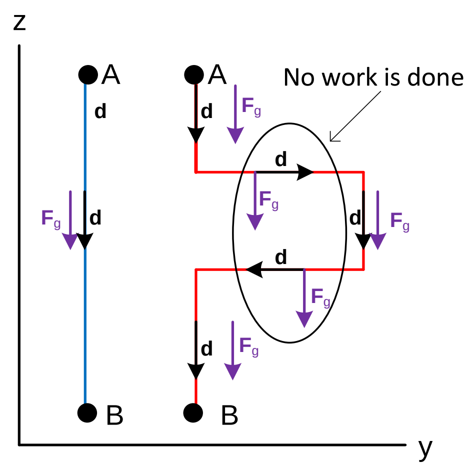
Line and Surface Integrals – Conservative Vector Fields

For the first lesson in this series we introduced vector fields. In that lesson we briefly discussed conservative vector fields where we mentioned a key property of conservative vector fields - If a scalar potential function exists for a given vector field that vector field is conservative. Most notably, the gradient vector field is a conservative vector field, $\boldsymbol{F} = \nabla f$, where f is the potential function. We also discussed the path independence property of conservative vector fields. However, at the time we did not yet have the tools to explore this property in detail. In this lesson, using the knowledge gained in our lesson on line integrals, we can study conservative vector fields in more depth.

Path Independence

We originally used the example of the work done when an object moves under the force of gravity to introduce path independence. We use the figure below to illustrate how the same amount of work is done for the direct path (in blue) or the indirect path (in red). It should also be obvious from the blue path that if the object instead moved from B to A the amount of work would be equal to the negative of the amount of work done from A to B. In other words when traveling in a closed loop, e.g. $A \rightarrow B \rightarrow A$, the total work is zero.



These two properties are defined in the *Fundamental Theorem for Conservative Vector Fields*.

The Fundamental Theorem for Conservative Vector Fields
Assume $\boldsymbol{F} = \nabla f$ on a domain D. 1. If $\boldsymbol{r}$ is a path along a curve C from A to B in D, then $$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = f(B) - f(A)$$ In other words, $\boldsymbol{F}$ is path-independent 2. The circulation around a closed curve C, (i.e. $A = B$) is zero $$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$$

Proof: Assume a path $\boldsymbol{r}(t)$ is defined in a conservative vector field, $\boldsymbol{F}$, for $a \leq t \leq b$, corresponding to the points, $\boldsymbol{r}(a) = A$ and $\boldsymbol{r}(b) = B$**.** Since $\boldsymbol{F}$ is conservative, i.e. $\boldsymbol{F} = \nabla f$, The vector line integral is then given as

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b \boldsymbol{F}\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{r}'(t)dt = \int_a^b \nabla f\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{r}'(t)dt$$

Next, using the multivariate chain rule we can write

$$\frac{d}{dt} f\big(\boldsymbol{r}(t)\big) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \cdot \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle = \nabla f \cdot \boldsymbol{r}'(t)$$

Therefore,

$$\int_a^b \nabla f\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{r}'(t)dt = \int_a^b \frac{d}{dt} f\big(\boldsymbol{r}(t)\big)\, dt$$

Finally, using the Fundamental Theorem of Calculus we can write

$$\boxed{\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b \frac{d}{dt} f\big(\boldsymbol{r}(t)\big)\, dt = f\big(\boldsymbol{r}(b)\big) - f\big(\boldsymbol{r}(a)\big) = f(B) - f(A)}$$

Note: This also proves the 2nd statement from the Theorem since if $A = B$ we have

$$\boxed{\int_a^a \frac{d}{dt} f\big(\boldsymbol{r}(t)\big)\, dt = f\big(\boldsymbol{r}(a)\big) - f\big(\boldsymbol{r}(a)\big) = f(A) - f(A) = 0}$$

Example 1: Let $\boldsymbol{F}(x, y, z) = \langle 2xy + z, x^2, x \rangle$

1. Evaluate the vector line integral for straight line path from $A = (1, -1, 2)$ to $B = (2, 2, 3)$.
2. Verify that $f(x, y, z) = x^2 y + xz$ is a potential function for $\boldsymbol{F}$.
3. Evaluate the vector line integral again using the conservative vector field theorem.

Solution:

1. The parameterization of the straight line path is given as

$$\boldsymbol{r}_{AB}(t) = \langle 1, -1, 2\rangle + \langle 1, 3, 1\rangle t = \langle 1+t, 3t-1, 2+t\rangle, \quad 0 \le t \le 1$$

Therefore,

$$\begin{aligned}\boldsymbol{F}\big(\boldsymbol{r}(t)\big) &= \langle 2(1+t)(3t-1) + (2+t), (1+t)^2, 1+t\rangle \\ &= \langle 6t^2 + 5t, t^2 + 2t + 1, 1 + t\rangle\end{aligned}$$

The line integral is then evaluated as follows

$$\begin{aligned}\int_0^1 \boldsymbol{F}\left(\boldsymbol{r}(t)\right) \cdot \boldsymbol{r}'(t)dt &= \int_0^1 \langle 6t^2 + 5t, t^2 + 2t + 1, 1 + t\rangle \cdot \langle 1,3,1\rangle dt \\ &= \int_0^1 6t^2 + 5t + 3(t^2 + 2t + 1) + \ (1+t)\, dt \\ &= \int_0^1 (9t^2 + 12t + 4)\, dt \\ &= \ 3t^3 + 6t^2 + 4t|_0^1 \\ &= \ 13\end{aligned}$$

2. We can verify the potential function using the gradient relationship.

$$\begin{aligned}\boldsymbol{F} &= \nabla f \\ \langle 2xy + z, x^2, x\rangle &= \ \langle \frac{\partial}{\partial x}(x^2 y + xz), \frac{\partial}{\partial y}(x^2 y + xz), \frac{\partial}{\partial z}(x^2 y + xz)\rangle \\ \langle 2xy + z, x^2, x\rangle &= \ \langle 2xy + z, x^2, x\rangle\end{aligned}$$

3. Since $\boldsymbol{F}$ has a potential function it is conservative, and we can use the path independence property to more easily evaluate the vector line integral from part 1.

$$\begin{aligned}\int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \ f(B) - f(A) \\ &= \ (x^2 y + xz)|_{(2,2,3)} - (x^2 y + xz)|_{(1,-1,2)} \\ &= \ (4 \cdot 2 + 2 \cdot 3) - (1 \cdot -1 + 1 \cdot 2) \\ &= \ 14 - 1 \\ &= \ 13\end{aligned}$$

Remember, this answer does not depend on the particular path that was taken from A to B.

Example 2: Given the vector field, $\boldsymbol{F}(x, y, z) = \langle ye^z, xe^z, xye^z \rangle$, and a corresponding potential function, $f(x, y, z) = xye^z$, evaluate the vector line integral over the given path

$$\boldsymbol{r}(t) = \langle t^2, t^3, t - 1 \rangle, 1 \le t \le 2$$

Solution: Since $\boldsymbol{F}$ is conservative only the end points of the path are needed and not the exact path.

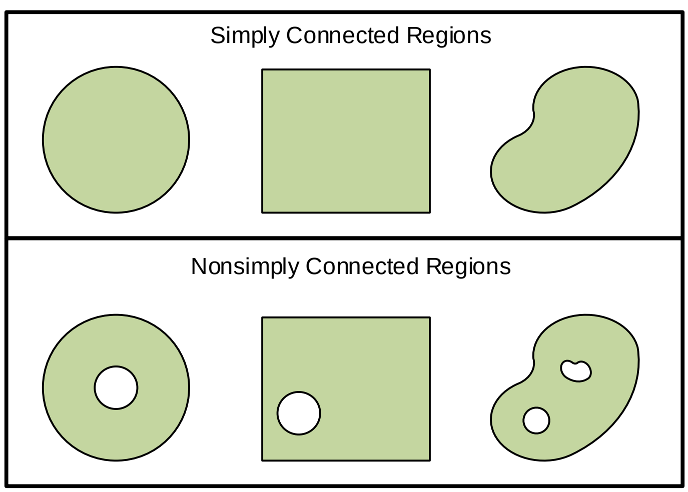
$$\begin{aligned}
\int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= f\big(\boldsymbol{r}(2)\big) - f\big(\boldsymbol{r}(1)\big) \\
&= f\big(\boldsymbol{r}(2)\big) - f\big(\boldsymbol{r}(1)\big) \\
&= (xye^z)|_{(4,8,1)} - (xye^z)|_{(1,1,0)} \\
&= (32e^1) - (1e^0) \\
&= 32e - 1
\end{aligned}$$

Finding Potential Functions

In the first lesson in this series we showed that the curl of a conservative vector field is zero. We did this by computing the curl of a gradient vector field, which is by definition conservative. This allowed us to write the so-called '*cross-partials condition'* for conservative vector fields.

$$curl(\boldsymbol{F}) = 0 \qquad \rightarrow \qquad \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \qquad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}, \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

It turns out that we must qualify this condition based on the domain over which we define the vector field. The qualifier is that the domain must be '*simply connected*'. Simply stated a simply connected domain must not have any holes. We illustrate below in two dimensions.



Conservative Vector Field Criteria
The vector field $\boldsymbol{F}$ is conservative on a simply connected domain, D, if $\boldsymbol{F}$ satisfies the cross-partials conditions.

We previously found potential functions for conservative vector fields in a somewhat ad-hoc fashion. The example illustrates a more algorithmic method.

Example 3: Find the potential function for the given the vector field

$$\boldsymbol{F}(x,y) = \langle 2xy + y^3, x^2 + 3xy^2 + 2y\rangle$$

Solution: In this case we have only one cross partial condition that must be satisfied

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

$$\frac{\partial}{\partial x}(x^2 + 3xy^2 + 2y) = \frac{\partial}{\partial y}(2xy + y^3)$$

$$2x + 3y^2 = 2xy + y^3$$

And since $\boldsymbol{F}$ is defined on all of R^2, which is a simply connected, $\boldsymbol{F}(x,y)$ is conservative.

To find the potential function we start by assuming the vector field has a potential function, f.

$$\boldsymbol{F}(x,y) = \nabla f$$

$$\langle 2xy + y^3, x^2 + 3xy^2 + 2y\rangle = \langle\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$$

For which we can extract the following equations.

$$\frac{\partial f}{\partial x} = 2xy + y^3 \qquad\qquad \frac{\partial f}{\partial y} = x^2 + 3xy^2 + 2y$$

We can solve these equations using the Fundamental Theorem of Calculus. For the first equation we assume f is a function of x only, i.e. treat the variable y as a constant.

$$f(x,y) = \int (2xy + y^3)dx$$

$$f(x,y) = x^2y + y^3x + g(y)$$

Note the integration 'constant' is a function of y since we assumed it was constant during the integration.

The second equation is similarly evaluated.

$$f(x,y) = \int (x^2 + 3xy^2 + 2y)dy$$

$$f(x,y) = x^2y + xy^3 + y^2 + h(x)$$

The two expressions for $f(x,y)$ of course must be equal.

$$x^2y + y^3x + g(y) = x^2y + xy^3 + y^2 + h(x)$$
$$g(y) = y^2 + h(x)$$

Which can be satisfied if $h(x) = 0$ and $g(y) = y^2$. Therefore, after adding another arbitrary constant we can write the potential function as

$$f(x,y) = x^2y + xy^3 + y^2 + C$$

Of course, the same method can be used in R^3.

Example 4: Find the potential function for the given the vector field

$$\boldsymbol{F}(x,y,z) = \langle 2xy + 5, x^2 - 4z, -4y \rangle$$

Solution: We start by checking the cross partial conditions.

$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$	$\frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$	$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$
$\frac{\partial}{\partial y}(-4y) = \frac{\partial}{\partial z}(x^2 - 4z)$	$\frac{\partial}{\partial x}(-4y) = \frac{\partial}{\partial z}(2xy + 5)$	$\frac{\partial}{\partial x}(x^2 - 4z) = \frac{\partial}{\partial y}(2xy + 5)$
$-4 = -4$	$0 = 0$	$2x = 2x$

In this case $\boldsymbol{F}$ is defined on all of R^3, which is also simply connected. Therefore, $\boldsymbol{F}$ is conservative.

To find the potential function we start by assuming the vector field has a potential function, f.

$$\boldsymbol{F}(x,y,z) = \nabla f$$
$$\langle 2xy + 5, x^2 - 4z, -4y \rangle = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$$

Which gives us three equations

1.

$$\frac{\partial f}{\partial x} = 2xy + 5$$
$$f(x,y,z) = \int (2xy + 5)dx$$
$$f(x,y,z) = x^2y + 5x + g(y,z)$$

2.

$$\frac{\partial f}{\partial y} = x^2 - 4z$$
$$f(x,y,z) = \int (x^2 - 4z)dy$$
$$f(x,y,z) = x^2y - 4zy + h(x,z)$$

3.

$$\frac{\partial f}{\partial z} = -4y$$
$$f(x,y,z) = \int (-4y)dz$$
$$f(x,y,z) = -4yz + w(x,y)$$

Equating the three expressions we have

$$x^2y + 5x + g(y,z) = x^2y - 4zy + h(x,z) = -4yz + w(x,y)$$

Which can be satisfied with

$$g(y,z) = -4yz \qquad h(x,z) = 5x \qquad w(x,y) = x^2y + 5x$$

$$x^2y + 5x + (-4yz) = x^2y - 4zy + (5x) = -4yz + (x^2y + 5x)$$

Therefore, we can write the potential function as

$$f(x,y,z) = x^2y + 5x - 4yz + C$$

Example 5: Find the potential function for the given the vector field

$$\boldsymbol{F}(x,y,z) = \langle cos(xz)\,, sin(yz)\,, xy\,sin(z)\,\rangle$$

Solution: The first partial condition is not satisfied; therefore, the vector field is not conservative and does not have a potential function.

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$$
$$\frac{\partial}{\partial y}(xy\,sin(z)) = \frac{\partial}{\partial z}(sin(yz))$$
$$xy\,sin(z) \neq y\,cos(yz)$$

Example 6: A vortex field is one that revolves around an axis, for example as in a tornado or water circling a drain. A vortex field that swirls around the origin in R^2 is defined as

$$\boldsymbol{F}(x, y) = \langle\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle$$

Determine if the field is conservative. Then find the vector line integral along a unit circle.

Solution: We check the cross partial condition first

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)$$

$$\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2}$$

$$\frac{y^2-x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Although the cross partial condition is satisfied the domain is the R^2 plane with the origin removed, ($\boldsymbol{F}(0,0) = \langle\frac{-0}{0}, \frac{0}{0}\rangle$,i.e. the origin is undefined), which is ***not*** a simply connected region. Therefore, the vortex field is not a conservative vector field.

If the field were conservative the vector line integral along the unit circle, i.e. a closed loop, would de zero. In this case we need to perform the integration. We start by parameterizing the circular path as

$$\boldsymbol{r}(t) = \langle cos(t), sin(t)\rangle$$

Therefore,

$$\int_0^{2\pi} \boldsymbol{F}(\boldsymbol{r}(t)) \cdot r'(t)dt = \int_0^{2\pi} \langle\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle\Big|_{(cos(t),sin(t))} \cdot \langle -sin(t), cos(t)\rangle dt$$

$$= \int_0^{2\pi} \langle -sin(t), cos(t)\rangle \cdot \langle -sin(t), cos(t)\rangle dt$$

$$= \int_0^{2\pi} (sin^2(t) + cos^2(t))\, dt = 2\pi$$

This results hints at a general formula for the line integral of a vortex field for a closed path that winds around the origin n times.

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 2\pi n$$

Where, n is called the *winding number* and is negative if the if the path winds around in the clockwise direction.

Conservative Fields in Physics

The conservation of energy principle is one of the most important laws in all of physics. It can be stated as follows:

In a closed system with only conservative forces present; the kinetic energy, K, and potential energy, U, may change but the total energy, E, remains constant. $$E = K + U$$

You may recall from basic physics that the kinetic energy is $K = 1/2\, mv^2$. Therefore, if we assume an object is moving along a path, $\boldsymbol{r}(t)$, under the influence a conservative force we can write the conservation of energy equation as follows.

$$E = (1/2\, m\boldsymbol{v} \cdot \boldsymbol{v}) + U\big(\boldsymbol{r}(t)\big)$$

Where, we wrote the velocity as a vector to be more precise.

Next, we can take the time derivative of both sides, knowing that E is a constant.

$$\frac{d}{dt}(E) = \frac{d}{dt}\left(\frac{1}{2}m\boldsymbol{v} \cdot \boldsymbol{v} + U\big(\boldsymbol{r}(t)\big)\right)$$

$$0 = \frac{1}{2}m\frac{d}{dt}(\boldsymbol{v} \cdot \boldsymbol{v}) + \frac{d}{dt}\Big(U\big(\boldsymbol{r}(t)\big)\Big)$$

For the first derivative we use the product rule for dot products and for the second we use the results from above where we showed the following

$$\frac{d}{dt}f\big(\boldsymbol{r}(t)\big) = \nabla f \cdot \boldsymbol{r}'(t)$$

Therefore,

$$0 = \frac{1}{2}m\left(\frac{d\boldsymbol{v}}{dt} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \frac{d\boldsymbol{v}}{dt}\right) + \nabla U \cdot \boldsymbol{r}'(t)$$

$$0 = m(\boldsymbol{a} \cdot \boldsymbol{v}) + \nabla U \cdot \boldsymbol{v}$$

$$0 = \boldsymbol{v} \cdot (m\boldsymbol{a} + \nabla U)$$

Next, we use the fact that potential functions associated with conservative vector fields are equal to the gradient. We will also use the physics convention where the potential function is written with a negative sign, i.e. $\boldsymbol{F} = -\nabla U$.

$$0 = \boldsymbol{v} \cdot (m\boldsymbol{a} - \boldsymbol{F})$$

Finally, using Newtons second law, $\boldsymbol{F} = m\boldsymbol{a}$, we see the conservation of energy equation is satisfied when using conservative forces! This explains why "*conservative*" is used to describe vector fields that have potential functions.

Two of the most fundamental forces in physics are the gravitational and the electrostatic. They are both given by inverse square laws as shown below.

Gravitational Force	***Electrostatic Force***
$\boldsymbol{F}_G = -\frac{GMm}{r^2}\boldsymbol{e}_r$	$\boldsymbol{F}_{\boldsymbol{E}} = \frac{kQq}{r^2}\boldsymbol{e}_r$

Where, G is the gravitational constant $G = 6.67E^{-11}$, k is the electrostatic constant $k = 8.9E^9$, M and m are the two masses in kilograms, Q and q are the two charges in Coulombs, r is the distance between the two objects or charges in meters, and $\boldsymbol{e}_r = \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \rangle$.

Combining the constant terms into one value, C, we write a general inverse square law force as

$$\boldsymbol{F} = \frac{C}{r^2}\boldsymbol{e}_r$$

Let us now find the potential function for this force. We will skip checking the partial conditions and assume the following relationship.

$$\boldsymbol{F}(x,y,z) = -\nabla f$$
$$C\,\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \rangle = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$$

The three integrals are evaluated in a similar fashion. We show the first one for illustration.

$$\frac{\partial f}{\partial x} = -C\frac{x}{r^3}$$
$$f(x,y,z) = -C\int\left(\frac{x}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\right)dx$$
$$= -\frac{C}{2}\int u^{-3/2}du$$
$$= \frac{C}{\sqrt{x^2+y^2+z^2}}$$
$$= \frac{C}{r} + g(y,z)$$

Where we used the substitution, $u = x^2 + y^2 + z^2 \rightarrow\ du = 2xdx$

The other two integrals give similar answers, leading to the following relationship.

$$\frac{GM}{r} + g(y,z) = \frac{GM}{r} + h(x,z) = \frac{GM}{r} + w(x,y)$$

Which is satisfied if $g(y,z) = h(x,z) = w(x,y) = 0$. Therefore, a conservative inverse square law force and its potential function can be written as shown.

Inverse Square Law Force and its Potential Function
$\boldsymbol{F}(x,y,z) = \frac{C}{r^2}\boldsymbol{e}_r \quad \rightarrow \quad f(x,y,z) = \frac{C}{r}$

Example 8: Compute the work required to move a satellite of mass $1000\ kg$ along any path from an altitude of $4E^6\ km$ to $6E^6\ km$.

Solution: The work required to move the satellite along a curve, C, is given as

$$W = -\int_C \boldsymbol{F}_G \cdot d\boldsymbol{r} = -\int_{r_1}^{r_2} \boldsymbol{F}_G(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)dt$$

And since the gravitational force is conservative with $\boldsymbol{F}_G = -\nabla U_G$, we have

$$\begin{aligned} W &= \int_{r_1}^{r_2} \nabla U_G(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)dt \\ &= \int_{r_1}^{r_2} \frac{d}{dt} U_G(\boldsymbol{r}(t))\, dt \\ &= U_G(r_2) - U_G(r_1) \\ &= \frac{-GMm}{r_2} - \frac{-GMm}{r_1} \\ &= GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \end{aligned}$$

The distances given are from the surface of the earth, however r is measured from the center of the earth. With the radius of the earth $6.4E^6$ we have $r_1 = 10.4E^6$ and $r_2 = 12.4E^6$. Finally, using the value of G from above and the mass of the earth, $M = 5.98E^{24}\ kg$, we find

$$W = (6.67E^{-11})(5.98E^{24})(1000)\left(\frac{1}{10.4E^6} - \frac{1}{12.4E^6}\right) \cong\ 6.2E^9\ Joules$$

Example 9: An electron and a proton are initially 1 nanometer apart. Compute the work required to move the proton so that they are separated by 1 centimeter. Ignore gravitational effects.

Solution: We take a similar approach using $\boldsymbol{F}_E = -\nabla U_E$, we have

$$\begin{aligned} W &= \int_{r_1}^{r_2} \nabla U_E(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)dt \\ &= \int_{r_1}^{r_2} \frac{d}{dt} U_E(\boldsymbol{r}(t))\, dt \\ &= U_E(r_2) - U_E(r_1) \\ &= \frac{kqq}{r_2} - \frac{kqq}{r_1} \\ &= kqq\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \end{aligned}$$

The charge of an electron and proton have the same magnitude, $1.6E^{-19}$, but opposite signs. Therefore, after using the k value from above we find

$$W = (8.9E^9)(-1.6E^{-19})(1.6E^{-19})\left(\frac{1}{1E^{-2}} - \frac{1}{1E^{-9}}\right) \cong\ 2.3E^{-19}\ Joules$$

Final Summary for Line and Surface Integrals – Conservative Vector Fields

The Fundamental Theorem for Conservative Vector Fields
Assume $\boldsymbol{F} = \nabla f$ on a domain D. 1. If $\boldsymbol{r}$ is a path along a curve C from A to B in D, then $$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b \frac{d}{dt} f(\boldsymbol{r}(t))\, dt = f(\boldsymbol{r}(b)) - f(\boldsymbol{r}(a)) = f(B) - f(A)$$ In other words, $\boldsymbol{F}$ is path-independent 2. The circulation around a closed curve C, (i.e. $A = B$) is zero $$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$$
Conservative Vector Field Criteria
The vector field $\boldsymbol{F}$ is conservative on a simply connected domain, D, if $\boldsymbol{F}$ satisfies the cross-partials conditions derived from the fact that the curl is zero. $curl(\boldsymbol{F}) = 0 \quad \rightarrow \quad \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$
Conservative Fields in Physics
The gravitational force and the electrostatic forces are conservative forces. They are both governed by the inverse square law.

Inverse Square Law Force and its Potential Function
$\boldsymbol{F}_C(x, y, z) = \frac{C}{r^2}\boldsymbol{e}_r \quad \rightarrow \quad f(x, y, z) = \frac{C}{r}$

Specifically,

Gravitational Force	***Electrostatic Force***
$\boldsymbol{F}_G = -\frac{GMm}{r^2}\boldsymbol{e}_r$	$\boldsymbol{F}_{\boldsymbol{E}} = \frac{kQq}{r^2}\boldsymbol{e}_r$

Where, $G = 6.67E^{-11}$, $k = 8.9E^9$, m_1 and m_2 are the two masses in kilograms, q_1 and q_2 are the two charges in Coulombs, and $\boldsymbol{e}_r = \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \rangle$.

By: ferrantetutoring