partial f(x_0, y)}{\partial x} \cong \frac{f(x_0 + \Delta x, y) - f(x_0, y)}{\Delta x}}$$

And use it to rewrite the vectors using the approximation as follows:

$$\boldsymbol{a} = \boldsymbol{G}(u_0, v_0 + \Delta v) - \boldsymbol{G}(u_0, v_0) \cong \frac{\partial \boldsymbol{G}(u_0, v_0)}{\partial u}\Delta u$$

$$\boldsymbol{b} = \boldsymbol{G}(u_0, v_0 + \Delta v) - \boldsymbol{G}(u_0, v_0) \cong \frac{\partial \boldsymbol{G}(u_0, v_0)}{\partial v}\Delta v$$

Furthermore, as we illustrate in the figure below, the partials above are equivalent to the tangent vectors to the grid curves, $\boldsymbol{G}(u, v_0)$, and $\boldsymbol{G}(u_0, v)$, which we represent as $\boldsymbol{T}_u$ and $\boldsymbol{T}_v$ .




$$\boldsymbol{T}_u(u_0, v_0) = \frac{\partial \boldsymbol{G}(u_0, v_0)}{\partial u} = \langle \frac{\partial}{\partial u} x(u_0, v_0), \frac{\partial}{\partial u} y(u_0, v_0), \frac{\partial}{\partial u} z(u_0, v_0) \rangle$$

$$\boldsymbol{T}_v(u_0, v_0) = \frac{\partial \boldsymbol{G}(u_0, v_0)}{\partial v} = \langle \frac{\partial}{\partial v} x(u_0, v_0), \frac{\partial}{\partial v} y(u_0, v_0), \frac{\partial}{\partial v} z(u_0, v_0) \rangle$$

Additionally, the normal vector to the tangent plane formed by $\boldsymbol{T}_u(u_0, v_0)$ and $\boldsymbol{T}_v(u_0, v_0)$ is given by the cross product as follows

$$\boldsymbol{N}(u_0, v_0) = \boldsymbol{T}_u(u_0, v_0) \times \boldsymbol{T}_v(u_0, v_0)$$

The area of a small parallelogram formed by $\boldsymbol{T}_u(u_0, v_0)$ and $\boldsymbol{T}_v(u_0, v_0)$ is then given by

$$\Delta S_{xyz} = \|\boldsymbol{a} \times \boldsymbol{b}\| \; = \|\boldsymbol{N}(u_0, v_0)\| \Delta u \Delta v$$

And if we allow $\Delta S_{xyz} \to dS$, we can write an expression for an infinitesimal surface area as

$$dS = \|\boldsymbol{N}(u, v)\| du dv$$

Finally, returning to the surface integral we can replace $dS$, giving us an integral that can more readily be evaluated.

$$\iint_S f(x, y, z) dS = \iint_D f(\boldsymbol{G}(u, v)) \, \|\boldsymbol{N}(u, v)\| du dv$$

| *Scalar Surface Integral* |
|---|
| Let $\boldsymbol{G}(u,v)$ be a parameterization of a surface, $\mathcal{S}$, on the domain. The scalar surface integral of the function $f(x,y,z)$ over the surface on the given domain is<br>$$\iint_{\mathcal{S}} f(x,y,z)dS = \iint_D f(\boldsymbol{G}(u,v))\,\|\boldsymbol{N}(u,v)\|dudv$$<br>For $f(x,y,z)=1$, we obtain the surface area on the domain $D$.<br>$$Area(\mathcal{S}) = \iint_D \|\boldsymbol{N}(u,v)\|dudv$$ |

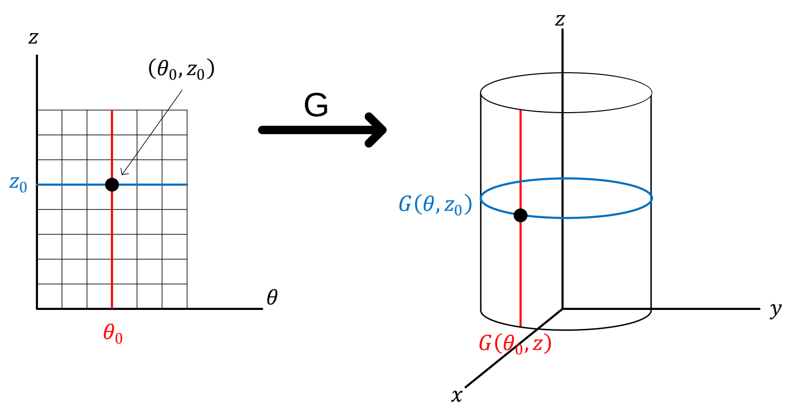
**Example 4:** Using the parameterization of the cylinder from example 1,

1. Describe the grid curves
2. Compute $\boldsymbol{T}_\theta$, $\boldsymbol{T}_z$, and $\boldsymbol{N}(\theta,z)$
3. Find the equation of the tangent plane at $P=(\pi/4\,,5)$
4. Compute the surface area of the cylinder for $0\le z\le 5$
5. Find the total charge on the surface if the charge density is $\delta(\theta,z)=0.003z\,sin^2(\theta)$.

Solution:

1. The parameterization from example 1 was given as shown below.

$$\boldsymbol{G}(\theta,z)=\langle 2\,cos(\theta)\,,2\,sin(\theta)\,,z\rangle$$
$$0\le\theta\le 2\pi,\quad -\infty<z<\infty$$




There are two grid curves, each representing a ‘freezing’ of one of the parameters.

| | | |
|---|---|---|
| $\theta$-Grid curve | $\boldsymbol{G}(\theta,z_0)=\langle 2\,cos(\theta)\,,2\,sin(\theta)\,,z_0\rangle$ | Circle of radius 2 |
| $z$-Grid curve | $\boldsymbol{G}(\theta_0,z)=\langle 2\,cos(\theta_0)\,,2\,sin(\theta_0)\,,z\rangle$ | Vertical line |

2. The tangent vector corresponding to the $\theta$-Grid curve is $\boldsymbol{T}_\theta$ is and $\boldsymbol{T}_z$ is the tangent vector corresponding to the $z$-Grid curve.

$$\boldsymbol{T}_\theta = \frac{\partial \boldsymbol{G}(\theta, z_0)}{\partial \theta}$$
$$= \langle \frac{\partial}{\partial \theta} x(\theta, z_0), \frac{\partial}{\partial \theta} y(\theta, z_0), \frac{\partial}{\partial \theta} z(\theta, z_0) \rangle$$
$$= \langle \frac{\partial}{\partial \theta}(2\, cos(\theta)), \frac{\partial}{\partial \theta}(2\, sin(\theta)), \frac{\partial}{\partial \theta} z_0 \rangle$$
$$\boldsymbol{T}_\theta = \langle -2\, sin(\theta)\, , 2\, cos(\theta)\, , 0 \rangle$$

$$\boldsymbol{T}_z = \frac{\partial \boldsymbol{G}(\theta_0, z)}{\partial z}$$
$$= \langle \frac{\partial}{\partial z} x(\theta_0, z), \frac{\partial}{\partial z} y(\theta_0, z), \frac{\partial}{\partial z} z(\theta_0, z) \rangle$$
$$= \langle \frac{\partial}{\partial z}(2\, cos(\theta)), \frac{\partial}{\partial z}(2\, sin(\theta)), \frac{\partial}{\partial z} z \rangle$$
$$\boldsymbol{T}_z = \langle 0, 0, 1 \rangle$$

The normal vector is then given by the cross product

$$\boldsymbol{N}(\theta, z) = \boldsymbol{T}_\theta \times \boldsymbol{T}_z$$
$$\boldsymbol{N}(\theta, z) = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ -2\, sin(\theta) & 2\, cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
$$\boldsymbol{N}(\theta) = \langle 2\, cos(\theta)\, , 2\, sin(\theta)\, , 0 \rangle$$

As expected, the normal vector is not dependent on $z$.

3. Recall from an earlier lesson that the tangent plane is found by noting that the dot product between the normal vector and an arbitrary vector in the plane the plane is zero. The vector in the plane we use begins at the point $(\theta_0, z_0) = (\pi/4\, , 5)$, which corresponds to

$$(x_0, y_0, z_0) = (2\, cos(\theta_0)\, , 2\, sin(\theta_0)\, , z_0) = (2\, cos(\pi/4)\, , 2\, sin(\pi/4)\, , 5) = \left(\sqrt{2}, \sqrt{2}, 5\right)$$

Therefore, the equation of the tangent plane is

$$\boldsymbol{N}(\theta_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$
$$\langle \sqrt{2}, \sqrt{2}, 0 \rangle \cdot \langle x - \sqrt{2}, y - \sqrt{2}, z - 5 \rangle = 0$$
$$\sqrt{2}x - 2 + \sqrt{2}y - 2 = 0$$
$$x + y = 2\sqrt{2}$$

4. The surface area of the cylinder with a height of 5 is found as follows

$$Area(\mathcal{S}) = \iint_D \|\mathbf{N}(\theta, z)\| d\theta dz$$

$$= \int_0^5 \int_0^{2\pi} \|\langle 2\cos(\theta), 2\sin(\theta), 0\rangle\| d\theta dz$$

$$= \int_0^5 \int_0^{2\pi} \sqrt{4\cos^2(\theta) + 4\sin^2(\theta)}\, d\theta dz$$

$$= \int_0^5 \left(\int_0^{2\pi} 2d\theta\right) dz$$

$$= \int_0^5 (4\pi)\, dz = 20\pi$$

5. The total charge is found using the scalar surface integral just as we did in a previous lesson with the scalar line integral for total charge of a thin rod. In this case, since the charge density is already given in cylindrical coordinates, we can use it directly in the integral

$$Q = \iint_D \delta(\theta, z)\, \|\mathbf{N}(\theta, z)\| d\theta dz$$

$$= \int_0^5 \int_0^{2\pi} \big((0.003z\sin^2(\theta)) \cdot 2\big) d\theta dz$$

$$= 0.006 \int_0^5 z\left(\int_0^{2\pi} \left(\frac{1 - \cos(2\theta)}{2}\right) d\theta\right) dz$$

$$= 0.006 \int_0^5 z\left(\frac{1}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi}\right) dz$$

$$= 0.006 \int_0^5 z(\pi)\, dz$$

$$= 0.006\pi \int_0^5 z\, dz$$

$$= 0.006\pi\left(\frac{25}{2}\right)$$

$$Q = 0.075\pi\ C$$

**Example 5:** Compute the scalar surface integral of $f(x,y,z) = z(x^2+y^2)$ over the parameterized surface

$$\boldsymbol{G}(u,v) = \langle u\cos(v), u\sin(v), u\rangle, \qquad 0 \le u \le 1, \qquad 0 \le v \le 1$$

Solution: Since the surface is already parameterized, we can directly use the surface integral

$$\iint_D f(\boldsymbol{G}(u,v))\,\|\boldsymbol{N}(u,v)\|dudv$$

The first term in the integrand is

$$f(\boldsymbol{G}(u,v)) = u(u^2\cos^2(v) + u^2\sin^2(v)) = u\cdot u^2(\cos^2(v)+\sin^2(v)) = u^3$$

Next, we find $\|\boldsymbol{N}(u,v)\|$

$$\begin{aligned}\boldsymbol{T}_u &= \frac{\partial \boldsymbol{G}(u,v)}{\partial u}\\ &= \langle\frac{\partial}{\partial u}(u\cos(v)), \frac{\partial}{\partial u}(u\sin(v)), \frac{\partial}{\partial u}u\rangle\\ &= \langle\cos(v), \sin(v), 1\rangle\end{aligned}$$

$$\begin{aligned}\boldsymbol{T}_v &= \frac{\partial \boldsymbol{G}(u,v)}{\partial v}\\ &= \langle\frac{\partial}{\partial v}(u\cos(v)), \frac{\partial}{\partial v}(u\sin(v)), \frac{\partial}{\partial v}u\rangle\\ &= \langle -u\sin(v), u\cos(v), 0\rangle\end{aligned}$$

$$\begin{aligned}\boldsymbol{N}(u,v) &= \boldsymbol{T}_u \times \boldsymbol{T}_v\\ &= \begin{vmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}}\\ \cos(v) & \sin(v) & 1\\ -u\sin(v) & u\cos(v) & 0\end{vmatrix}\\ &= \langle -u\cos(v), u\sin(v), u\cos^2(v) + u\sin^2(v)\rangle\\ &= \langle -u\cos(v), u\sin(v), u\rangle\end{aligned}$$

$$\|\boldsymbol{N}(u,v)\| = \sqrt{u^2\cos^2(v) + u^2\sin^2(v) + u^2} = \sqrt{u^2+u^2} = \sqrt{2}u$$

Finally, the integral is evaluated as follows.

$$\begin{aligned}\iint_D f(\boldsymbol{G}(u,v))\,\|\boldsymbol{N}(u,v)\|dudv &= \int_0^1\int_0^1 (u^3)(\sqrt{2}u)dudv\\ &= \sqrt{2}\int_0^1 u^4\left(\int_0^1 dv\right)du = \sqrt{2}\int_0^1 u^4 du = \frac{\sqrt{2}}{5}\end{aligned}$$

**Example 6:** Compute the scalar surface integral of $f(x, y, z) = x^2$ over the surface given by

$$x^2 + y^2 + z^2 = 1, \qquad x, y, z \geq 0$$

Solution: The surface is a unit sphere; therefore, we parameterize with spherical coordinates.

$$\boldsymbol{G}(\theta, \phi) = \langle x(\theta, \phi), y(\theta, \phi), z(\theta, \phi)\rangle = \langle sin(\phi)\, cos(\theta)\, , sin(\phi)\, sin(\theta)\, , cos(\phi)\rangle$$

Therefore,

$$f\big(\boldsymbol{G}(\theta, \phi)\big) = sin^2(\phi)\, cos^2(\theta)$$

Next, we find $\|\boldsymbol{N}(\theta, \phi)\|$

$$\begin{aligned} \boldsymbol{T}_\theta &= \frac{\partial \boldsymbol{G}(\theta, \phi)}{\partial \theta} \\ &= \langle \frac{\partial}{\partial \theta}(sin(\phi)\, cos(\theta)), \frac{\partial}{\partial \theta}(sin(\phi)\, sin(\theta)), \frac{\partial}{\partial \theta}(cos(\phi))\rangle \\ &= \langle - sin(\phi)\, sin(\theta)\, , sin(\phi)\, cos(\theta)\, , 0\rangle \end{aligned}$$

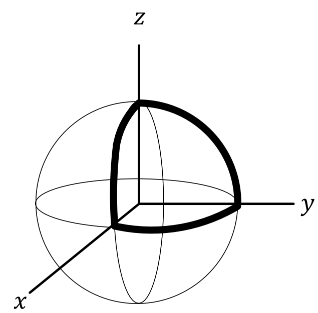
$$\begin{aligned} \boldsymbol{T}_\phi &= \frac{\partial \boldsymbol{G}(\theta, \phi)}{\partial \phi} \\ &= \langle \frac{\partial}{\partial \phi}(sin(\phi)\, cos(\theta)), \frac{\partial}{\partial \phi}(sin(\phi)\, sin(\theta)), \frac{\partial}{\partial \phi}(cos(\phi))\rangle \\ &= \langle cos(\phi)\, cos(\theta)\, , sin(\theta)\, cos(\phi)\, , - sin(\phi)\rangle \end{aligned}$$

$$\begin{aligned} \boldsymbol{N}(u, v) &= \boldsymbol{T}_u \times \boldsymbol{T}_v \\ &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ - sin(\phi)\, sin(\theta) & sin(\phi)\, cos(\theta) & 0 \\ cos(\phi)\, cos(\theta) & sin(\theta)\, cos(\phi) & - sin(\phi) \end{vmatrix} \\ &= \langle - sin^2(\phi)\, cos(\theta)\, , - sin^2(\phi)\, sin(\theta)\, , -(sin^2(\theta)\, cos(\phi)\, sin(\phi) \\ &\qquad + cos^2(\theta)\, sin(\phi)\, cos(\phi))\rangle \\ &= \langle - sin^2(\phi)\, cos(\theta)\, , - sin^2(\phi)\, sin(\theta)\, , - cos(\phi)\, sin(\phi)\rangle \end{aligned}$$

$$\begin{aligned} \|\boldsymbol{N}(u, v)\| &= \sqrt{sin^4(\phi)\, cos^2(\theta) + sin^4(\phi)\, sin^2(\theta) + cos^2(\phi)\, sin^2(\phi)} \\ &= \sqrt{sin^4(\phi) + cos^2(\phi)\, sin^2(\phi)} \\ &= \sqrt{sin^2(\phi)\, (sin^2(\phi) + cos^2(\phi))} \\ &= sin(\phi) \end{aligned}$$

The domain of integration, shown below, corresponds to the following parameterized variables.

$$0 \leq \theta \leq \pi/2, \qquad 0 \leq \phi \leq \pi/2$$




$$\iint_D f(\boldsymbol{G}(\theta,\phi))\,\|\boldsymbol{N}(\theta,\phi)\| d\theta d\phi = \int_0^{\pi/2}\int_0^{\pi/2} (sin^2(\phi)\,cos^2(\theta))(sin(\phi))d\theta d\phi$$
$$= \int_0^{\pi/2} cos^2(\theta)\,d\theta \int_0^{\pi/2} sin^3(\phi)\,d\phi$$

We solve each integral separately below, using substitution for the second.

$$\int_0^{\pi/2} cos^2(\theta)\,d\theta = \frac{1}{2}\int_0^{\pi/2}(1 + cos(2\theta))d\theta$$
$$= \frac{1}{2}\left(\theta + \frac{1}{2}sin(2\theta)\Big|_0^{\pi/2}\right)$$
$$= \frac{1}{2}(\pi/2) = \frac{\pi}{4}$$

$$\int_0^{\pi/2} sin^3(\phi)\,d\phi = \int_0^{\pi/2} sin(\phi)\,(1 - cos^2(\phi))d\phi$$
$$= -\int_1^0 (1 - u^2)du$$
$$= \int_0^1 (1 - u^2)du$$
$$u - \frac{1}{3}u^3\Big|_0^1 = \frac{2}{3}$$

Finally, we have

$$\iint_D f(\boldsymbol{G}(\theta,\phi))\,\|\boldsymbol{N}(\theta,\phi)\| d\theta d\phi = \int_0^{\pi/2} cos^2(\theta)\,d\theta \int_0^{\pi/2} sin^3(\phi)\,d\phi = \left(\frac{\pi}{4}\right)\left(\frac{2}{3}\right) = \frac{\pi}{6}$$

**Example 7:** Compute the scalar surface integral of $f(x,y,z) = z - x$ over the portion of the surface given by the graph $z = x + y^2$ where $0 \leq x \leq y,\ \ 0 \leq y \leq 1$.

Solution: Recall from example 3 if the surface can be represented as $z = g(x,y)$, we can use a simple parameterization as follows

$$\boldsymbol{G}(x,y) = \langle x, y, g(x,y)\rangle$$

In this case the normal vector is found as follows

$$\begin{aligned}\boldsymbol{T}_x &= \frac{\partial \boldsymbol{G}}{\partial x}\\ &= \langle \frac{\partial}{\partial x}x, \frac{\partial}{\partial x}y, \frac{\partial}{\partial x}g(x,y)\rangle\\ &= \langle 1,0,g_x\rangle\end{aligned}$$

$$\begin{aligned}\boldsymbol{T}_y &= \frac{\partial \boldsymbol{G}}{\partial y}\\ &= \langle \frac{\partial}{\partial y}x, \frac{\partial}{\partial y}y, \frac{\partial}{\partial y}g(x,y)\rangle\\ &= \langle 0,1,g_y\rangle\end{aligned}$$

$$\boldsymbol{N}(x,y) = \boldsymbol{T}_x \times \boldsymbol{T}_y = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{vmatrix} = \langle -g_x, -g_y, 1\rangle$$

Therefore,

$$\boldsymbol{N}(x,y) = \sqrt{{g_x}^2 + {g_y}^2 + 1}$$

With this we can state the following

| ***Scalar Surface Integral over a Surface $z = g(x,y)$*** |
|---|
| The scalar surface integral of the function $f(x,y,z)$ over a portion of a surface that can be represented as $z = g(x,y)$, is given as $$\iint_S f(x,y,z)dS = \iint_D f\big(x,y,g(x,y)\big)\left(\sqrt{{g_x}^2 + {g_y}^2 + 1}\right)dxdy$$ |

For the problem at hand we have

$$\iint_D f(x,y,g(x,y))\left(\sqrt{{g_x}^2+{g_y}^2+1}\right)dxdy = \int_0^1\int_0^y ((x+y^2)-x)\left(\sqrt{1+4y^2+1}\right)dxdy$$

$$= \int_0^1\left(\int_0^y \left(y^2\sqrt{2+4y^2}\right)dx\right)dy$$

$$= \int_0^1 \left(y^3\sqrt{2+4y^2}\right)dy$$

While a substitution may not look obvious at first, we can evaluate this integral as follows

$$u = 2+4y^2 \qquad\qquad du = 8ydy$$

$$\int_0^1\left(y^3\sqrt{2+4y^2}\right)dy = \frac{1}{8}\int_2^6 (y^2\sqrt{u})du$$

$$= \frac{1}{8}\int_2^6\left(\left(\frac{u-2}{4}\right)\sqrt{u}\right)du$$

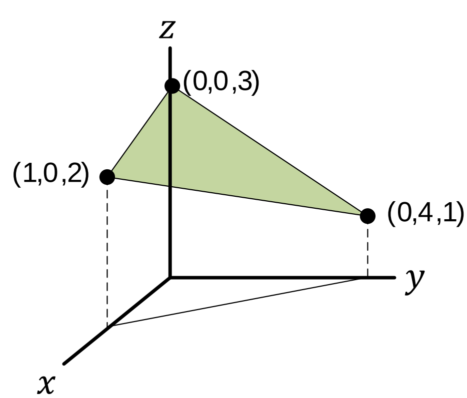
$$= \frac{1}{32}\int_2^6 (u^{3/2}-2u^{1/2})du$$

$$= \frac{1}{32}\left(\frac{2}{5}u^{5/2}-\frac{4}{3}u^{3/2}\Big|_2^6\right)$$

$$\cong 0.54$$

**Example 7:** Compute the scalar surface integral of $f(x,y,z) = xy + e^z$ over the surface of a triangle with vertices $(0,0,3)$, $(1,0,2)$, and $(0,4,1)$

Solution: The surface is shown below.



We can find an equation for the plane by first solving for a normal vector as follows

$$\begin{aligned}\boldsymbol{n} &= \langle(0,4,1)-(1,0,2)\rangle \times \langle(0,0,3)-(1,0,2)\rangle \\ &= \langle -1,4,-1\rangle \times \langle -1,0,1\rangle \\ &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ -1 & 4 & -1 \\ -1 & 0 & 1 \end{vmatrix} = \langle 4,2,4\rangle\end{aligned}$$

The equation of the plane is then

$$\begin{aligned}\boldsymbol{n}\cdot\overrightarrow{PP_0} &= 0 \\ \langle 2,1,2\rangle\cdot\langle(x,y,z)-(0,0,3)\rangle &= 0 \\ \langle 2,1,2\rangle\cdot\langle x,y,z-3\rangle &= 0 \\ 2x+y+2z-6 &= 0\end{aligned}$$

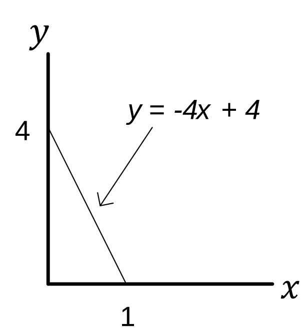
$$z=-x-\frac{1}{2}y+3$$

Since the surface can be represented as $z=g(x,y)$, the surface integral is

$$\iint_D f(x,y,g(x,y))\left(\sqrt{{g_x}^2+{g_y}^2+1}\right)dxdy=\iint_D \left(xy+e^{\left(-x-\frac{1}{2}y+3\right)}\right)\left(\sqrt{1+\frac{1}{4}+1}\right)dxdy$$

$$=\frac{3}{2}\iint_D \left((xy)+e^{\left(-x-\frac{1}{2}y+3\right)}\right)dxdy$$

The domain in the $xy$ plane is found as follows.




$$0\le x\le 1,\qquad 0\le y\le -4x+4$$

Therefore,

$$\frac{3}{2}\iint_D \left((xy) + e^{\left(-x-\frac{1}{3}y+3\right)}\right) dxdy = \frac{3}{2}\int_0^1 \left(\int_0^{4-4\text{x}} \left(xy + e^3 e^{-x} e^{-\frac{1}{2}y}\right) dy\right) dx$$

Let's evaluate the inner integral first

$$\int_0^{4-4x} \left(xy + e^3 e^{-x} e^{-\frac{1}{2}y}\right) dy = \left(x\int_0^{4-4x} y\, dy\right) + \left(e^3 e^{-x}\int_0^{4-4x} e^{-\frac{1}{2}y}\, dy\right)$$

$$= \left(\frac{x}{2}(4-4x)^2\right) + \left(-2e^3 e^{-x}\left(e^{-\frac{1}{2}(-4x+4)} - 1\right)\right)$$

$$= 8x - 16x^2 + 8x^3 - 2e^1 e^x + 2e^3 e^{-x}$$

Therefore,

$$\frac{3}{2}\int_0^1 \left(\int_0^{-4x+4} \left(xy + e^3 e^{-x} e^{-\frac{1}{2}y}\right) dy\right) dx = \frac{3}{2}\int_0^1 (8x - 16x^2 + 8x^3 - 2e^1 e^x + 2e^3 e^{-x})\, dx$$

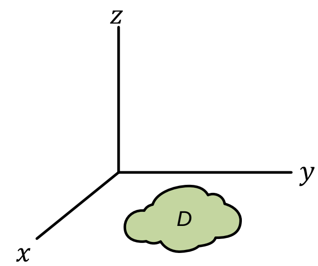
$$= \frac{3}{2}\left(4x^2 - \frac{16}{3}x^3 + 2x^4 - 2e^1 e^x - 2e^3 e^{-x}\right)\Big|_0^1$$

$$= \frac{3}{2}\left(4 - \frac{16}{3} + 2 - 2e^2 - 2e^2\right) - (-2e^1 - 2e^3 e^{-x})$$

$$= \frac{3}{2}\left(\frac{2}{3} - 4e^2 + 2e^1 + 2e^3\right)$$

$$\cong 25.08$$

**Example 9:** What is the area of the portion of the plane $2x + 3y + 4z = 28$ lying above the domain $D$ in the $xy$-plane shown below, if $Area(D) = 5$?




Solution: As shown earlier, the surface area for a surface in $xyz$ space on a domain $D$ in the $uv$-plane is given as

$$Area(\mathcal{S}) = \iint_D \|\boldsymbol{N}(u,v)\| du dv$$

When the mapping is linear, as is the case for the surface of a plane, the integral can be removed, and we can write the following

$$Area\big(G(D)\big) = \|\boldsymbol{N}(u,v)\| Area(D)$$

We can find the normal vector by first expressing the plane in the form: $z = g(x,y)$

$$2x + 3y + 4z = 28$$
$$z = 7 - \frac{1}{2}x - \frac{3}{4}y = g(x,y)$$

Therefore,

$$\|\mathbf{N}\| = \sqrt{g_x^2 + g_y^2 + 1} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2 + 1} = \frac{\sqrt{29}}{4}$$

And

$$Area\big(G(D)\big) = \|\boldsymbol{N}(u,v)\| Area(D) = \frac{\sqrt{29}}{4} \cdot 5 \cong 6.73$$

## *Final Summary for Line and Surface Integrals – Scalar Surface Integrals*

| ***Scalar Surface Integral*** |
|---|
| Let $\boldsymbol{G}(u,v)$ be a parameterization of a surface, $\mathcal{S}$, on the domain. The scalar surface integral of the function $f(x,y,z)$ over the surface on the given domain is<br><br>$$\iint_{\mathcal{S}} f(x,y,z)dS = \iint_D f(\boldsymbol{G}(u,v))\,\|\boldsymbol{N}(u,v)\|dudv$$<br><br>For $f(x,y,z)=1$, we obtain the surface area on the domain $D$.<br><br>$$Area(\mathcal{S}) = \iint_D \|\boldsymbol{N}(u,v)\|dudv$$ |
| ***Scalar Surface Integral over a Surface*** $\boldsymbol{z = g(x,y)}$ |
| The scalar surface integral of the function $f(x,y,z)$ over a portion of a surface that can be represented as $z=g(x,y)$, is given as<br><br>$$\iint_{\mathcal{S}} f(x,y,z)dS = \iint_D f\big(x,y,g(x,y)\big)\left(\sqrt{{g_x}^2+{g_y}^2+1}\right)dxdy$$ |

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