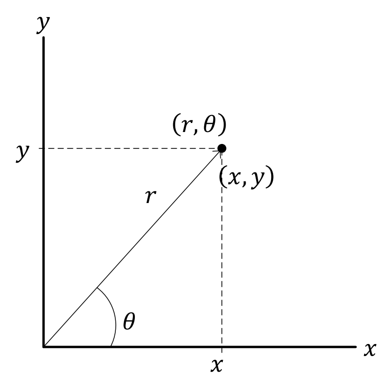
Multiple Integration – Polar, Cylindrical, and Spherical Coordinates

Up to this point we have been performing integration on functions described in the rectangular coordinate system, which as we have seen in a previous lesson, is not always the most convenient coordinate system to work with. In this earlier lesson we showed how some problems can be more easily described using an alternate coordinate system. In this lesson we will show how integration also becomes much simpler for certain functions when they are described in one of these alternate coordinate systems. From a geometric standpoint when we use one of these alternate coordinate systems, we essentially change the variables that are used to describe a location in space, e.g. $(x, y) \rightarrow (r, \theta)$. This process can be described with a general, and very important, formula called the *Change of Variables Formula*, which we will cover in an upcoming lesson. The change of variables we perform in this lesson are specific cases of this general process. As we showed in an earlier lesson, using one of these alternate coordinate systems allows certain functions to be more easily described. However, for multiple integration even more important is that fact that these coordinate systems can also be used to simplify the description of the integration region.

Double Integrals in Polar Coordinates

Polar coordinates are especially convenient when the domain of integration has radial symmetry in the x-y plane. As shown in the figure below, each point is identified by its distance from the origin, r, and the angular value, θ, between the positive x-axis and a line that connect the origin to the point. The relationship between rectangular and polar coordinate variables is also shown below.

Rectangular vs Polar Coordinates



Change of Coordinate Variables

$$x = r\,cos(\theta)$$

$$y = r\,sin(\theta)$$

Using the above change of variable formulas, we can write the double integral of a two variable function over a region, R, as

$$\iint_R f(x, y)dA = \iint_R f(r\,cos(\theta), r\,sin(\theta))dA$$

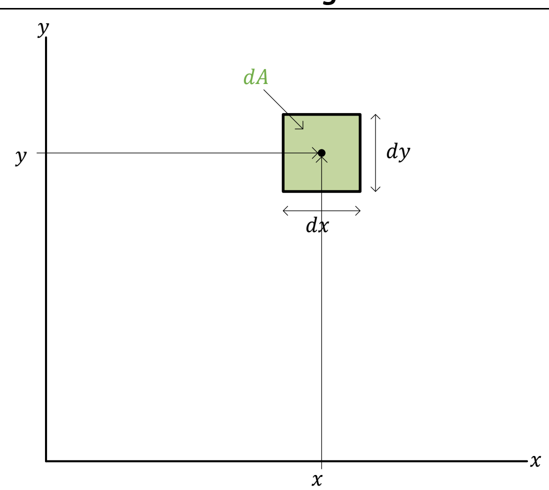
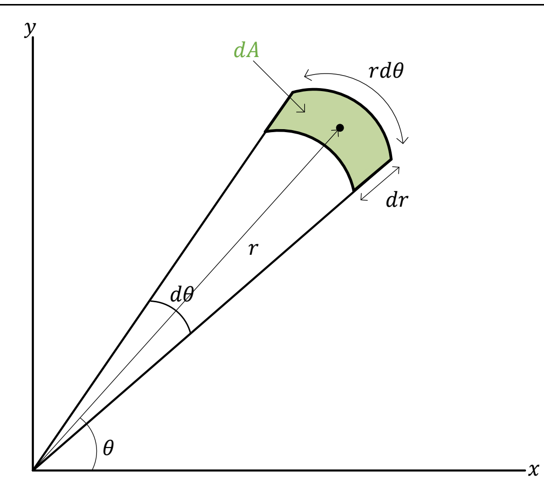
Where, we used the change of variable formulas $x = r\,cos(\theta)$ and $y = r\,sin(\theta)$.

However, as you will recall, for rectangular coordinates the area element is the rectangle shown in the figure below and defined as $dA = dxdy$. In polar coordinates the area element is a so-called *polar rectangle,* which is created from an infinitesimal change in the two polar variables. Using the figure, we see that a small change in r results in this polar rectangle having one side length of dr. The other side length is equal to an arc length, which is created by a small change in θ, $d\theta$. The formula for arc length can be derived using the following proportion

$$\frac{l}{\theta} = \frac{2\pi r}{2\pi} \rightarrow l = r\theta$$

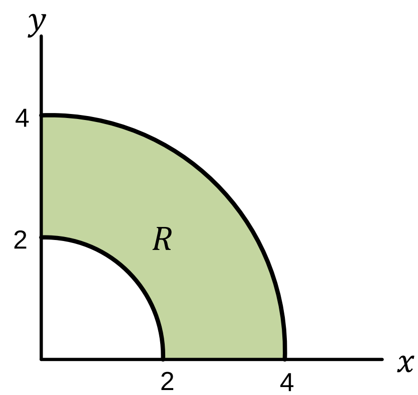
With regard to the figure below we have $dl = rd\theta$. Therefore, for polar coordinates the area element can be defined as $dA = rdrd\theta$. The double integral in polar coordinates can then be written as shown.

Double Integral in Polar Coordinates
For a continuous function, f, on the domain, D $D = \{(r,\theta)\vert\ r_1 \le r \le r_2, \theta_1 \le \theta \le \theta_2\}$ $\iint_D f(x,y)dA = \int_{\theta=\theta_1}^{\theta_2} \int_{r=r_1}^{r_2} f(r\cos(\theta), r\sin(\theta))\, rdrd\theta$

Area Element in Rectangular Coordinates	***Area Element in Polar Coordinates***
	

Let’s do some examples using polar coordinates.

Example 1: Compute the double integral of $f(x, y) = x + y$, over the region, R, shown below.



Solution: Evaluating this integral using rectangular coordinates would require the region to decomposed, (We did a similar example in a previous lesson). The region displays a radial symmetry and therefore lends itself to the use of polar coordinates. In polar coordinates the region can be defined as follows:

$$R = \left\{(r, \theta)\middle|\ 2 \le r \le 4, 0 \le \theta \le \frac{\pi}{2}\right\}$$

In other words, the region is the equivalent of a boxed region in rectangular coordinates. The integral is then

$$\int_{\theta=0}^{\frac{\pi}{2}} \int_{r=2}^{4} f(r\,cos(\theta), r\,sin(\theta))\, r dr d\theta = \int_{0}^{\frac{\pi}{2}} \int_{2}^{4} (r\,cos(\theta) + r\,sin(\theta))\, r dr d\theta$$

$$= \int_{0}^{\frac{\pi}{2}} (cos(\theta) + sin(\theta)) \left(\int_{2}^{4} r^2\, dr \right) d\theta$$

$$= \int_{0}^{\frac{\pi}{2}} (cos(\theta) + sin(\theta)) \left(\frac{4^3 - 2^3}{3} \right) d\theta$$

$$= \frac{56}{3} \int_{0}^{\frac{\pi}{2}} (cos(\theta) + sin(\theta))\, d\theta$$

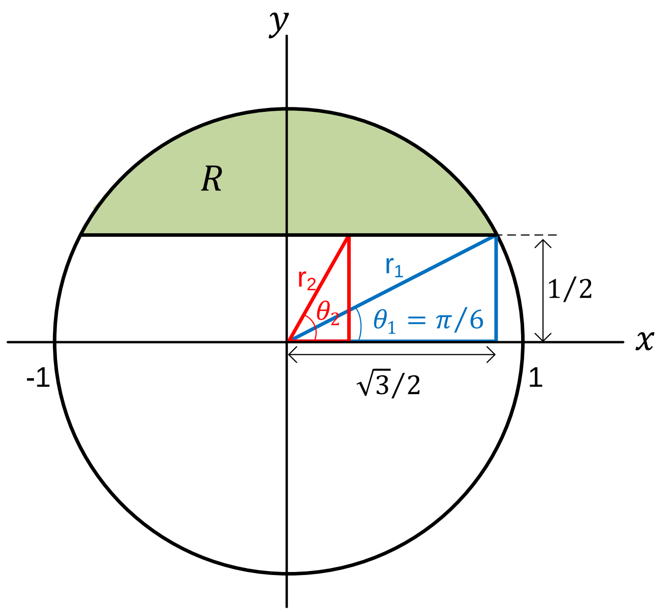
$$= \frac{56}{3} (sin(\theta) - cos(\theta)) \Big|_{0}^{\frac{\pi}{2}}$$

$$= \frac{56}{3} \left(sin\left(\frac{\pi}{2}\right) - cos\left(\frac{\pi}{2}\right) - sin(0) + cos(0) \right) = \frac{112}{3}$$

Example 2: Evaluate the following double integral using polar coordinates.

$$\int_{x=-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \int_{y=\frac{1}{2}}^{\sqrt{1-x^2}} \left(\frac{y}{x^2+y^2}\right) dydx$$

Solution: We start by sketching the region of integration in an attempt define it using polar coordinates. In rectangular coordinates the y integration upper limit describes the top portion of a unit circle, while the remaining limits are fixed values. The region is shown below.



Describing the region in polar coordinates requires us to determine how θ and r varies. From the figure we see that the angle varies from $\pi/6$ to $5\pi/6$. The upper limit of the r value is fixed to the outer rim of the circle, i.e. 1. However, the lower limit varies as the hypotenuse of a triangle that has a fixed height of $1/2$. This is illustrated in the figure which shows the hypotenuse changing from r_1 to r_2 as the angle varies from θ_1 to θ_2. Using right triangle trigonometry, the lower limit of r can be written as

$$r = \frac{1/2}{\sin(\theta)} = \frac{1}{2\sin(\theta)}$$

Therefore, the region can be described in polar coordinates as

$$R = \left\{(r,\theta)| \ \frac{1}{2\,sin(\theta)} \leq r \leq 1, \pi/6 \leq \theta \leq 5\pi/6\right\}$$

The integrand also needs to undergo a change of variables.

$$\frac{y}{x^2+y^2} = \frac{r\,sin(\theta)}{r^2} = \frac{sin(\theta)}{r}$$

Finally, the double integral in polar coordinates is given as

$$\int_{\theta=\pi/6}^{5\pi/6}\int_{r=1/2\,sin(\theta)}^{1}\left(\frac{sin(\theta)}{r}\right)rdrd\theta = \int_{\pi/6}^{5\pi/6}\int_{1/2\,sin(\theta)}^{1} sin(\theta)\,drd\theta$$

Evaluating we have

$$\int_{\pi/6}^{5\pi/6}\int_{1/2\,sin(\theta)}^{1} sin(\theta)\,drd\theta = \int_{\pi/6}^{5\pi/6} sin(\theta)\left(\int_{1/2\,sin(\theta)}^{1} 1dr\right)d\theta$$

$$= \int_{\pi/6}^{5\pi/6} sin(\theta)\left(1-\frac{1}{2\,sin(\theta)}\right)d\theta$$

$$= \int_{\pi/6}^{5\pi/6}\left(sin(\theta)-\frac{1}{2}\right)d\theta$$

$$= -\left(cos(\theta)+\frac{1}{2}\theta\Big|_{\pi/6}^{5\pi/6}\right)$$

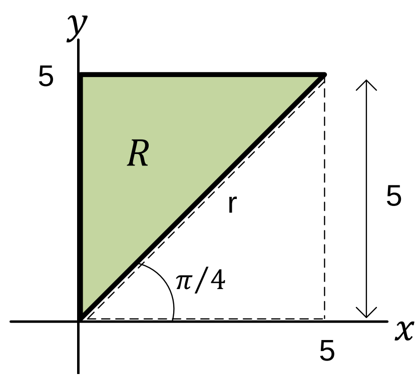
$$= -\left(\left(cos(5\pi/6)+\frac{1}{2}\cdot\frac{5\pi}{6}\right)-\left(cos(\pi/6)+\frac{1}{2}\cdot\frac{\pi}{6}\right)\right)$$

$$= \sqrt{3}-\frac{\pi}{3}$$

Example 3: Evaluate the following double integral using polar coordinates.

$$\int_{y=0}^{5}\int_{x=0}^{y}(x)dxdy$$

Solution: We again start by sketching the region of integration in an attempt define it using polar coordinates.



In this case, θ varies from $\pi/4$ to $\pi/2$. As the angle varies the r variable is equal to the hypotenuse of the dotted line triangle with a constant height of 5.

$$r = \frac{5}{\sin(\theta)}$$

Therefore, r varies from 0 to $5/sin(\theta)$. The double integral in polar coordinates is then

$$\int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{5/sin(\theta)} (r\,cos(\theta))\,rdrd\theta = \int_{\pi/4}^{\pi/2} cos(\theta)\left(\int_0^{5/sin(\theta)} r^2 dr\right) d\theta$$

$$= \int_{\pi/4}^{\pi/2} cos(\theta)\left(\frac{125}{3}\left(\frac{1}{sin^3(\theta)}\right)\right) d\theta$$

$$= \frac{125}{3}\int_{\pi/4}^{\pi/2} \frac{cos(\theta)}{sin^3(\theta)} d\theta$$

At this point we use the following substitution.

$$u = sin(\theta) \qquad\qquad du = cos(\theta)d\theta$$

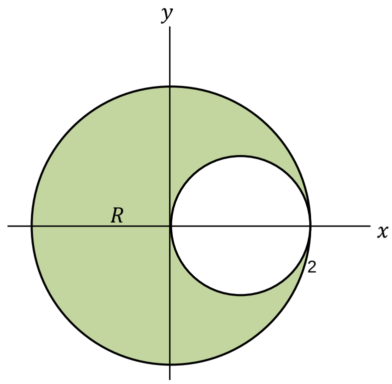
$$\frac{125}{3}\int_{\pi/4}^{\pi/2} cos(\theta)\,sin^3(\theta)\,d\theta = \frac{125}{3}\int_{sin(\pi/4)}^{sin(\pi/2)} u^{-3} du$$

$$= -\frac{125}{6} u^{-2}\Big|_{\sqrt{2}/2}^{1}$$

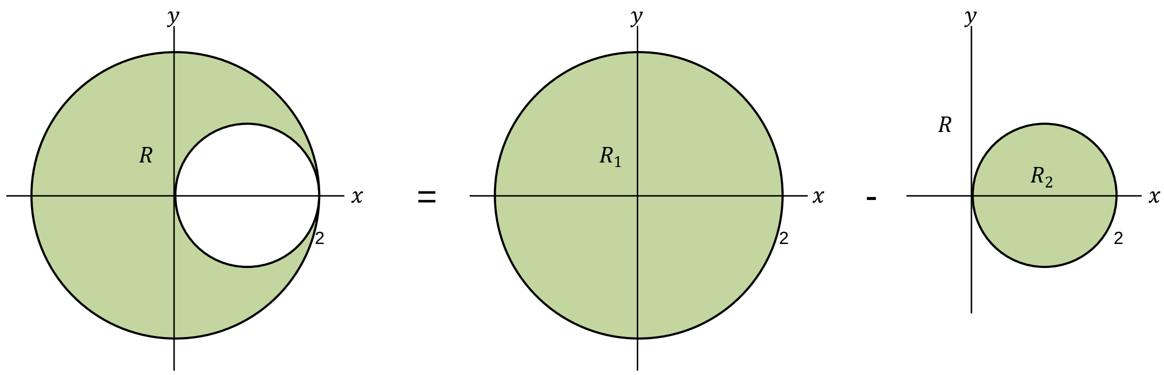
$$= -\frac{125}{6}(1-2)$$

$$= \frac{125}{6}$$

Example 4: Compute the double integral of $f(x,y) = \sqrt{x^2 + y^2}$, over the region, R, shown.



Solution: In this case we can decompose the region as shown below.



Where,

$$R_1 = \{(r,\theta)|\ 0 \le r \le 2, 0 \le \theta \le 2\pi\}$$

To determine R_2, recall the polar equation for a circle or radius a centered at $(a, 0)$ is given as

$$r = 2a\, cos(\theta)$$

In our case $a = 1$, therefore r is related to θ as

$$r = 2\, cos(\theta)$$

Note this shifted circle is created when θ changes by π. To keep r positive we let θ vary from $-\pi/2$ to $\pi/2$. In this case, the circle is created staring at $r = 0$ and moving counterclockwise drawing the bottom half of the circle first.

$$R_2 = \{(r,\theta)|\ 0 \le r \le 2\, cos(\theta), -\pi/2 \le \theta \le \pi/2\}$$

The double integral for the first region is

$$\int_{r=0}^{2}\int_{\theta=0}^{2\pi}(r)\, r d\theta dr = \int_0^2 r^2\left(\int_0^{2\pi} d\theta\right)dr = 2\pi\int_0^2 r^2 dr = \frac{16\pi}{3}$$

The double integral for the second region is

$$\int_{\theta=-\pi/2}^{\pi/2}\int_{r=0}^{2\,cos(\theta)}(r)\,rdrd\theta = \int_{-\pi/2}^{\pi/2}\left(\int_{0}^{2\,cos(\theta)} r^2dr\right)d\theta$$

$$= \frac{8}{3}\int_{-\pi/2}^{\pi/2} cos^3(\theta)\,d\theta$$

$$= \frac{8}{3}\int_{-\pi/2}^{\pi/2} (1 - sin^2(\theta))\,cos(\theta)\,d\theta$$

At this point we use the following substitution.

$$u = sin(\theta) \qquad\qquad du = cos(\theta)d\theta$$

$$\frac{8}{3}\int_{sin(-\pi/2)}^{sin(-\pi/2)} (1 - u^2)du = \frac{8}{3}\left(u - \frac{1}{3}u^3\Big|_{-1}^{1}\right)$$

$$= \frac{8}{3}\left(\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)\right)$$

$$= \frac{32}{9}$$

Finally, the double integral is

$$\iint\limits_{R} f(x,y)dA = \iint\limits_{R_1} f(x,y)dA - \iint\limits_{R_2} f(x,y)dA$$

$$= \left(\frac{16\pi}{3}\right) - \left(\frac{32}{9}\right)$$

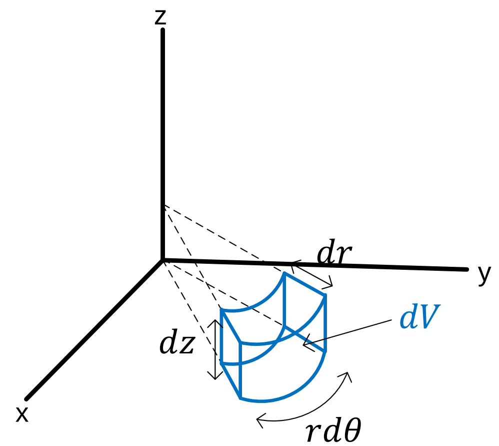
$$= \frac{48\pi - 32}{9}$$

Triple Integrals in Cylindrical Coordinates

Cylindrical coordinates are identical to polar coordinates with an added third dimension that is mapped one-to-one to the z variable in rectangular coordinates. The coordinate system is useful for problems that display axial symmetry. The triple integral is a straightforward extension of the double integral in polar coordinates. It is shown below along with the volume element, dV, used.

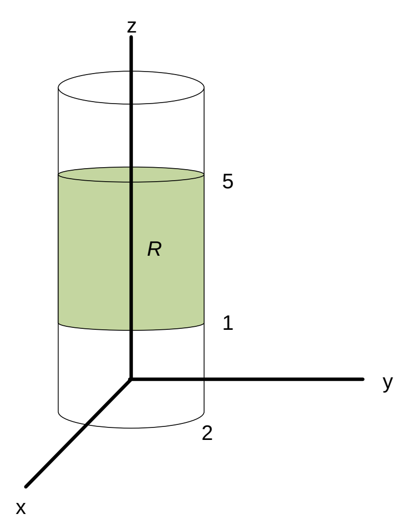
Triple Integral in Cylindrical Coordinates
For a continuous function, f, on the domain, D $D = \{(r,\theta,z)\mid\ r_1 \le r \le r_2, \theta_1 \le \theta \le \theta_2, z_1 \le z \le z_2\}$ $\iiint\limits_D f(x,y,z)dV = \int_{\theta=\theta_1}^{\theta_2}\int_{r=r_1}^{r_2}\int_{z=z_1}^{z_2} f(r\cos(\theta), r\sin(\theta), z)\, r dz dr d\theta$

Volume Element in Cylindrical Coordinates



Let's do some examples using cylindrical coordinates.

Example 5: Compute the triple integral of $f(x,y,z) = z\sqrt{x^2+y^2}$, over the region, R, shown.



Solution: Since the region is a cylinder it lends itself perfectly to modeling in cylindrical coordinates. The region can be defined as

$$R = \{(r, \theta, z) |\ 0 \le r \le 2, 0 \le \theta \le 2\pi, 1 \le z \le 5\}$$

And the integral is computed as follows

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=1}^{5} f(r\cos(\theta), r\sin(\theta), z)\, r dz dr d\theta = \int_{0}^{2\pi} \int_{0}^{2} \int_{1}^{5} (zr)\, r dz dr d\theta$$

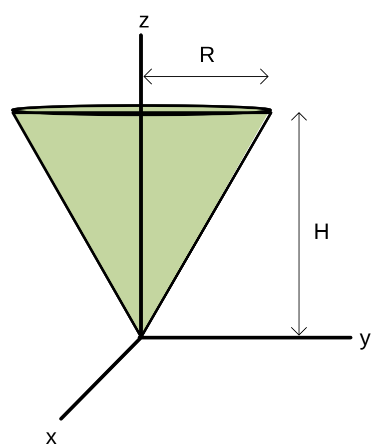
$$= \int_{0}^{2\pi} \int_{0}^{2} r^2 \left(\int_{1}^{5} z\, dz \right) dr d\theta$$

$$= \int_{0}^{2\pi} \left(\int_{0}^{2} 12r^2\, dr \right) d\theta$$

$$= \int_{0}^{2\pi} 32 d\theta$$

$$= 64\pi$$

Example 6: Derive the formula for the volume of the right-circular cone shown below.



Solution: Recall from the previous lesson that we can express the volume of a solid region as

$$V = \iiint_R 1 dV$$

Since the region, i.e. the cone, display radial symmetry, it can most easily be described using cylindrical coordinates. The variables, θ and z, vary over fixed values as

$$0 \le \theta \le 2\pi \qquad\qquad 0 \le z \le H$$

In addition, r can be written as a function of z using the slope of the line in the y-z plane.

$$z = \frac{H}{R} r \rightarrow r(z) = \frac{R}{H} z$$

The region is then

$$R = \left\{ (r, \theta, z) \middle| \ 0 \leq r \leq \frac{R}{H} z, 0 \leq \theta \leq 2\pi, 0 \leq z \leq H \right\}$$

Therefore, we have

$$\iiint_R 1 dV = \int_{\theta=0}^{2\pi} \int_{z=0}^{H} \left(\int_{r=0}^{\frac{R}{H}z} r\, dr \right) dz d\theta$$

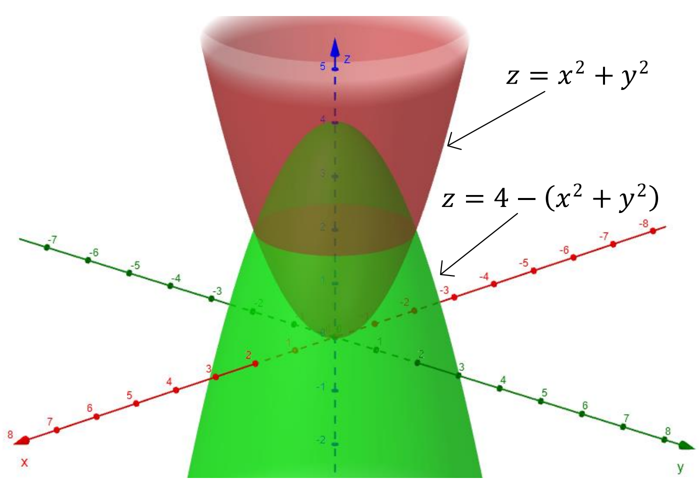
$$= \int_{\theta=0}^{2\pi} \int_{z=0}^{H} \left(\frac{1}{2} \left(\frac{R}{H} z \right)^2 \right) dz d\theta$$

$$= \frac{1}{2} \cdot \frac{R^2}{H^2} \int_{\theta=0}^{2\pi} \left(\int_{z=0}^{H} z^2 \, dz \right) d\theta$$

$$= \frac{1}{6} \cdot \frac{R^2}{\cancel{H^2}} \cdot \cancel{H^3} \int_{\theta=0}^{2\pi} d\theta$$

$$V = \frac{\pi}{3} R^2 H$$

Example 7: Find the volume of the region between the two surfaces shown in the figure.



Solution: In this case we decompose the region into a bottom and top half. In both cases θ varies from 0 to π. For the bottom half z varies from 0 to 2 and for the top from 2 to 4. Next, we can change the variables in both equations for the following cylindrical equations.

$$z_B = r^2 \qquad\qquad z_T = 4 - r^2$$

With these relationships we can state the interval of r as a function of z.

Bottom Region	***Top Region***
$0 \le r \le \sqrt{z}$	$0 \le r \le \sqrt{4-z}$

The volume is then

$$V = \int_{\theta=0}^{2\pi}\int_{z=0}^{2}\left(\int_{r=0}^{\sqrt{z}} r\,dr\right)dzd\theta + \int_{\theta=0}^{2\pi}\int_{z=2}^{4}\left(\int_{r=0}^{\sqrt{4-z}} r\,dr\right)dzd\theta$$

$$= \frac{1}{2}\int_{0}^{2\pi}\left(\int_{0}^{2} z\,dz\right)d\theta + \frac{1}{2}\int_{0}^{2\pi}\left(\int_{2}^{4}(4-z)\,dz\right)d\theta$$

$$= \frac{1}{4}\int_{0}^{2\pi} 4d\theta + \frac{1}{2}\int_{0}^{2\pi}\left(4z - \frac{1}{2}z^2\Big|_{2}^{4}\right)d\theta$$

$$= 2\pi + \frac{1}{2}\int_{0}^{2\pi} 2d\theta$$

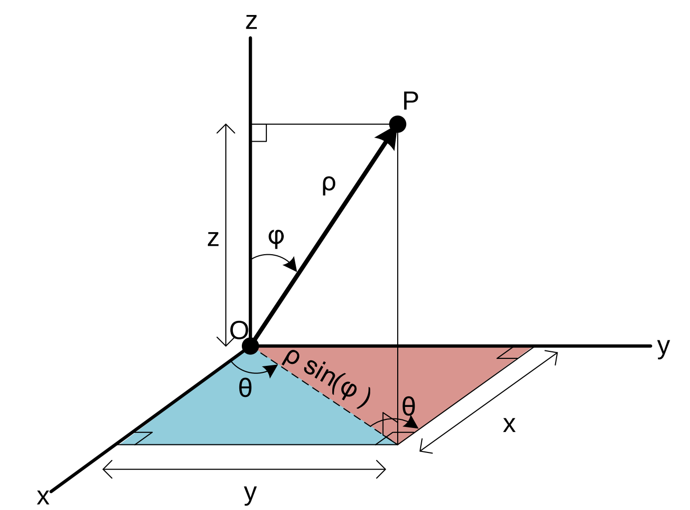
$$= 2\pi + 2\pi$$

$$= 4\pi$$

Note, we could have also used the symmetry of the two half compute one of the integrals and multiply the answer by two.

Triple Integrals in Spherical Coordinates

Spherical coordinates are convenient when the domain features radial symmetric in three dimensions. The spherical coordinate system was introduced in a previous lesson and is shown below for review.

<table>
<tr><th><u>Spherical Coordinates</u></th><th><u>Change of Coordinate Variables</u></th></tr>
<tr><td>
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$x = \rho \, sin(\phi) \, cos(\theta)$

$y = \rho \, sin(\phi) \, sin(\theta)$

$z = \rho \, cos(\phi)$

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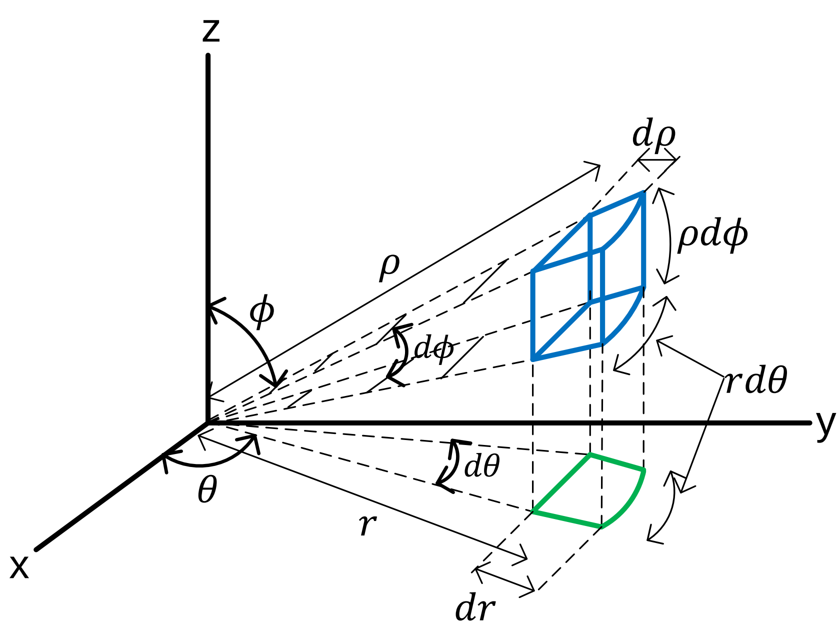
The volume element is illustrated in the figure below. The volume of the blue wedge is

$$dV = (rd\theta)(\rho d\phi)(d\rho)$$

From the figure above, we see that $r = \rho \, sin(\phi)$. Substituting this we have

$$dV = \rho^2 \, sin(\phi) \, d\rho d\phi d\theta$$



The triple integral in spherical coordinates is given below.

Triple Integral in Spherical Coordinates
For a continuous function, f, on the domain, D $D = \{(\rho,\phi,\theta)\mid\ \rho_1 \le \rho \le \rho_2, \phi_1 \le \phi \le \phi_2, \theta_1 \le \theta \le \theta_2,\}$ $\iiint_D f(x,y,z)dV$ $= \int_{\theta=\theta_1}^{\theta_2}\int_{\phi=\phi_1}^{\phi_2}\int_{\rho=\rho_1}^{\rho_2} f(\rho\, sin(\phi)\, cos(\theta)\,, \rho\, sin(\phi)\, sin(\theta)\,, \rho\, cos(\phi))\, \rho^2\, sin(\phi)\, d\rho d\phi d\theta$

Let's do some examples.

Example 8: Derive the formula for the volume of a sphere.

Solution: In spherical coordinates the region of a sphere can be given as

$$R = \{(\rho,\phi,\theta)\mid\ 0 \le \rho \le R, 0 \le \phi \le \pi, \pi \le \theta \le \pi,\}$$

Therefore, we have

$$V = \int_{\theta=-\pi}^{\pi}\int_{\phi=0}^{\pi}\left(\int_{\rho=0}^{R}\rho^2 d\rho\right) sin(\phi)\, d\phi d\theta$$

$$= \frac{1}{3}R^3\int_{-\pi}^{\pi}\left(\int_0^{\pi} sin(\phi)\, d\phi\right) d\theta$$

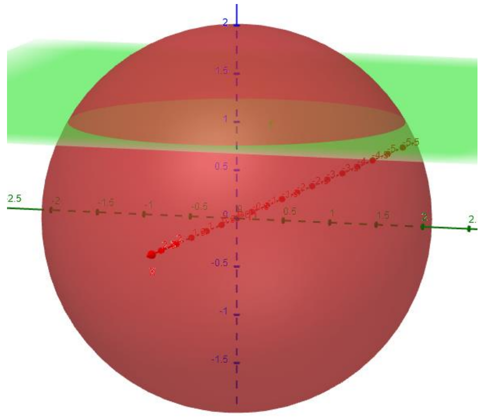
$$= \frac{1}{3}R^3\int_{-\pi}^{\pi} -(cos(\pi) - cos(0))d\theta$$

$$= \frac{1}{3}R^3\int_{-\pi}^{\pi} 2d\theta$$

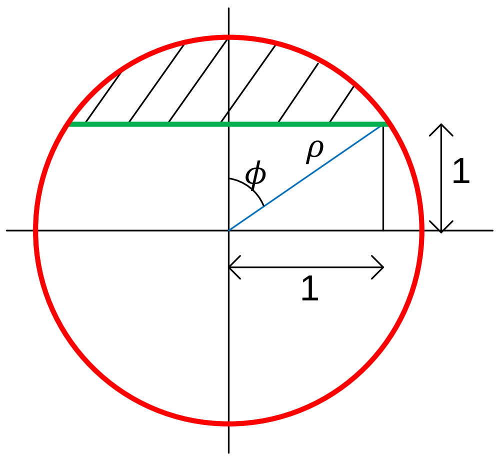
$$= \frac{2}{3}R^3(2\pi)$$

$$V = \frac{4}{3}\pi R^3$$

Example 9: Find the volume of the region bounded below by $z = 1$ and above by a sphere of radius 2.



Solution: We first need to define the region in terms of the spherical coordinate variables, (ρ, ϕ, θ). The angle, θ, lies in the x-y plane and varies over its entire domain, i.e. $[-\pi, \pi]$. To determine the two other variables, we draw the projection in the y-z plane below.



The initial angle, ϕ, can be found using the triangle shown, which has a height of one and a hypotenuse equal to the radius of the sphere, 2. Therefore, the initial angle is

$$\phi = cos^{-1}(1/2) = \frac{\pi}{3}$$

Note since θ varies from $-\pi$ to π the entire region is covered by having ϕ vary from 0 to $\frac{\pi}{3}$. As we cover this region the upper limit of ρ is fixed to 2 and the lower limit is related to ϕ as

$$cos(\phi) = \frac{1}{\rho} \rightarrow \rho = \frac{1}{cos(\phi)}$$

The volume is then equal to the following triple integral.

$$V = \int_{\phi=0}^{\frac{\pi}{3}} \int_{\theta=-\pi}^{\pi} \int_{\rho=\frac{1}{cos(\phi)}}^{2} 1\, \rho^2 d\rho\, sin(\phi)\, d\theta d\phi$$

$$= \int_{0}^{\frac{\pi}{3}} \int_{-\pi}^{\pi} \left(\int_{\rho=\frac{1}{cos(\phi)}}^{2} \rho^2 d\rho \right) sin(\phi)\, d\theta d\phi$$

$$= \frac{1}{3} \int_{0}^{\frac{\pi}{3}} \int_{-\pi}^{\pi} \left(8 - \frac{1}{cos^3(\phi)}\right) sin(\phi)\, d\theta d\phi$$

$$= \frac{2\pi}{3} \int_{0}^{\frac{\pi}{3}} \left(8\, sin(\phi) - \frac{sin(\phi)}{cos^3(\phi)}\right) d\phi$$

$$= \frac{2\pi}{3} \left(8 \int_{0}^{\frac{\pi}{3}} sin(\phi)\, d\phi + \int_{1}^{\frac{1}{2}} u^{-3} du \right)$$

$$= \frac{2\pi}{3} \left(8 \left(-cos\left(\frac{\pi}{3}\right) + cos(0) \right) + -\frac{1}{2}(4-1) \right)$$

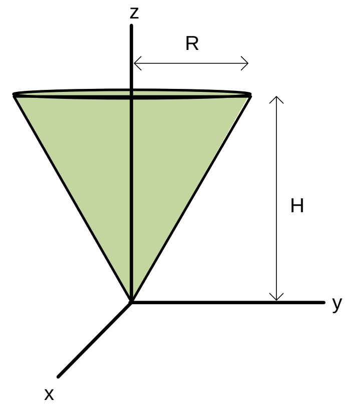
$$= \frac{2\pi}{3} \left(8\left(\frac{1}{2}\right) + -\frac{3}{2} \right)$$

$$V = \frac{5\pi}{3}$$

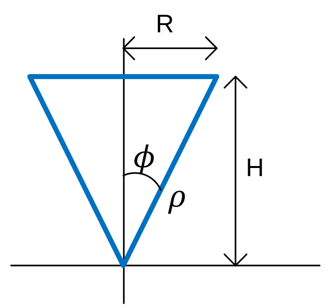
Where, we used the following substitution after splitting the ϕ integral.

$$u = cos(\phi) \qquad\qquad du = -\,sin(\phi)\, d\phi$$

Example 10: In example 6 we derived the formula for the volume of the right-circular cone using cylindrical coordinates. Re-derive the formula using spherical coordinates.



Solution: Similar to the previous problem, the angle, θ, lies in the x-y plane and varies over its entire domain, i.e. $[-\pi, \pi]$. Also similar to the previous problem we draw the y-z plane below to help determine the two other variables.



With this we can use basic trigonometric relationships to find how ϕ and ρ vary. The angle, ϕ, varies from zero to $tan^{-1}(R/H)$, while ρ varies from zero to $H/cos(\phi)$. The volume can now be found using the integral below.

$$V = \int_{\phi=0}^{tan^{-1}(R/H)} \int_{\theta=-\pi}^{\pi} \int_{\rho=0}^{\frac{H}{cos(\phi)}} 1\,\rho^2 d\rho\, sin(\phi)\, d\theta d\phi$$

$$= \int_0^{tan^{-1}(R/H)} sin(\phi) \int_{-\pi}^{\pi} \left(\int_0^{\frac{H}{cos(\phi)}} \rho^2 d\rho \right) d\theta d\phi$$

$$= \frac{1}{3} \int_0^{tan^{-1}(R/H)} sin(\phi) \frac{H^3}{cos^3(\phi)} \left(\int_{-\pi}^{\pi} 1\, d\theta \right) d\phi$$

$$= \frac{2\pi}{3} H^3 \int_0^{tan^{-1}(R/H)} \frac{sin(\phi)}{cos^3(\phi)} d\phi$$

Next, we use the substitution below.

$$u = cos(\phi) \qquad du = -\,sin(\phi)\,d\phi$$

Continuing, we have

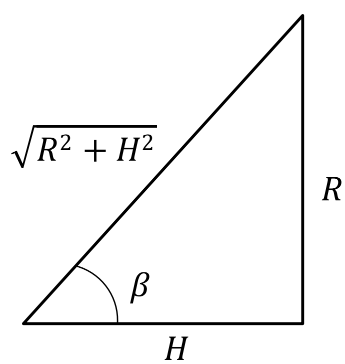
$$= -\frac{2\pi}{3}H^3 \int_1^{cos\left(tan^{-1}(R/H)\right)} u^{-3} du$$

$$= \frac{\pi}{3}H^3\left(\frac{1}{cos^2(tan^{-1}(R/H))} - 1\right)$$

To rewrite the $cos(tan^{-1}(R/H))$ term we let $\beta = tan^{-1}(R/H)$. Then we can write

$$\tan(\beta) = R/H$$

For which we can draw the following triangle.



Then we have

$$cos(\beta) = \frac{H}{\sqrt{R^2 + H^2}}$$

Substituting we find the same volume formula from example 6, admittingly with a lot more effort.

$$V = \frac{\pi}{3}H^3\left(\frac{R^2 + H^2}{H^2} - 1\right)$$

$$V = \frac{\pi}{3}(HR^2 + H^3 - H^3)$$

$$V = \frac{\pi}{3}R^2H$$

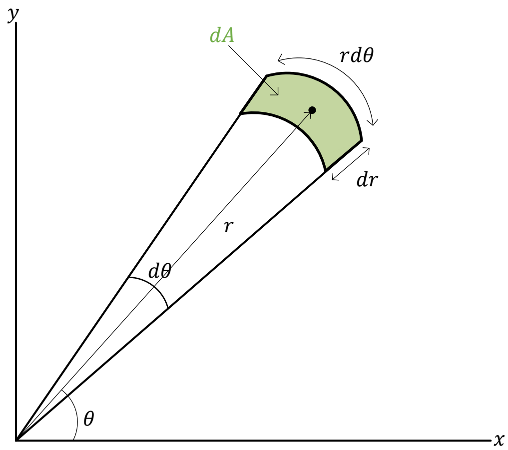
Final Summary for Multiple Integration – Polar, Cylindrical, and Spherical Coordinates

Double Integral in Polar Coordinates

For a continuous function, f, on the domain, D

$$D = \{(r,\theta)|\ r_1 \le r \le r_2, \theta_1 \le \theta \le \theta_2\}$$

$$\iint_D f(x,y)dA = \int_{\theta=\theta_1}^{\theta_2}\int_{r=r_1}^{r_2} f(r\cos(\theta), r\sin(\theta))\, r dr d\theta$$



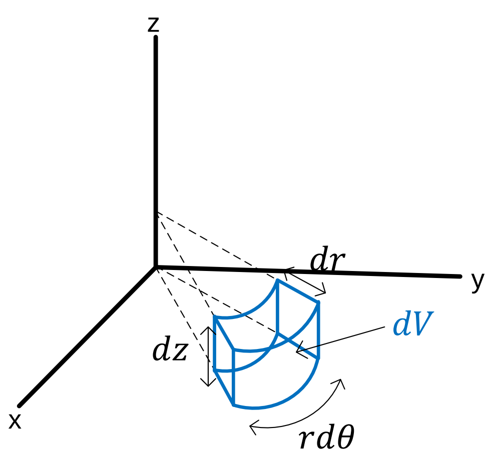
$$dA = rdrd\theta$$

Triple Integral in Cylindrical Coordinates

For a continuous function, f, on the domain, D

$$D = \{(r,\theta,z)|\ r_1 \le r \le r_2, \theta_1 \le \theta \le \theta_2, z_1 \le z \le z_2\}$$

$$\iiint_D f(x,y,z)dV = \int_{\theta=\theta_1}^{\theta_2}\int_{r=r_1}^{r_2}\int_{z=z_1}^{z_2} f(r\cos(\theta), r\sin(\theta), z)\, rdzdrd\theta$$



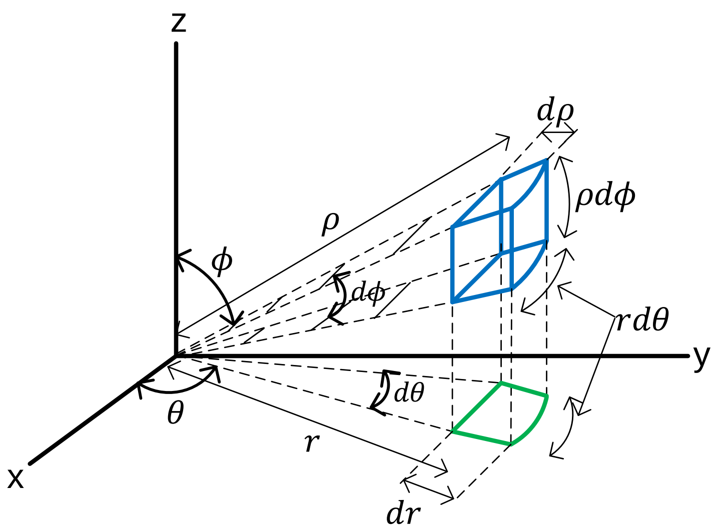
$$dV = rdzdrd\theta$$

Triple Integral in Spherical Coordinates

For a continuous function, f, on the domain, D

$$D = \{(\rho, \phi, \theta) |\ \rho_1 \le \rho \le \rho_2, \phi_1 \le \phi \le \phi_2, \theta_1 \le \theta \le \theta_2, \}$$

$$\iiint_D f(x, y, z) dV$$
$$= \int_{\theta=\theta_1}^{\theta_2} \int_{\phi=\phi_1}^{\phi_2} \int_{\rho=\rho_1}^{\rho_2} f(\rho\, sin(\phi)\, cos(\theta)\,, \rho\, sin(\phi)\, sin(\theta)\,, \rho\, cos(\phi))\, \rho^2\, sin(\phi)\, d\rho d\phi d\theta$$



$$dV = \rho^2\, sin(\phi)\, d\rho d\phi d\theta$$

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