# Multiple Integration – Double Integrals over Rectangular Regions

Similar to multivariable differentiation, multiple integration, i.e. integration applied to multivariable functions, is a natural extension of single variable integrals. Also similar is the myriad of additional scenarios to which multiple integration applies. We begin our study with the integration of functions with two variables, i.e. double integrals. In this lesson we focus on double over rectangular regions. In the next lesson we extend this to arbitrary regions.

*Single Variable Integration Review*

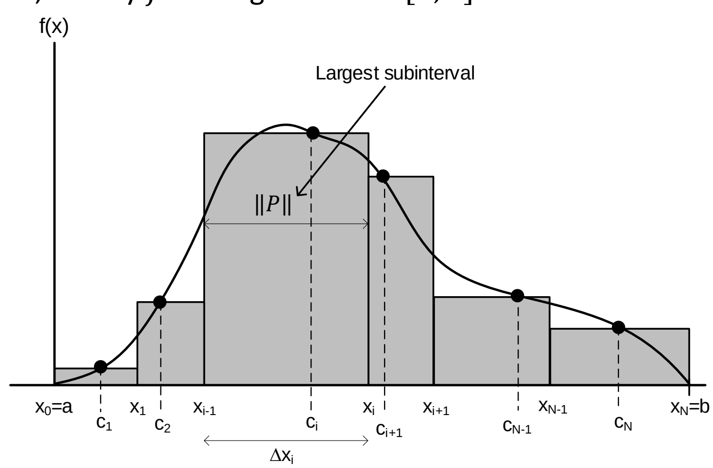
Let's start by briefly reviewing integrals for functions of a single variable. In calculus we defined the definite integral as the limit of the Riemann sum as shown below.

**Definite Integral Definition**

The definite integral of $f$ over $[a, b]$, is the limit of the Riemann Sum.

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0} \left\{ \sum_{i=1}^{N} f(c_i)\Delta x_i \right\}$$

When this limit exists, we say $f$ is integrable over $[a, b]$.




As $\|P\|$, the largest subinterval, approaches zero the number of rectangles must approach infinity and the Riemann sum tends to exact area under the graph of $y = f(x)$.

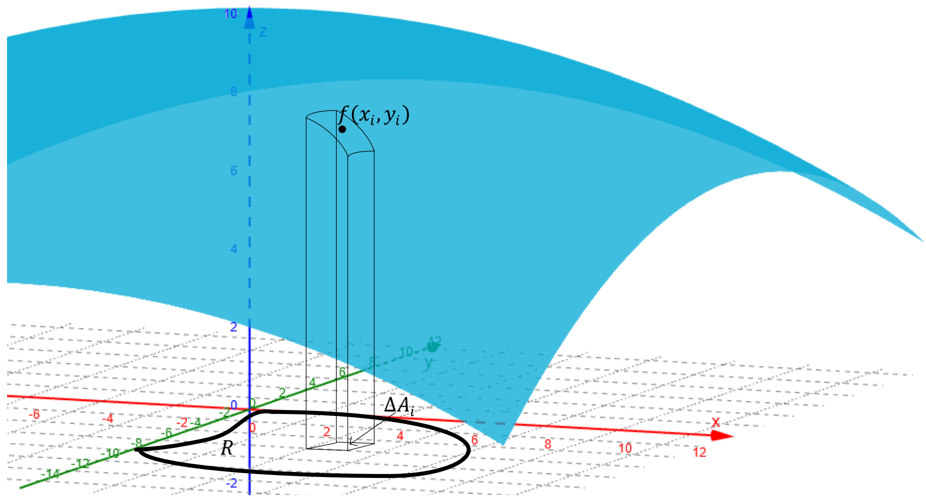
Therefore, performing single variable integration can be interpreted as computing the area under the curve, $y = f(x)$, by summing an infinite number of rectangles with height, $f(x)$ and infinitesimal width, $dx$.

$$A = \int_a^b f(x)dx$$

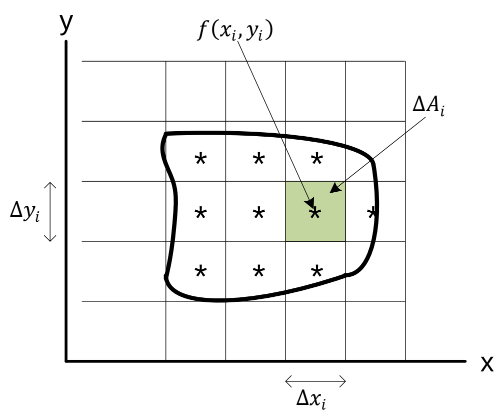
By direct analogy, integration for functions of two variables can be interpreted as computing the *volume* under the *surface*, $z = f(x, y)$, by summing an infinite number of *rectangular prisms* with height, $f(x, y)$ and infinitesimal *base area*, $dA$.

*Double Integrals*

The figure below shows a region, $R$, in the $x$-$y$ plane over which we would like to compute the volume contained under the surface, $f(x,y)$. To do so we divide the region into rectangular prisms of height $f(x_i, y_i)$ and base area, $\Delta A_i$, one of which is shown.




To more conveniently illustrate we redraw the $x$-$y$ plane only, showing the desired region and how it may be divided to approximate the volume.




The volume of the $i^{th}$ rectangular prism is equal to the height multiplied by the area of the base, i.e. $f(x_i, y_i)\Delta A_i$. Dividing the region into $N$ rectangles, we can again use the Reiman sum to approximate the total volume

$$V \cong \sum_{i=1}^{N} f(x_i, y_i)\Delta A_i = \sum_{i=1}^{N} f(x_i, y_i)\Delta x_i \Delta y_i$$

We can then define $\|P\|$ as the largest diagonal distance of all the subregions. Then, similar to the single variable case, as this value approaches zero the number of rectangular prisms must approach infinity and the Riemann sum tends to exact *volume* under the graph of $z = f(x,y)$. The definite double integral over a region, $R$, can then be defined as follows:

$$\iint_R f(x,y)dA = \lim_{\|P\|\to 0} \left\{ \sum_{i=1}^{N} f(x_i, y_i)\Delta A_i \right\}$$

| ***Double Integral over a Rectangular Region*** |
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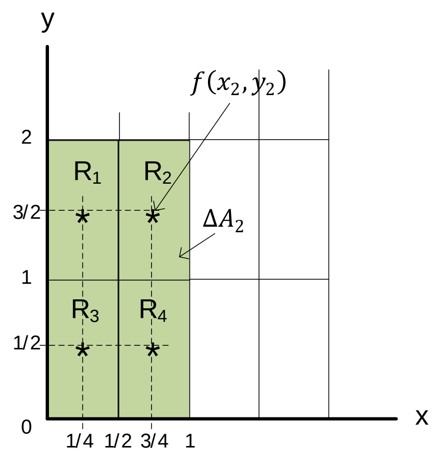
| The definite double integral of $f(x,y)$ over a rectangular region, $R$, is the limit of the Riemann Sum. $$\iint_R f(x,y)dA = \lim_{\|P\|\to 0}\left\{\sum_{i=1}^{N} f(x_i,y_i)\Delta A_i\right\}$$ When this limit exists, we say $f(x,y)$ is integrable over $R$. |

**Example 1:** Approximate the double integral of $f(x,y) = 10 - (x^2 + y^2)$ over the region, $R$.

$$R = \{(x,y)|0 \le x \le 1, 0 \le y \le 2\}$$

Solution: Since the region is rectangular, we may can choose the subregions so that they all fit within the overall region. We divide the region as shown below.




Each subregion has the same base area

$$\Delta A_i = \Delta A = \Delta x \Delta y = \left(\frac{1}{2}\right)(1) = \frac{1}{2}$$

The two variable function, $f(x,y)$, evaluated at the center of each region is used for the rectangular prism heights. The volume can then be approximated as follows

$$V \cong \sum_{i=1}^{4} f(x_i,y_i)\,\Delta A_i$$

$$\cong \Delta A \sum_{i=1}^{4} f(x_i,y_i)$$

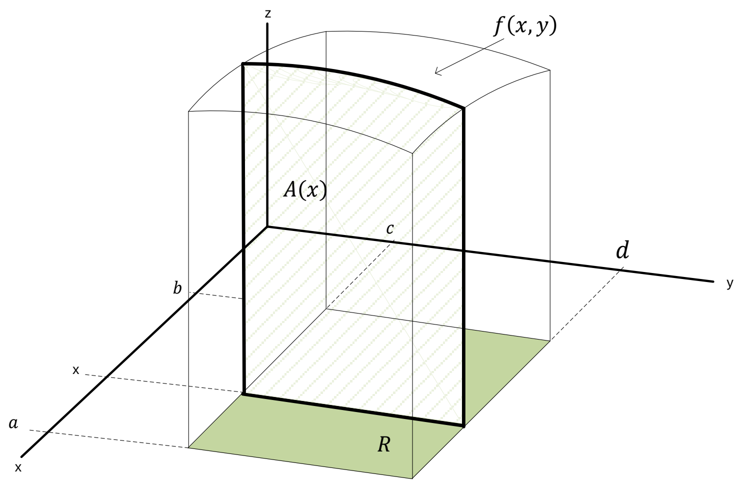
$$\cong \frac{1}{2}\left(f\left(\frac{1}{4},\frac{3}{2}\right) + f\left(\frac{3}{4},\frac{3}{2}\right) + f\left(\frac{1}{4},\frac{1}{2}\right) + f\left(\frac{3}{4},\frac{1}{2}\right)\right)$$

$$\cong \frac{1}{2}\left(\frac{123}{16} + \frac{115}{16} + \frac{155}{16} + \frac{147}{16}\right) = \frac{135}{8} = 16.875$$

*Evaluation of Double Integrals (Fubini's Theorem)*

In example 1 we used the Reiman sum to approximate the given double integral. However, we would like to use a much more efficient method for computing double integrals. For single variable integrals the Fundamental Theorem of Calculus provided us with an efficient method for this task. Unfortunately, there is no direct analog of the Fundamental Theorem of Calculus for double integrals. Nevertheless, we can use the volume interpretation to develop a similar procedure. To do so we start by considering the following rectangular region

$$R = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$$




The volume of the solid formed under the graph of $f(x, y)$ in the region, $R$, is given by

$$V = \iint_R f(x, y) dA$$

Ae can also compute the volume by first considering the area of the vertical slice at a distance $x$ in the $y$-$z$ plane, $A(x)$. This area can then be represented as the area under the graph of $f(x, y)$ as shown below. Note since $x$ is fixed we only need to integrate with respect to $y$.

$$A(x) = \int_c^d f(x, y) dy$$

The volume of the solid can then be found by summing these area slices as $x$ varies from $a$ to $b$.

$$V = \int_a^b A(x) dx$$

$$V = \int_a^b \left[ \int_c^d f(x, y) dy \right] dx$$

Therefore, we have derived an alternate formula that can be used to compute double integrals using two successive single variable integrals.

$$\iint_R f(x, y) dA = \int_a^b \left[ \int_c^d f(x, y) dy \right] dx$$

You can imagine if we instead first created the area slice in the $x$-$z$ plane represented as $B(y)$, the above formula can then be written as

$$\iint_R f(x,y)dA = \int_c^d B(y)dy = \int_c^d \left[\int_a^b f(x,y)dx\right] dy$$

These results were first proven by the Italian mathematician Guido Fubini and they allow for the computation of double integrals using known single variable integration techniques.

| ***Fubini's Theorem*** |
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| The double integral of a continuous function $f(x,y)$ over the rectangular region, $R = \{(x,y) \mid a \le x \le b, c \le y \le d\}$, is equal to the iterated single integrals (in either order). $$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dx\,dy = \int_a^b \int_c^d f(x,y)dy\,dx$$ |

Let's do some examples to illustrate the above theorem.

**Example 2:** Use Fubini's Theorem to compute the double integral from example 1.

$$f(x,y) = 10 - (x^2 + y^2) \qquad\qquad R = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 2\}$$

Solution: We'll solve using both orders. Similar to partial differentiation, when integrating with respect to one of the variables, e.g. $x$, the second variable, e.g. $y$, is considered as a constant.

| ***Integrating with respect to x first*** |
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| $$\begin{aligned}\int_c^d \int_a^b f(x,y)dx\,dy &= \int_0^2 \int_0^1 (10 - x^2 - y^2)dx\,dy \\ &= \int_0^2 \left(10x - \frac{1}{3}x^3 - y^2x\right)\Big\|_{x=0}^{x=1} dy \\ &= \int_0^2 \left(\frac{29}{3} - y^2\right) dy \\ &= \left(\frac{29}{3}y - \frac{1}{3}y^3\right)\Big\|_{y=0}^{y=2} = \frac{58}{3} - \frac{8}{3} = \frac{50}{3} = 16.\bar{6}\end{aligned}$$ |
| ***Integrating with respect to y first*** |
| $$\begin{aligned}\int_a^b \int_c^d f(x,y)dy\,dx &= \int_0^1 \int_0^2 (10 - x^2 - y^2)dy\,dx \\ &= \int_0^1 \left(10y - x^2y - \frac{1}{3}y^3\right)\Big\|_{y=0}^{y=2} dx \\ &= \int_0^1 \left(\frac{52}{3} - 2x^2\right) dx \\ &= \left(\frac{52}{3}x - \frac{2}{3}x^3\right)\Big\|_{x=0}^{x=1} = \frac{50}{3} = 16.\bar{6}\end{aligned}$$ |

Next, let's consider if the two variable function can be expressed as $f(x,y) = g(x)h(y)$.

$$\int_c^d \int_a^b f(x,y)dx\,dy = \int_c^d \left[\int_a^b g(x)h(y)dx\right] dy$$

Since $h(y)$ is constant with respect to the first integral, we can move it to the outside as follows

$$= \int_c^d h(y) \left[\int_a^b g(x)dx\right] dy$$

Furthermore, since $\left[\int_a^b g(x)dx\right]$ is constant, we can move it to the outside of the $y$ integral

$$= \left[\int_a^b g(x)dx\right] \int_c^d h(y)dy$$

Therefore, when $f(x,y) = g(x)h(y)$, the double integral can be expressed as the product of two integrals as shown below.

$$\boxed{\int_c^d \int_a^b f(x,y)dx\,dy = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right)}$$

**Example 3:** Evaluate the double integral of $f(x,y) = xy^2$ over the region, $R$.

$$R = \{(x,y)|0 \le x \le 4, 1 \le y \le 3\}$$

Solution: Since $f(x,y)$ can be written as the product of two single variable functions we can use the results from above.

$$\int_1^3 \int_0^4 (xy^2)dx\,dy = \left(\int_0^4 xdx\right)\left(\int_1^3 y^2dy\right)$$

$$= \left(\frac{1}{2}x^2\Big|_0^4\right)\left(\frac{1}{3}y^3\Big|_1^3\right)$$

$$= (8-0)\left(9-\frac{1}{3}\right)$$

$$= \frac{208}{3} = 69.\overline{3}$$

According to Fubini's Theorem, double integrals can be computed in either order, i.e. $dxdy$ or $dydx$. However, some double integrals are much simpler to perform in a specific order. This concept is illustrated in the next example.

**Example 4:** Evaluate the following double integral.

$$\int_0^1 \int_0^1 (xe^{xy})dx\,dy$$

Solution: Integrating as given we need to use integration by parts for the first integral.

$$u = x \qquad du = dx \qquad dv = e^{xy}dx \qquad v = \frac{1}{y}e^{xy}$$

$$\int_0^1 \left[\int_0^1 (xe^{xy})dx\right]dy = \int_0^1 \left[x\frac{1}{y}e^{xy} - \int \frac{1}{y}e^{xy}dx\right]dy$$

$$= \int_0^1 \left[\left(x\frac{1}{y}e^{xy} - \frac{1}{y^2}e^{xy}\right)\Big|_0^1\right]dy$$

$$= \int_0^1 \left(\frac{1}{y}e^{y} - \frac{1}{y^2}e^{y} + \frac{1}{y^2}\right)dy$$

Which leads us to a very complex integral. Instead of attempting to work through this integral, let's change the order of the original double integral and try again.

$$\int_0^1 \int_0^1 (xe^{xy})dx\,dy = \int_0^1 \left[x\int_0^1 (e^{xy})dy\right]dx$$

$$= \int_0^1 \left[\cancel{x}\left(\frac{1}{\cancel{x}}e^{xy}\right)\Big|_0^1\right]dx$$

$$= \int_0^1 (e^{x} - 1)dx$$

$$= e^{x} - x|_0^1 = (e^1 - 2)$$

**Example 5:** Evaluate the following double integral.

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} cos(x+2y)\, dy\, dx$$

Solution:

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} cos(x+2y)\, dy\, dx = \int_0^{\frac{\pi}{2}}\left(\frac{1}{2}sin(x+2y)\Big|_0^{\frac{\pi}{2}}\right)dx$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}(sin(x+\pi)-sin(x))dx$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}(-sin(x)-sin(x))dx$$

$$=\int_0^{\frac{\pi}{2}}-sin(x)\, dx$$

$$=\left(cos\left(\frac{\pi}{2}\right)-cos(0)\right)=-1$$

Where, we used a simple substitution for the first integral and for the second we use the fact that $sin(x+\pi)=-sin(x)$.

**Example 6:** Evaluate the following double integral.

$$\int_{-1}^{1}\int_0^1 (1+2x+2y)^4 dx\, dy$$

Solution: We start with the following $u$-substitution for the first integral.

$$u = 1+2x+2y \qquad\qquad du = 2dx$$

Therefore

$$\int_{-1}^{1}\left(\int_0^1 (1+2x+2y)^4 dx\right)dy = \int_{-1}^{1}\left(\frac{1}{2}\int_{1+2y}^{3+2y} u^4 du\right)dy$$

$$=\frac{1}{10}\int_{-1}^{1}(3+2y)^5-(1+2y)^5 dy$$

$$=\frac{1}{10}\left(\int_{-1}^{1}(3+2y)^5 dy-\int_{-1}^{1}(1+2y)^5 dy\right)$$

Applying a similar $u$-substitution a second time to both integrals we find

$$\frac{1}{20}\left(\int_{1}^{5} u^5 dy - \int_{-1}^{3} u^5 dy\right) = \frac{1}{120}\left((5^6 - 1) - (3^6 - 1)\right)$$

$$= \frac{1}{120}(5^6 - 3^6) = 124.1\overline{3}$$

**Example 7:** Evaluate the following double integral.

$$\int_0^1 \int_0^1 \sqrt{x + y + 1}\, dxdy$$

Solution: This integral also requires $u$-substitution. The first one being

$$u = x + y + 1 \qquad\qquad du = dx$$

$$\int_0^1 \int_0^1 \sqrt{x + y + 1}\, dxdy = \int_0^1 \left(\int_{y+1}^{y+2} \left(\sqrt{u} du\right)\right) dy$$

$$= \frac{2}{3}\int_0^1 (y + 2)^{3/2} - (y + 1)^{3/2} dy$$

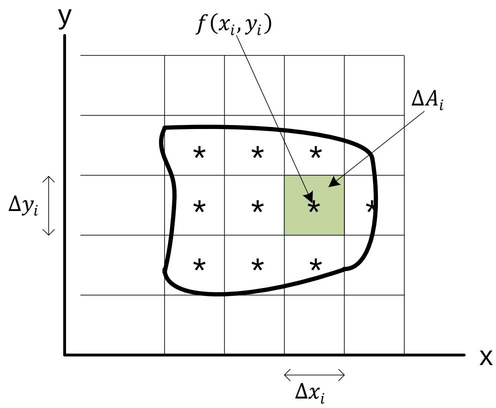
$$= \frac{2}{3}\left(\int_0^1 (y + 2)^{3/2} dy - \int_0^1 (y + 1)^{3/2} dy\right)$$

$$= \frac{2}{3}\left(\int_2^3 (u)^{3/2} du - \int_1^2 (u)^{3/2} du\right)$$

$$= \frac{4}{15}\left(\left(3^{5/2} - 2^{5/2}\right) - \left(2^{5/2} - 1^{5/2}\right)\right)$$

$$= \frac{4}{15}\left(3^{5/2} - 2^{7/2} + 1\right) \cong 1.41$$

## *Final Summary for Multiple Integration – Double Integrals over Rectangular Regions*

| ***Double Integral over a Rectangular Region*** |
| --- |
| The definite double integral of $f(x, y)$ over a rectangular region, $R$, is the limit of the Riemann Sum. $$\iint_R f(x,y)dA = \lim_{\|P\| \to 0} \left\{ \sum_{i=1}^{N} f(x_i, y_i)\Delta A_i \right\} = \lim_{\|P\| \to 0} \left\{ \sum_{i=1}^{N} f(x_i, y_i)\Delta x_i \Delta y_i \right\}$$ When this limit exists, we say $f(x, y)$ is integrable over $R$.   |
| ***Fubini's Theorem*** |
| The double integral of a continuous function $f(x, y)$ over the rectangular region, $R = \{(x, y) \mid a \le x \le b, c \le y \le d\}$, is equal to the iterated single integral (in either order). $$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dx\,dy = \int_a^b \int_c^d f(x,y)dy\,dx$$ When $f(x, y) = g(x)h(y)$, the double integral can be expressed as the product of two integrals as shown below. $$\int_c^d \int_a^b f(x,y)dx\,dy = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right)$$ |

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