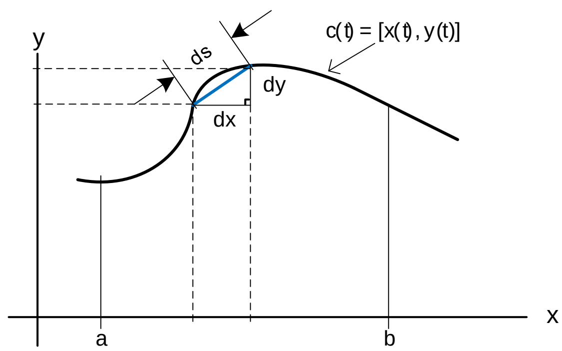
Vector Calculus – Arc Length and Speed

In the previous section we introduced the mechanics associated with computing the derivative of vector-valued functions. In the next three sections we'll look more closely at how we can use what we learned to study motion of objects in a very general sense. This section starts by introducing two key concepts; arc length, i.e. distance along a path, and speed. We also introduce what is called an *arc length parameterization*, which is a very useful concept we will use in the remaining two sections.

Arc Length

In calculus 2 we derived the expression for the arc length of a parametric curve as shown below.



The infinitesimal distance, ds, can be expressed using the Pythagorean theorem as

$$ds = \sqrt{dx^2 + dy^2}$$

Next, we multiple the right-hand side by $\sqrt{\frac{dt^2}{dt^2}}$, resulting in the following

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}\, dt$$

Integrating, we find the arc length.

$$s = \int_a^b \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}\, dt$$

Note the parametric curve can also expressed as a vector-valued function, $\boldsymbol{r}(t)$, along with its derivative.

$$\boldsymbol{r}(t) = \langle x(t), y(t) \rangle \qquad\qquad \boldsymbol{r}'(t) = \langle x'(t), y'(t) \rangle$$

Finally, we see that the magnitude of $\boldsymbol{r}'(t)$, $\|\boldsymbol{r}'(t)\|$, is exactly equal to our expression for the arc length, s, above. Therefore, we can write the arc-length, i.e. the length of a path, as follows:

$$s = \int_a^b \|\boldsymbol{r}'(t)\|\, dt$$

The result extends to general space curves in R^3 and is formally stated below.

Arc Length - The length of a Path (Distance Traveled)
Assume $\boldsymbol{r}(t)$ is differentiable and $\boldsymbol{r}'(t)$ is continuous on $[a, b]$. Then the distance, s, a particle travels along the path, $\boldsymbol{r}(t)$, for $a \le t \le b$ is equal to $$s = \int_a^b \|\boldsymbol{r}'(t)\|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$ The distance traveled as a function of t can also be written as $$s(t) = \int_a^t \|\boldsymbol{r}'(u)\|\, du$$ Which, we sometimes refer to as the ***arc length function***.

Speed, by definition, is defined as the rate of change of the distance with respect to time. Furthermore, since speed is equal to the magnitude of velocity, we can write the following relationships.

$$\|\boldsymbol{v}(\boldsymbol{t})\| = \frac{d}{dt}\big(s(t)\big)$$
$$\|\boldsymbol{v}(\boldsymbol{t})\| = \frac{d}{dt}\left(\int_a^t \|\boldsymbol{r}'(u)\|\, du\right)$$
$$\|\boldsymbol{v}(\boldsymbol{t})\| = \|\boldsymbol{r}'(t)\|$$

Where, in the last step we used the fundamental theorem of calculus. We summarize the relationships between speed, distance, velocity, and position below.

Position, Velocity, Distance, and Speed Relationships	
Given the following:	
$\boldsymbol{v}(t)$: The velocity of a particle at time t.	
$v(t)$: The speed of a particle at time t.	
$\boldsymbol{r}(t)$: The position of a particle at time t.	
$s(t)$: The distance a particle has traveled at time t.	
We can write the following relationships:	
The velocity is the time derivative of position:	$\boldsymbol{v}(t) = \boldsymbol{r}'(t)$
The speed is the magnitude of velocity:	$v(t) = \|\boldsymbol{v}(t)\| = \|\boldsymbol{r}'(t)\|$
The position is the time integral of velocity:	$\boldsymbol{r}(t) = \int \boldsymbol{v}(t)\, dt + \boldsymbol{r}(a)$
The distance traveled, arc length, is the time integral of speed:	$s(t) = \int_a^t \|\boldsymbol{r}'(u)\|\, du$

Before we introduce the so-called *arc length parametrization*, let's do a few examples using what we have learned above.

Example 1: Compute the arc length of the following curves over the given time interval.

a. $\boldsymbol{r}(t) = \langle 3t, 4t - 3, 6t + 1 \rangle, \;\; 0 \leq t \leq 3$

b. $\boldsymbol{r}(t) = \langle cos(t)\, , sin(t)\, , t \rangle, \;\; 0 \leq t \leq 6\pi$

Solution: We can directly use the arc length formula shown below.

$$s = \int_a^b \|\boldsymbol{r}'(t)\|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

a.

$$\begin{aligned} s &= \int_0^3 \sqrt{\left(\frac{d}{dt}(3t)\right)^2 + \left(\frac{d}{dt}(4t - 3)\right)^2 + \left(\frac{d}{dt}(\,6t + 1)\right)^2}\, dt \\ &= \int_0^3 \sqrt{3^2 + 4^2 + 6^2}\, dt \\ &= \sqrt{61} \int_0^3 1 dt \\ &= 3\sqrt{61} \end{aligned}$$

b.

$$\begin{aligned} s &= \int_0^{6\pi} \sqrt{\left(\frac{d}{dt}(cos(t))\right)^2 + \left(\frac{d}{dt}(\,sin(t))\right)^2 + \left(\frac{d}{dt}(\,t)\right)^2}\, dt \\ &= \int_0^{6\pi} \sqrt{sin^2(t) + cos^2(t) + 1^2}\, dt \\ &= \int_0^{6\pi} \sqrt{2}\, dt \\ &= 6\pi\sqrt{2} \end{aligned}$$

Example 2: Compute the arc length function, $s(t)$, where $\boldsymbol{r}(0) = \langle 0, 0, 0\rangle$

$$\boldsymbol{r}(t) = \langle t^2, 2t^2, t^3\rangle$$

Solution: We can directly use the relationship in the summary above for $s(t)$, with $a = 0$.

$$\begin{aligned} s(t) &= \int_0^t \|\boldsymbol{r}'(u)\|\, du \\ &= \int_0^t \sqrt{\left(\frac{d}{du}(u^2)\right)^2 + \left(\frac{d}{du}(\,2u^2)\right)^2 + \left(\frac{d}{du}(\,u^3)\right)^2}\, du \\ &= \int_0^t \sqrt{4u^2 + 16u^2 + 9u^4}\, du \\ &= \int_0^t u\sqrt{9u^2 + 20}\, du \end{aligned}$$

Which, can be evaluated with substitution as follows:

$$v = 9u^2 + 20 \qquad\qquad dv = 18udu$$

$$s(t) = \frac{1}{18}\int_{20}^{9t+20} \sqrt{v}\, dv$$

$$s(t) = \frac{1}{27}\left((9t^2 + 20)^{3/2} - (20)^{3/2}\right)$$

Example 3: Find the speed at the given value of t.

$$\boldsymbol{r}(t) = \langle t, ln(t)\,, (ln(t))^2\,\rangle, \qquad t = 4$$

Solution: The speed is equal to the magnitude of the velocity function, which is in turn equal to the derivative of the position function, $\boldsymbol{r}(t)$.

$$v(t) = \|\boldsymbol{v}(t)\| = \|\boldsymbol{r}'(t)\|$$

$$v(t) = \sqrt{\left(\frac{d}{dt}(t)\right)^2 + \left(\frac{d}{dt}(\ ln(t))\right)^2 + \left(\frac{d}{dt}((ln(t))^2)\right)^2}$$

$$v(t) = \sqrt{1 + \frac{1}{t^2} + \frac{4\,ln^2(t)}{t^2}}$$

The speed at $t = 4$ is then

$$v(4) = \sqrt{1 + \frac{1}{4^2} + \frac{4\,ln^2(4)}{4^2}} \cong 1.24$$

Example 4: A particle moves in R^3 with a velocity given as $\boldsymbol{v}(t) = \langle t, t^2, t^3 \rangle$. If the initial position of the particle at $t = 0$ is $(1, 2, 3)$, what is the particles position when $t = 1$?

Solution: We can start by finding the position function given the initial position, $\boldsymbol{r}(0) = \langle 1, 2, 3 \rangle$

$$\boldsymbol{r}(t) = \int \boldsymbol{v}(t)\,dt + \boldsymbol{r}(0)$$

$$\boldsymbol{r}(t) = \langle \int t\,dt, \int t^2\,dt, \int t^3\,dt \rangle + \langle 1, 2, 3 \rangle$$

$$\boldsymbol{r}(t) = \langle \frac{1}{2}t^2, \frac{1}{3}t^3, \frac{1}{4}t^4 \rangle + \langle 1, 2, 3 \rangle$$

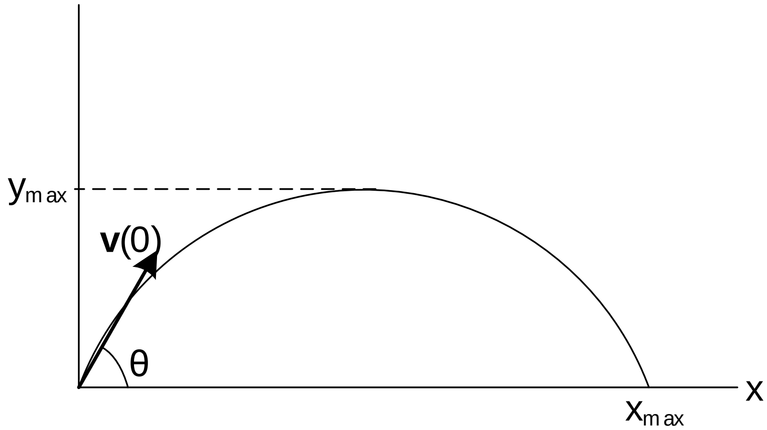
$$\boldsymbol{r}(t) = \langle \frac{1}{2}t^2 + 1, \frac{1}{3}t^3 + 2, \frac{1}{4}t^4 + 3 \rangle$$

The position at $t = 1$ is then

$$\boldsymbol{r}(1) = \langle \frac{1}{2} + 1, \frac{1}{3} + 2, \frac{1}{4} + 3 \rangle = \langle \frac{3}{2}, \frac{7}{3}, \frac{13}{4} \rangle$$

Example 5: A projectile with a mass m is shot upward at an angle of θ, and at an initial speed of $v(0)$ meters per second. Answer the following questions.

a. Find the vector-valued function, $\boldsymbol{r}(t)$, that gives the position as a function of time.
b. Verify that the trajectory is a parabola.
c. What is the total flight time of the projectile?
d. What is the range of the projectile?
e. What angle will maximize the range?



Solution:

a. The velocity vector can be written in component form as follows:

$$\boldsymbol{v}(t) = \langle v_x(0) + a_x t, v_y(0) + a_y t \rangle$$

Where, $v_x(0) = \|v(0)\| \, cos(\theta)$, $v_y(0) = \|v(0)\| \, sin(\theta)$, $a_x = 0$, and $a_y = -g$. Therefore

$$\boldsymbol{v}(t) = \langle \|v(0)\| \, cos(\theta) \, , \|v(0)\| \, sin(\theta) - gt \rangle$$

The position function is found by integrating with $\boldsymbol{r}(0) = \langle 0,0 \rangle$.

$$\boldsymbol{r}(t) = \int \boldsymbol{v}(t) \, dt + \boldsymbol{r}(0)$$

$$\boldsymbol{r}(t) = \langle \int \|v(0)\| \, cos(\theta) \, dt, \int (\|v(0)\| \, sin(\theta) - gt) \, dt \rangle + \langle 0,0 \rangle$$

$$\boldsymbol{r}(t) = \langle \|v(0)\| \, cos(\theta) \, t, \|v(0)\| \, sin(\theta) t - \frac{1}{2} g t^2 \rangle$$

b. To verify the trajectory is a parabola we eliminate the parameter, t, by solving $x(t)$ for t and substituting this into $y(t)$ as shown below.

$$x = \|v(0)\| \, cos(\theta) \, t$$

$$t = \frac{x}{\|v(0)\| \, cos(\theta)}$$

$$y(t) = \|v(0)\| \, sin(\theta)t - \frac{1}{2}gt^2$$

$$y(x) = \cancel{\|v(0)\|} sin(\theta) \left(\frac{x}{\cancel{\|v(0)\|} cos(\theta)}\right) - \frac{1}{2} g \left(\frac{x}{\|v(0)\| \, cos(\theta)}\right)^2$$

$$y(x) = -\left(\frac{g}{2\|v(0)\|^2 cos^2(\theta)}\right) x^2 + tan(\theta)x$$

Which, is a parabola of the form: $y(x) = -ax^2 + bx$

c. The range of the projectile, x_{max}, is found by setting $y(x) = 0$.

$$0 = -\left(\frac{g}{2\|v(0)\|^2 cos^2(\theta)}\right) x^2 + tan(\theta)x$$

$$0 = (-x)\left(\frac{gx}{2\|v(0)\|^2 cos^2(\theta)} - tan(\theta)\right)$$

The first solution, $x = 0$, corresponds to $t = 0$, and the second solution corresponds to x_{max}.

$$\frac{gx}{2\|v(0)\|^2 cos^2(\theta)} - tan(\theta) = 0$$

$$x_{max} = \frac{sin(\theta) \, 2\|v(0)\|^2 cos^{\cancel{2}}(\theta)}{g \, \cancel{cos(\theta)}}$$

$$x_{max} = \frac{\|v(0)\|^2}{g} sin(2\theta)$$

d. Since the maximum range is given in terms of the angle, we maximize $x_{max}(\theta)$.

$$\theta_{max} = \underset{0 \le \theta \le 90^{\circ}}{argmax}\big(x_{max}(\theta)\big)$$

Which is found by differentiating $x_{max}(\theta)$ and setting the result to zero.

$$\frac{d}{dx}\big(x_{max}(\theta)\big) = 0$$

$$\frac{d}{dx}\left(\frac{\|v(0)\|^2}{g} sin(2\theta)\right) = 0$$

$$\cancel{2}\frac{\cancel{\|v(0)\|^2}}{\cancel{g}}cos(2\theta) = 0$$

$$cos(2\theta) = 0$$

$$\theta_{max} = \frac{cos^{-1}(0)}{2}$$

$$\theta_{max} = \frac{90^{\circ}}{2} = 45^{\circ}$$

Arc Length (Unit Speed) Parameterization

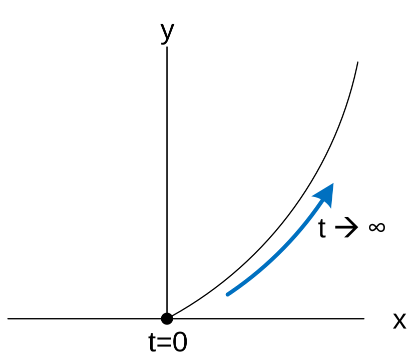
Recall the distinction we made in an earlier lesson between a curve and a path.

Curve	Represents the set of all points, (x, y, z), as the parameter, e.g. t, varies over its domain.
Path: Parameterization of a curve $\boldsymbol{r}(t) = \langle x(t), y(t), z(t) \rangle$	Represents the particular way a curve is traversed, e.g. it may traverse the curve several times, reverse direction, move back and forth, etc.

Therefore, any particular parameterization of a curve is *not* unique. Take for example the curve represented by $y = x^2$. One particular parameterization of this curve is:

$$\boldsymbol{r}(t) = \langle t, t^2 \rangle$$

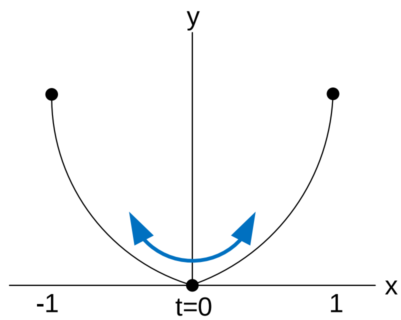
Which, if we let t vary from zero to infinity describes a particle traversing the curve as shown.



We can obtain a new parameterization by making t a function of a new parameter, e.g. u. In other words, we let $t = g(u)$, e.g. $g(u) = sin(u)$. The new parameterization is then given as

$$\boldsymbol{r}\big(g(u)\big) = \boldsymbol{r}(sin(u)) = \langle sin(u)\,, sin^2(u)\rangle$$

Resulting in a very different path, along the same underlying curve, than the previous parameterization. In this case the particle oscillates along the bottom of the curve as shown below.



Let's look at an example to clarify.

Example 6: Parameterize the path $\boldsymbol{r}(t) = \langle t^2, sin(t)\,, t\rangle$ for $3 \le t \le 9$ using the parameter, u, where $t = g(u) = e^u$.

Solution: Substituting $t = e^u$ in $\boldsymbol{r}(t)$, we obtain a new parameterization.

$$\boldsymbol{r}\big(g(u)\big) = \boldsymbol{r}(e^u) = \langle e^{2u}, sin(e^u), e^u\ \rangle$$

Furthermore, since $u = ln(t)$, the new parameter, u, varies from $ln(3)$ to $ln(9)$.

Given the innumerable ways that any particular curve can be parameterized, how exactly do know which to choose? The answer, of course, is that it depends on the type of information one would like to extract from the analysis.

As we have seen above, the way in which a particle moves along a curve changes the parameterization of that curve. What if we wanted to analyze the properties of the underlying curve itself, irrespective of how the curve is traversed? One way to do this is to develop a parameterization that ensures the particle does not change its speed while traversing the curve. In particular, we can force the particle to have a unit speed, i.e. $\|\boldsymbol{v}(t)\| = \|\boldsymbol{r}'(t)\| = 1$. This restriction will allow us to create a unique parameterization that focusing on the shape of the curve only and not on the particular way in which it is traversed. The parametrization is often referred to as an *arc length parameterization*, however a much more precise name is a *unit speed parameterization*. Below we outline a general procedure to create an arc length parametrization of a generic curve, $\boldsymbol{r}(t)$.

Starting with any parameterization, $\boldsymbol{r}(t)$, we proceed as follows:

Step 1: Find the arc length function.

$$s = g(t) = \int_a^t \|\boldsymbol{r}'(u)\|\, du$$

Step 2: Compute the following inverse function.

$$t = g^{-1}(s)$$

Step 3: Create the new unit speed parameterization as follows:

$$\boldsymbol{r}(s) = \boldsymbol{r}\left(g^{-1}(s)\right)$$

In example 6 we arbitrarily defined the mapping from t to u, i.e. $t = e^u$. The difference in this case is that we are choosing a specific mapping that ensures $\|\boldsymbol{v}(t)\| = 1$.

Let's show this is the case with the following example.

Example 7: A line that is parallel with the x-y plane can be parameterized as follows:

$$\boldsymbol{r}(t) = \langle t, mt, c\rangle$$

Compute the arc length parameterization of this line and show that $\|\boldsymbol{v}(t)\| = 1$.

Solution: We can directly use the procedure above.

Step 1: Find the arc length function, $s = g(t)$

$$g(t) = \int_0^t \|\boldsymbol{r}'(u)\|\, du$$

$$g(t) = \int_0^t \sqrt{\left(\frac{d}{du}(u)\right)^2 + \left(\frac{d}{du}(mu)\right)^2 + \left(\frac{d}{du}(c)\right)^2}\, du$$

$$g(t) = \int_0^t \sqrt{1 + m^2}\, du$$

$$g(t) = t\sqrt{1 + m^2}$$

Step 2: Compute inverse function, $t = g^{-1}(s)$, which is straightforward in this case

$$t = \frac{g(t)}{\sqrt{1 + m^2}}$$

$$t = \frac{s}{\sqrt{1 + m^2}}$$

Step 3: Create the new unit speed parameterization

$$\boldsymbol{r}(s) = \boldsymbol{r}\left(g^{-1}(s)\right)$$
$$\boldsymbol{r}(s) = \boldsymbol{r}\left(\frac{s}{\sqrt{1+m^2}}\right)$$
$$\boldsymbol{r}(s) = \langle \frac{s}{\sqrt{1+m^2}}, \frac{ms}{\sqrt{1+m^2}}, c\rangle$$

Finally, let's verify that $\|\boldsymbol{v}(s)\| = 1$.

$$\|\boldsymbol{v}(s)\| = \|\boldsymbol{r}'(s)\|$$
$$= \sqrt{\left(\frac{d}{ds}\left(\frac{s}{\sqrt{1+m^2}}\right)\right)^2 + \left(\frac{d}{ds}\left(\frac{ms}{\sqrt{1+m^2}}\right)\right)^2 + \left(\frac{d}{ds}(c)\right)^2}$$
$$= \sqrt{\frac{1}{1+m^2} + \frac{m^2}{1+m^2}} == \sqrt{\frac{1+m^2}{1+m^2}} = 1$$

Therefore, $\boldsymbol{r}(s) = \langle \frac{s}{\sqrt{1+m^2}}, \frac{ms}{\sqrt{1+m^2}}, 1\rangle$, is indeed the unique arc length parameterization for a line that is parallel with the x-y plane.

Finally, we note that in most cases we cannot evaluate the arc length integral $s = g(t)$ explicitly, and therefore cannot find a formula for its inverse, $t = g^{-1}(s)$. So, although the arc length parameterization does indeed exist, we can only find it *explicitly* in some special cases. Before we summarize this section, let's look at one last example where the arc length integral can be found.

Example 8: Compute the arc length parameterization for the helix given below.

$$\boldsymbol{r}(t) = \langle cos(t), \sin(t), t\rangle$$

Solution:

Step 1: Find the arc length function, $s = g(t)$

$$g(t) = \int_0^t \|\boldsymbol{r}'(u)\|\, du$$
$$g(t) = \int_0^t \sqrt{\left(\frac{d}{du}(cos(u))\right)^2 + \left(\frac{d}{du}(sin(u))\right)^2 + \left(\frac{d}{du}(u)\right)^2}\, du$$
$$g(t) = \int_0^t \sqrt{sin^2(u) + cos^2(u) + 1}\, du$$
$$g(t) = t\sqrt{2}$$

Step 2: Compute inverse function, $t = g^{-1}(s)$, which in this case is straightforward

$$t = \frac{s}{\sqrt{2}}$$

Step 3: Create the new unit speed parameterization

$$\boldsymbol{r}(s) = \boldsymbol{r}\big(g^{-1}(s)\big)$$
$$\boldsymbol{r}(s) = \boldsymbol{r}\left(\frac{s}{\sqrt{2}}\right)$$
$$\boldsymbol{r}(s) = \langle cos\left(\frac{s}{\sqrt{2}}\right), sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}}\rangle$$

Let's again verify that this represents a unit speed parameterization.

$$\|\boldsymbol{v}(s)\| = \|\boldsymbol{r}'(s)\|$$
$$= \sqrt{\left(\frac{d}{ds}\left(cos\left(\frac{s}{\sqrt{2}}\right)\right)\right)^2 + \left(\frac{d}{ds}\left(sin\left(\frac{s}{\sqrt{2}}\right)\right)\right)^2 + \left(\frac{d}{ds}\left(\frac{s}{\sqrt{2}}\right)\right)^2}$$
$$= \sqrt{\frac{1}{2}sin^2(u) + \frac{1}{2}cos^2(u) + \frac{1}{2}}$$
$$= \sqrt{\frac{1}{2}(sin^2(u) + cos^2(u) + 1)} = 1$$

Final Summary for Vector Calculus – Arc Length and Speed

Arc Length - The length of a Path (Distance Traveled)
Assume $\boldsymbol{r}(t)$ is differentiable and $\boldsymbol{r}'(t)$ is continuous on $[a, b]$. Then the distance, s, a particle travels along the path, $\boldsymbol{r}(t)$, for $a \leq t \leq b$ is equal to $$s = \int_a^b \|\boldsymbol{r}'(t)\|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$ The distance traveled as a function of t can also be written as $$s(t) = \int_a^t \|\boldsymbol{r}'(u)\|\, du$$ Which, we sometimes refer to as the ***arc length function***.
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Given the following:

$\boldsymbol{v}(t)$: The velocity of a particle at time t.

$v(t)$: The speed of a particle at time t.

$\boldsymbol{r}(t)$: The position of a particle at time t.

$s(t)$: The distance a particle has traveled at time t.

We can write the following relationships:

The velocity is the time derivative of position:	$\boldsymbol{v}(t) = \boldsymbol{r}'(t)$
The speed is the magnitude of velocity:	$v(t) = \|\boldsymbol{v}(t)\| = \|\boldsymbol{r}'(t)\|$
The position is the time integral of velocity:	$\boldsymbol{r}(t) = \int \boldsymbol{v}(t)\, dt + \boldsymbol{r}(a)$
The distance traveled, arc length, is the time integral of speed:	$s(t) = \int_a^t \|\boldsymbol{r}'(u)\|\, du$

Arc Length (Unit Speed) Parameterization

The arc length parameterization of a curve is one in which the speed is unity, i.e. $\|\boldsymbol{v}(s)\| = 1$. This restriction, $\|\boldsymbol{v}(s)\| = 1$, allows for the creation of a unique parameterization that focusing on the shape of the curve only and not on the particular way in which it is traversed.

Starting with any parameterization, $\boldsymbol{r}(t)$, we proceed as follows:

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Step 3: Create the new unit speed parameterization as follows:

$$\boldsymbol{r}(s) = \boldsymbol{r}\left(g^{-1}(s)\right)$$

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