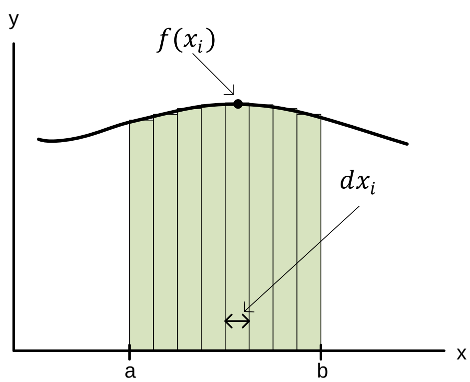
<u>Multiple Integration – Triple Integrals</u>

Triple integrals are completely analogous to double integrals we introduced in the previous lesson. Unfortunately, the geometric interpretation that we have become accustomed to for integrals is not directly available for triple integrals. Nonetheless, triple integrals can be used to represent a variety of physical quantities. In this lesson we will focus more on learning how to evaluate triple integrals and less so on any physics applications. However, since we cannot readily use a geometric interpretation, we will introduce triple integrals with one of the many physics applications.

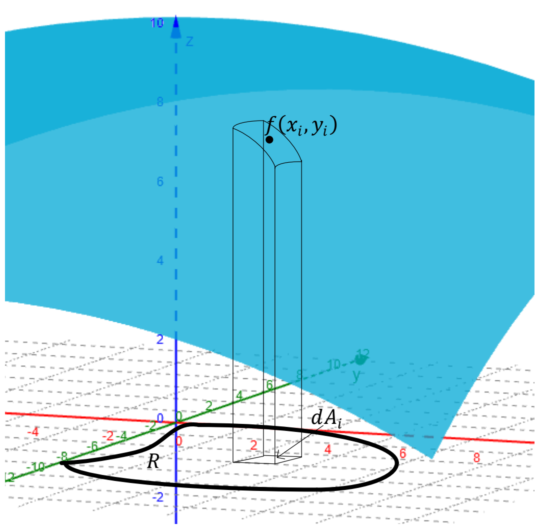
Triple Integral Physical Interpretation

The geometric interpretation for single variable integration makes use of *two dimensional infinitesimal rectangles*. Each rectangle has a height of $f(x)$, an infinitesimal width of dx, and therefore an infinitesimal area of $dA = f(x)dx$. The integral is then defined as an infinite sum of these rectangles over a certain interval, i.e. the area under the curve $f(x)$ over the interval $[a, b]$.



$$A = \int_a^b f(x)dx$$

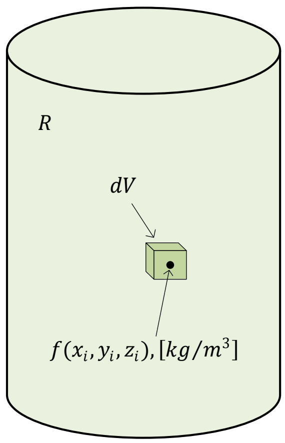
In a complementary way, the geometric interpretation for double integrals makes use of *three dimensional infinitesimal rectangular prisms* with a height of $f(x, y)$ and infinitesimal base area of dA. In this case, we can define the infinitesimal volume of each prism as $dV = f(x, y)dA$. The double integral is then defined as an infinite sum of these rectangular prisms over a certain region, i.e. the volume under the surface $f(x, y)$ over a region, R.



$$V = \iint_R f(x, y)dA$$

If we follow this geometric interpretation for triple integrals, we find ourselves trying to visualize *four dimensional objects*. These 'objects' would have an infinitesimal volume, dV, and would extend into a fourth dimension with a value of $f(x,y,z)$. Since these 'objects' are impossible to visualize we move to a different interpretation for triple integrals.

Assume the function, $f(x,y,z)$, represents the mass density of a three dimensional solid object in units of mass per volume, e.g. kg/m^3. For example, assume the cylinder shown below has a mass density at the point (x,y,z) of $f(x,y,z)$. Now, consider an infinitesimal volume element, dV, within the cylinder. Since $mass = volume\ x\ denisty$, we see that the mass of this infinitesimal volume element is $dM = f(x,y,z)dV$. With this we can define the triple integral as an infinite sum of these infinitesimal mass elements over a certain three dimensional region, i.e. the total mass of the solid object over a region, R.



$$M = \iiint_R f(x,y,z)dV$$

Computation of the triple integral also follows directly from double integrals as shown below.

Triple Integral Over a Boxed Region

The triple integral of a continuous function $f(x,y,z)$ over a box, R is:

$$\iiint_R f(x,y,z)dV = \int_{x=a}^{b}\int_{y=c}^{d}\int_{z=p}^{q} f(x,y,z)dzdydx$$

Where,

$$R = (x,y,z)|\ a \le x \le b, c \le y \le d, p \le z \le q$$

Furthermore, the integral can be evaluated in any order.

Let's do some examples.

Example 1: Evaluate the triple integral of $f(x, y, z) = x^2y + yz$ over R.

$$R = \{(x, y, z)|\ 0 \le x \le 1, 2 \le y \le 4, 1 \le z \le 2\}$$

Solution: For double integrals we could integrate in one of two orders, i.e. $dxdy$ or $dydx$. For triple integrals there are six possible orders to choose from.

$$dzdxdy \qquad dzdydx \qquad dydxdz \qquad dydzdx \qquad dxdydz \qquad dxdzdy$$

For illustration purposes we'll evaluate using two different orders, $dzdxdy$ and $dydzdx$.

$$\int_{y=2}^{4}\int_{x=0}^{1}\int_{z=1}^{2}(x^2y + yz)dzdxdy$$

$$= \int_{2}^{4}\int_{0}^{1}\left(\int_{1}^{2}(x^2y + yz)dz\right)dxdy$$

$$= \int_{2}^{4}\int_{0}^{1}\left(x^2yz + \frac{1}{2}yz^2\Big|_{1}^{2}\right)dxdy$$

$$= \int_{2}^{4}\left(\int_{0}^{1}x^2y + \frac{3}{2}y\,dx\right)dy$$

$$= \int_{2}^{4}\left(\frac{11}{6}y\right)dy$$

$$= \frac{11}{12}(16 - 4)$$

$$= 11$$

$$\int_{x=0}^{1}\int_{z=1}^{2}\int_{y=2}^{4}(x^2y + yz)dydzdx$$

$$= \int_{0}^{1}\int_{1}^{2}\left(\int_{2}^{4}y(x^2 + z)dy\right)dzdx$$

$$= \left(\frac{1}{2}y^2\Big|_{2}^{4}\right)\int_{0}^{1}\int_{1}^{2}(x^2 + z)dzdx$$

$$= 6\int_{0}^{1}\left(\int_{1}^{2}(x^2 + z)dz\right)dx$$

$$= 6\int_{0}^{1}\left(x^2z + \frac{1}{2}z^2\Big|_{1}^{2}\right)dx$$

$$= 6\int_{0}^{1}\left(x^2 + \frac{3}{2}\right)dx$$

$$= 6\left(\frac{1}{3} + \frac{3}{2}\right)$$

$$= 11$$

The additional complexity with triple integrals is primarily associated with the region for which the integral is computed. Single variable integrals are computed over a single dimension, which is simple to visualize. Double integrals are computed over a two dimensional region, which is also fairly straightforward to visualize. On the other hand, a triple integral is computed over a three dimensional region, which can be much more difficult to visualize. For a simple box the integration limits are simple to determine, and the order of integration can be freely changed. However, for general regions, similar to double integrals, the integration limits as well as the order needs to be chosen carefully. With this, we now look at triple integrals over more general regions.

Example 2: Evaluate the tripe integral $\iiint_D zdV$, where D is the region between the two planes, $z_1 = x + y$ and $z_2 = 3x + 5y$ lying over the rectangle, R, in the x-y plane.

$$R = \{(x, y)|\ 0 \le x \le 3, 0 \le y \le 2\}$$

Solution: The region of integration in the x-y plane is a simple rectangle. In the z direction the limits of integration are from the lower plane to the higher plane. For the rectangle region in the x-y plane $z_2 > z_1$, which determines the integration limits in z.

$$\iiint\limits_D zdV = \int_{y=0}^{2}\int_{x=0}^{3}\int_{z=z_1}^{z_2}(z)dzdxdy$$

$$= \frac{1}{2}\int_0^2\int_0^3((3x + 5y)^2 - (x + y)^2)\,dxdy$$

$$= \frac{1}{2}\int_0^2\left(\int_0^3(8x^2 + 24y^2 + 28xy)\,dx\right)dy$$

$$= \frac{1}{2}\int_0^2\left(\frac{8}{3}\cdot 27 + 24y^2\cdot 3 + 14y\cdot 9\right)dy$$

$$= \frac{1}{2}\int_0^2(72 + 72y^2 + 126y)\,dy$$

$$= \frac{1}{2}\left(72\cdot 2 + \frac{72}{3}\cdot 8 + \frac{126}{2}\cdot 4\right) = 294$$

Example 3: Evaluate the tripe integral of $f(x, y, z) = x$ over the region, D, bounded between the surface $z_1 = 4 - x^2 - y^2$ and $z_2 = x^2 + 3y^2$ and where $x \ge 0, y \ge 0$.

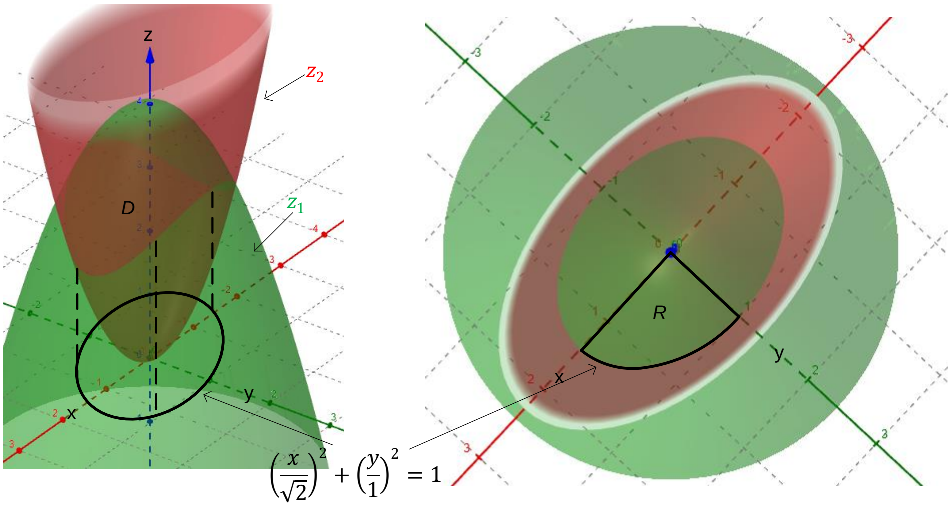
Solution: The region is bounded between two paraboloids. The z_1 paraboloid has a maximum value at $z = 4$ and opens down whereas z_2 has a minimum at $z = 0$ and opens up. They intersect when $z_1 = z_2$, which turns out to be an ellipse projected in the x-y plane as shown below.

$$4 - x^2 - y^2 = x^2 + 3y^2$$

$$2x^2 + 4y^2 = 4$$

$$\left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y}{1}\right)^2 = 1$$

However, we are interested only in the region from the first quadrant in the x-y plane. The entire region, D, as well as the region in the x-y plane, R, are shown in the figures below.



Treating the region in the x-y plane as vertically simple we have

$$R = \left\{(x, y) \middle|\ 0 \le x \le \sqrt{2 - 2y^2}, 0 \le y \le 1\right\}$$

The limits of integration in the z direction are from z_2 to z_1. The triple integral is then

$$\iiint_D z dV = \int_{y=0}^{1} \int_{x=0}^{\sqrt{2-2y^2}} \int_{z=x^2+3y^2}^{4-x^2-y^2} (x) dz dx dy$$

$$= \int_0^1 \int_0^{\sqrt{2-2y^2}} x\big((4 - x^2 - y^2) - (x^2 + 3y^2)\big)\, dx dy$$

$$= \int_0^1 \left(\int_0^{\sqrt{2-2y^2}} (-2x^3 - 4xy^2 + 4x)\, dx \right) dy$$

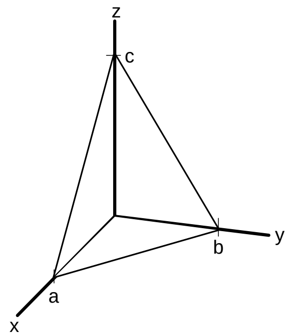
$$= \int_0^1 \left(-\frac{1}{2}\left(\sqrt{2 - 2y^2}\right)^4 - 2y^2 \left(\sqrt{2 - 2y^2}\right)^2 + 2\left(\sqrt{2 - 2y^2}\right)^2 \right) dy$$

$$= \int_0^1 (2y^4 - 4y^2 + 2)\, dy$$

$$= \left(\frac{2}{5} y^5 - \frac{4}{3} y^3 + 2y \Big|_0^1 \right) = \frac{16}{15}$$

In the previous two examples determining the integration order was fairly straightforward since the x-y plane regions were explicitly given or fairly straightforward to determine. However, in general the most difficult part of triple integrals is deciding on the best order of integration and determining the limits of integration for each of the three integrals. Recall that there are six possible orders to choose from. The next example illustrates a general method to determine integration limits for all six orders.

Example 4: Express the six possible triple integrals for the general function, $f(x,y,z)$, over the region enclosed by the tetrahedron, D, shown below.

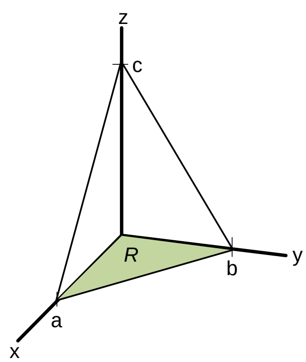


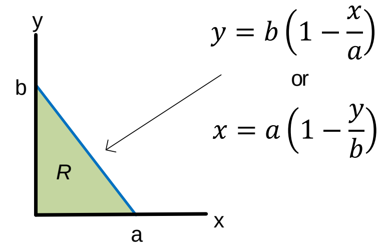
Solution: The general procedure uses the fact that the triple integral can be written as follows:

$$\iiint_D z\,dV = \iint_R \left(\int_{q_1}^{q_2} f(x,y,z)\,dq\right) dA$$

Where, the inner integral is with respect to any one of the three variables, i.e. we choose q to be one of the elements of the set $\{x,y,z\}$. The region, R, is the projection of the solid object in the plane defined by the two remaining variables. We can then express the dA in two different orders for each of the 3 possible projections as shown below.

1. *Projection in the x-y plane*





The front face of the tetrahedron is described by the equation of a plane, which is given by

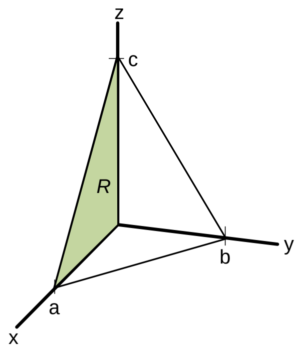
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

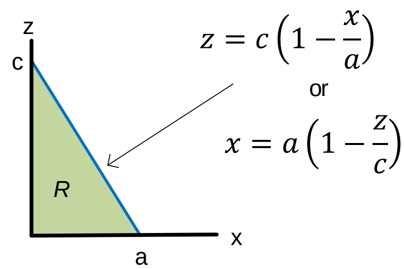
Note: We derive this equation in the next example.

- The solid above the x-y plane is bounded by $z = 0$ and $z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$.
- The region R can be defined with respect to $dA = dxdy$ or $dA = dydx$

1.	*2.*
$\int_{y=0}^{b} \int_{x=0}^{a\left(1-\frac{y}{b}\right)} \left(\int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} f(x,y,z)dz\right) dxdy$	$\int_{x=0}^{a} \int_{y=0}^{b\left(1-\frac{x}{a}\right)} \left(\int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} f(x,y,z)dz\right) dydx$

2. *Projection in the x-z plane*

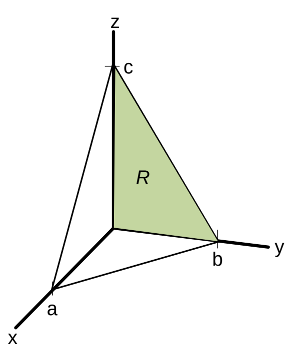


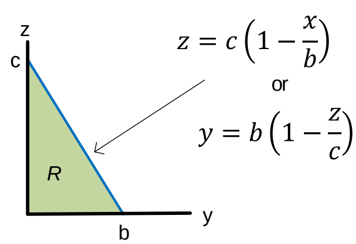


- The solid above the x-z plane is bounded by $y = 0$ and $y = b\left(1 - \frac{x}{a} - \frac{z}{c}\right)$.
- The region R can be defined with respect to $dA = dxdz$ or $dA = dzdx$

3.	*4.*
$\int_{z=0}^{c} \int_{x=0}^{a\left(1-\frac{z}{c}\right)} \left(\int_{y=0}^{b\left(1-\frac{x}{a}-\frac{z}{c}\right)} f(x,y,z)dy \right) dxdz$	$\int_{x=0}^{a} \int_{z=0}^{c\left(1-\frac{x}{a}\right)} \left(\int_{y=0}^{b\left(1-\frac{x}{a}-\frac{z}{c}\right)} f(x,y,z)dy \right) dzdx$

3. Projection in the y-z plane

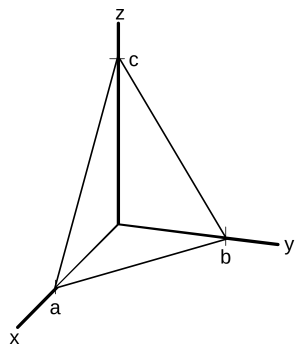




- The solid above the y-z plane is bounded by $x = 0$ and $x = a\left(1 - \frac{y}{b} - \frac{z}{c}\right)$.
- The region R can be defined with respect to $dA = dydz$ or $dA = dzdy$

5.	*6.*
$\int_{z=0}^{c} \int_{y=0}^{b\left(1-\frac{z}{c}\right)} \left(\int_{y=0}^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)} f(x,y,z)dx \right) dydz$	$\int_{y=0}^{b} \int_{z=0}^{c\left(1-\frac{x}{b}\right)} \left(\int_{y=0}^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)} f(x,y,z)dx \right) dzdy$

Example 5: Integrate $f(x,y,z) = x$ over the tetrahedron shown with $(a,b,c) = (4,4,6)$.



Solution: We can use the results from the previous example. First, as mentioned, we'll derive the equation of the plane for the front face of the tetrahedron.

The general equation of a plane is given as

$$n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) = 0$$

Where $(x_0, y_0, z_0) = (a, 0, 0)$ and $\boldsymbol{n} = \langle n_x, n_y, n_z \rangle$ is the normal vector, which is found as

$$n = \overrightarrow{ca} \times \overrightarrow{cb} = \langle a, 0, -c \rangle \times \langle 0, b, -c \rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\boldsymbol{k}} \\ a & 0 & -c \\ 0 & b & -c \end{vmatrix} = \langle bc, ac, ab \rangle$$

Therefore,

$$bc(x - a) + ac(y) + ab(z) = 0$$

$$bcx + acy + abz = abc$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Since the integrand is a function of z, the simplest order of integration is likely the one with the z integral on the outside, 2 or 4. We use the 2nd.

$$\int_{x=0}^{a} \int_{y=0}^{b\left(1-\frac{x}{a}\right)} \left(\int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} xdz \right) dydx = \int_0^4 \int_0^{4-x} \left(\int_0^{6\left(1-\frac{x}{4}-\frac{y}{4}\right)} xdz \right) dydx$$

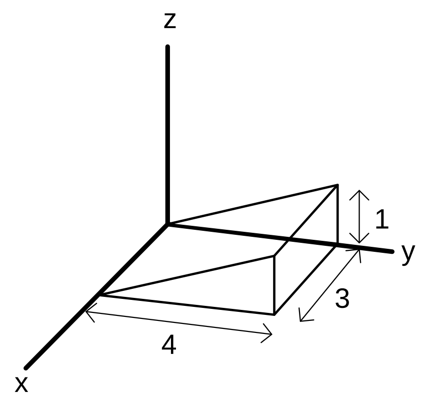
$$= \int_0^4 \left(6x \int_0^{4-x} \left(\left(1 - \frac{x}{4} - \frac{y}{4}\right) \right) dy \right) dx$$

$$= \int_0^4 6x \left((4 - x) - \frac{x}{4}(4 - x) - \frac{(4 - x)^2}{8} \right) dx$$

$$= \int_0^4 \left(12x - 6x^2 + \frac{3}{4}x^3 \right) dx$$

$$= \left(6x^2 - 2x^3 + \frac{3}{16}x^4 \Big|_0^4 \right) = 16$$

Example 6: Integrate $f(x, y, z) = z$ over the wedge shown below.



Solution: In this case, the top face of the wedge can be represented by the equation of the line in the z-y plane since x is constant.

$$z = \frac{1}{4}y$$

The projection on the x-y plane is a simple rectangular region, R.

$$R = \{(x, y)|\ 0 \le x \le 3, 0 \le y \le 4\}$$

Therefore,

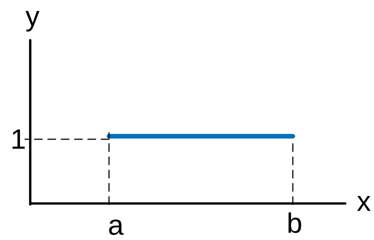
$$\int_{x=0}^{3}\int_{y=0}^{4}\left(\int_{z=0}^{\frac{1}{4}y} z\,dz\right)dydx = \int_0^3\left(\int_0^4\left(\frac{1}{32}y^2\right)dy\right)dx$$

$$= \int_0^3\left(\frac{1}{96}y^3\Big|_0^4\right)dx$$

$$= \frac{2}{3}\cdot 3 = 2$$

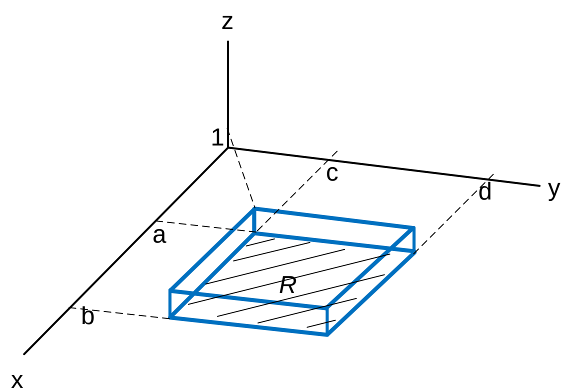
Lastly, we look at how we can use triple integrals to compute the volume of solid objects. To do so we can start with a simple example using single variable integrals. Integrating $f(x) = 1$ over an interval, I, e.g. $[a, b]$, gives the length of that interval.

$$\int_I 1dx = \int_a^b 1dx = (b - a) = \mathit{Length\ of\ Interval}$$



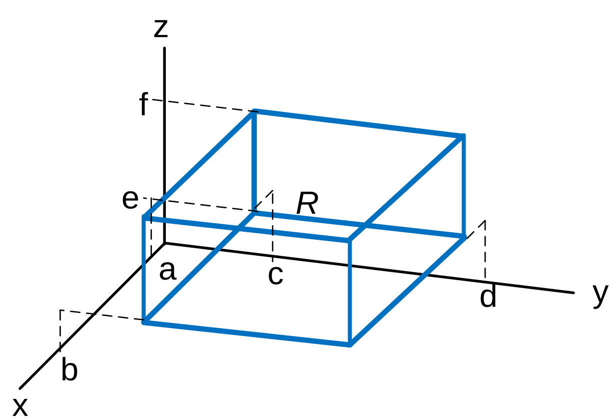
A similar argument for the double integral of $f(x, y) = 1$ over a region, R, can also be made. For example, with $R = (x, y)|\ \{a \leq x \leq b, c \leq y \leq d\}$

$$\iint_R 1dA = \int_c^d \int_a^b 1dxdy = (b - a) \cdot (d - c) = \mathit{Area\ of\ Region}$$

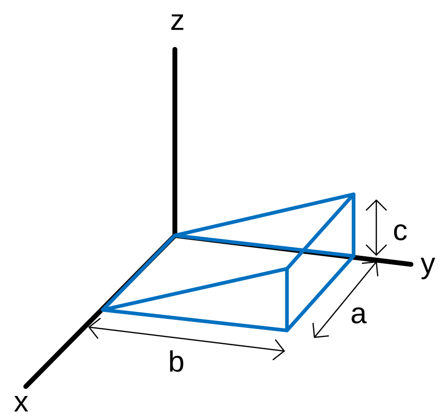


The argument holds for triple integrals, e.g. $R = \{(x, y, z)|\ a \leq x \leq b, c \leq y \leq d, e \leq z \leq f\}$

$$\iiint_R 1dV = \int_e^f \int_c^d \int_a^b 1dxdydz = (b - a) \cdot (d - c) \cdot (f - e) = \mathit{Volume\ of\ Region}$$



Example 7: Derive a formula for the volume of a general wedge from example 6.



Solution: We use the same integral, but with $f(x, y, z) = 1$

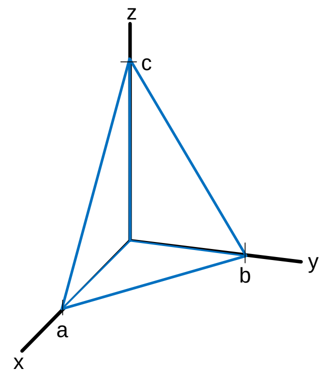
$$\int_{x=0}^{a}\int_{y=0}^{b}\left(\int_{z=0}^{\frac{c}{b}y} dz\right) dydx = \int_{0}^{a}\left(\int_{0}^{b}\frac{c}{b}ydy\right)dx$$

$$= \int_{0}^{a}\left(\frac{c}{2b}y^2\Big|_0^b\right)dx$$

$$= \int_{0}^{a}\frac{bc}{2}dx = \frac{abc}{2}$$

Which makes sense since two wedges placed on top of one another in opposite direction forms a rectangular box with volume, $V = abc$.

Example 7: Derive a formula for the volume of a general tetrahedron as given in example 4.



Solution: We use the same integral, but with $f(x,y,z) = 1$

$$V = \int_{x=0}^{a} \int_{y=0}^{b\left(1-\frac{x}{a}\right)} \left(\int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz \right) dy dx$$

$$= \int_{0}^{a} \left(\int_{0}^{b\left(1-\frac{x}{a}\right)} \left(c\left(1 - \frac{x}{a} - \frac{y}{b}\right)\right) dy \right) dx$$

$$= c \int_{0}^{a} \left(y - \frac{xy}{a} - \frac{y^2}{2b} \Big|_{0}^{b\left(1-\frac{x}{a}\right)} \right) dx$$

$$= bc \int_{0}^{a} \left(\left(1 - \frac{x}{a}\right) - \frac{x\left(1 - \frac{x}{a}\right)}{a} - \frac{\left(1 - \frac{x}{a}\right)^2}{2} \right) dx$$

$$= bc \int_{0}^{a} \left(\frac{1}{2} - \frac{x}{a} + \frac{x^2}{2a^2} \right) dx$$

$$= bc \left(\frac{x}{2} - \frac{x^2}{2a} + \frac{x^3}{6a^2} \Big|_{0}^{a} \right)$$

$$= bc \left(\frac{a}{2} - \frac{a^{\cancel{2}}}{2\cancel{a}} + \frac{a^{\cancel{3}}}{6\cancel{a^2}} \right)$$

$$= abc \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \frac{1}{6} \right)$$

$$V = \frac{abc}{6}$$

Final Summary for Multiple Integration – Triple Integrals

Triple Integral Over a Boxed Region

The triple integral of a continuous function $f(x, y, z)$ over a box, R is:

$$\iiint_R f(x, y, z)dV = \int_{x=a}^{b}\int_{y=c}^{d}\int_{z=p}^{q} f(x, y, z)dzdydx$$

Where,

$$R = (x, y, z)|\ a \le x \le b, c \le y \le d, p \le z \le q$$

Furthermore, the integral can be evaluated in any order.

Triple Integral Over a General Region

For a general region the triple integral is best written as follows:

$$\iiint_D zdV = \iint_R \left(\int_{q_1}^{q_2} f(x, y, z)dq\right)dA$$

Where, the inner integral is with respect to any one of the three variables, i.e. we choose q to be one of the elements of the set $\{x, y, z\}$. The region, R, is the projection of the solid object in the plane defined by the two remaining variables. We can then express the dA in two different orders for each of the 3 possible projections as shown below.

Area and Volume

The *area* of a general region can be found using the double integral of $f(x, y) = 1$ over a region, R. For example

$$R = \{(x, y)|\ a \le x \le b, c \le y \le d\}$$

$$\iint_R 1dA = \int_c^d\int_a^b 1dxdy = (b - a)\cdot(d - c) = \ Area\ of\ Region$$

The *volume* of a general region can be found using the triple integral of $f(x, y, z) = 1$ over a region, R. For example

$$R = \{(x, y, z)|\ a \le x \le b, c \le y \le d, e \le z \le f\}$$

$$\iiint_R 1dV = \int_e^f\int_c^d\int_a^b 1dxdydz = (b - a)\cdot(d - c)\cdot(f - e) = \ Volume\ of\ Region$$

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