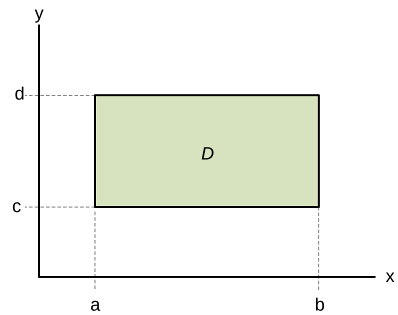
Multiple Integration – Double Integrals over Arbitrary Regions

In the previous lesson we learned how to compute double integrals over rectangular regions. In this lesson we extend this to computing double integrals over more general regions. Specifically, we will work with two different types of regions referred to as vertically simple and horizontally simple We define them below, starting the rectangular region as a baseline from the previous lesson.

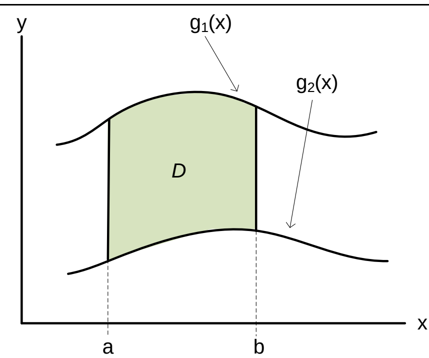
Rectangular Region: The region can be defined in the x and y directions using fixed values.

$$D = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$$



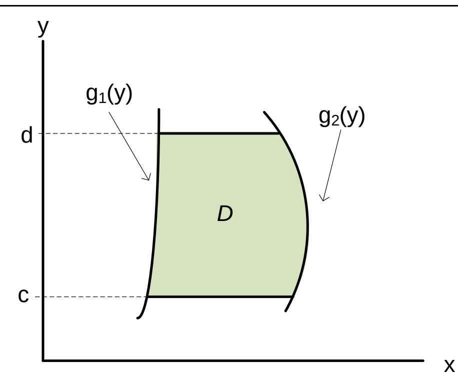
Vertically Simple Region: The region can be defined in the x direction using fixed values, and in the y direction by two continuous functions, e.g. $y = g_1(x)$, $y = g_2(x)$.

$$D = \{(x, y)|\ a \leq x \leq b, \qquad g_1(x) \leq y \leq g_2(x)\}$$



Horizontally Simple Region: The region can be defined in the y direction using fixed values, and in the x direction by two continuous functions, e.g. $x = g_1(y)$, x= $g_2(y)$.

$$D = \{(x, y)|\ g_1(y) \leq x \leq g_2(y), \qquad c \leq y \leq d\}$$



Recall that for a rectangular region the order of integration was interchangeable. In the case of vertically and horizontally simple regions the order of integration is not interchangeable and must be done so that the integral with the fixed value limits is on the outside.

Double Integral over Vertically Simple Regions

A vertically simple region is defined as:

$$D = (x, y)| \; a \le x \le b, \qquad g_1(x) \le y \le g_2(x)$$

And the double integral of $f(x, y)$ over D is

$$\iint\limits_D f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx$$

Double Integral over Horizontally Simple Regions

A horizontally simple region is defined as:

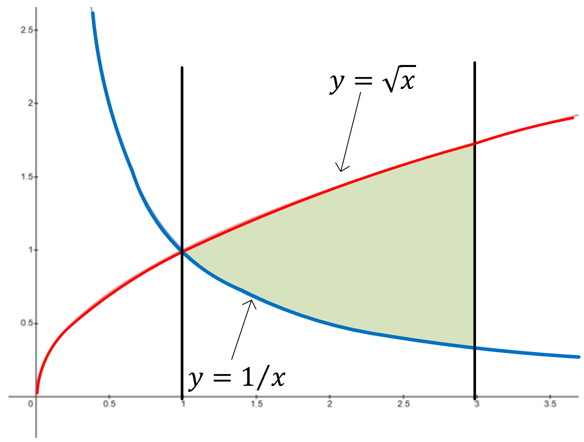
$$D = \{(x, y)| \; g_1(y) \le x \le g_2(y), \qquad c \le y \le d\}$$

And the double integral of $f(x, y)$ over D is

$$\iint\limits_D f(x, y)dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x, y)dx\,dy$$

Let's do some examples.

Example 1: Evaluate $\iint_R (x^2y)dA$ over the region shown below.



Solution: The region is vertically simple since can be defined by two vertical lines and two functions of x as defined below.

$$D = \{(x, y) |\ 1 \le x \le 3, \qquad 1/x \le y \le \sqrt{x}\}$$

The double integral is then given as

$$\iint_D f(x, y)dA = \int_1^3 \int_{1/x}^{\sqrt{x}} (x^2y)dydx$$

Solving we have

$$\int_1^3 \int_{1/x}^{\sqrt{x}} (x^2y)dydx = \int_1^3 x^2 \left(\frac{1}{2}y^2\Big|_{1/x}^{\sqrt{x}}\right) dx$$

$$= \frac{1}{2}\int_1^3 x^2\left(x - \frac{1}{x^2}\right)dx$$

$$= \frac{1}{2}\int_1^3 (x^3 - 1)dx$$

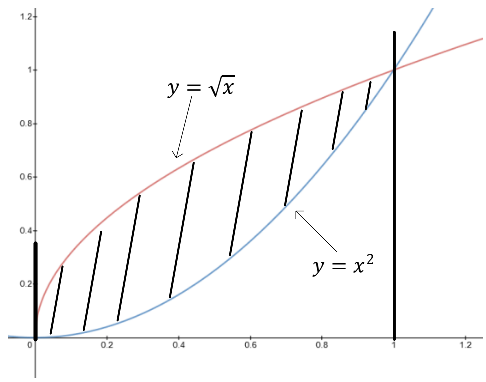
$$= \left(\frac{1}{8}x^4 - \frac{1}{2}x\right)\Big|_1^3$$

$$= \left(\left(\frac{81}{8} - \frac{3}{2}\right) - \left(\frac{1}{8} - \frac{1}{2}\right)\right) = 9$$

In some cases, the region can be defined as either vertically simple or horizontally simple. The next example illustrates such a case.

Example 2: Evaluate the double integral of $f(x, y) = x^2 y$ over the region enclosed by the functions, $y = x^2$, and $y = \sqrt{x}$.

Solution: The integral is first solved using a vertically simple region as shown below.



$$\iint_D f(x, y)dA = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 y)dydx$$

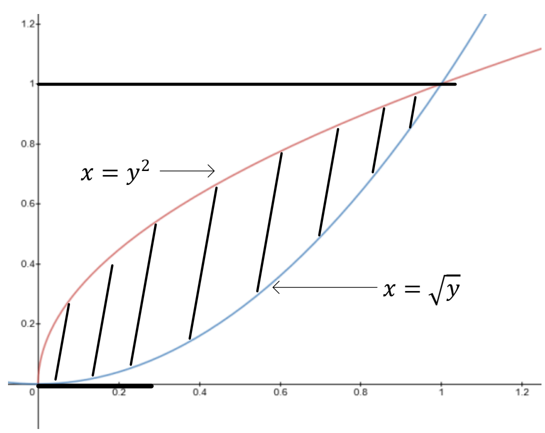
$$= \int_0^1 x^2 \left(\frac{1}{2} y^2 \Big|_{x^2}^{\sqrt{x}} \right) dx$$

$$= \frac{1}{2} \int_0^1 x^2 (x - x^4)dx$$

$$= \frac{1}{2} \int_0^1 (x^3 - x^6)dx$$

$$= \frac{1}{2} \left(\frac{1}{4} x^3 - \frac{1}{7} x^7 \right) \Big|_0^1 = \frac{3}{56}$$

We can also define the region as horizontally simple as shown below.



Where, we found the inverse of the functions to create functions of y.

In this case the integral is solved as shown. Note the limits of integration for the first integral are from left to right on the figure.

$$\iint_D f(x,y)dA = \int_0^1 \int_{y^2}^{\sqrt{y}} (x^2 y)dx\,dy$$

$$= \int_0^1 y\left(\frac{1}{3}x^3\Big|_{y^2}^{\sqrt{y}}\right)dy$$

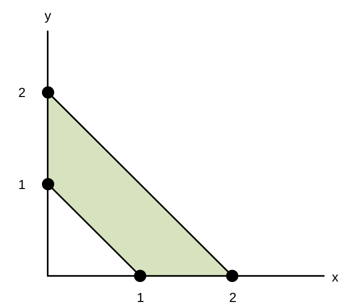
$$= \frac{1}{3}\int_0^1 y\left(y^{3/2} - y^6\right)dy$$

$$= \frac{1}{3}\int_0^1 \left(y^{5/2} - y^7\right)dy$$

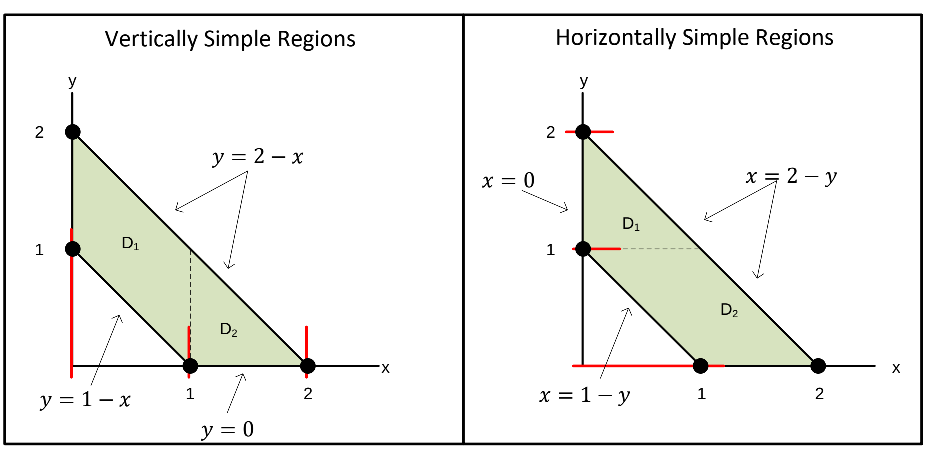
$$= \frac{1}{3}\left(\frac{2}{7}y^{7/2} - \frac{1}{8}y^8\right)\Big|_0^1 = \frac{3}{56}$$

Sometimes the region of integration is such that the entire region cannot be classified as either horizontally or vertically simple. In these cases, we usually attempt to decompose the region into two or more of these simple types. The next example illustrates this scenario.

Example 3: Calculate the double integral of $f(x,y) = xy$ over the region shown below.



Solution: Examining the region we notice it is neither vertically or horizontally simple. However, we can decompose the region as shown below.



As the figures show, we can decompose the region using both vertically and horizontally simple regions. Let's start with the vertically simple regions.

$$\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$$

$$= \int_0^1 \int_{1-x}^{2-x} (xy)dydx + \int_1^2 \int_0^{2-x} (xy)dydx$$

$$= \int_0^1 x\left(\left(\frac{1}{2}y^2\right)\Big|_{1-x}^{2-x}\right)dx + \int_1^2 x\left(\left(\frac{1}{2}y^2\right)\Big|_0^{2-x}\right)dx$$

$$= \frac{1}{2}\left(\int_0^1 x((2-x)^2 - (1-x)^2)dx + \int_1^2 x((2-x)^2 - 0)\,dx\right)$$

$$= \frac{1}{2}\left(\int_0^1 (-2x^2 + 3x)dx + \int_1^2 (x^3 - 4x^2 + 4x)\,dx\right)$$

$$= \frac{1}{2}\left(\left(-\frac{2}{3}x^3 + \frac{3}{2}x^2\Big|_0^1\right) + \left(\frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2\Big|_1^2\right)\right)$$

$$= \frac{1}{2}\left(\left(\frac{5}{6}\right) + \left(\frac{4}{3}\right) - \left(\frac{11}{12}\right)\right)$$

$$= \frac{5}{8}$$

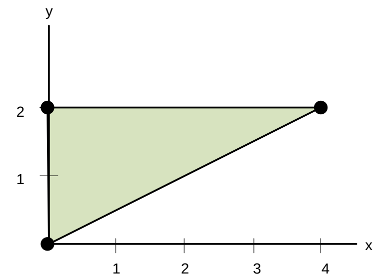
The integral for the horizontally simple regions is set up as follows

$$\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$$

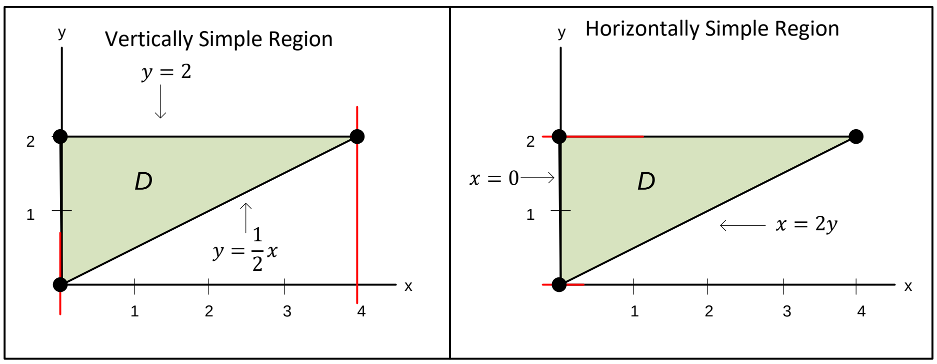
$$= \int_0^1 \int_{1-y}^{2-y} (xy)dx\,dy + \int_1^2 \int_0^{2-y} (xy)dx\,dy$$

Notice that the limits of integration are the same as the previous integral. Furthermore, since the function is $f(x,y) = xy$, evaluating this integral is essentially identical with only a switch in variables. Therefore, it will result in the same value.

Example 4: Calculate the double integral of $f(x,y) = e^{y^2}$ over the region shown below.



Solution: This region can be represented as either vertically or horizontally simple as shown.



Using the vertically simple region the double integral is

$$\iint_D f(x,y)dA = \int_0^4 \int_{\frac{1}{2}x}^2 \left(e^{y^2}\right)dydx$$

In this case there is not explicit integral expression for e^{y^2}. Therefore, let's try the integral using the horizontally simple region.

$$\iint_D f(x,y)dA = \int_0^2 \int_0^{2y} \left(e^{y^2}\right)dxdy$$

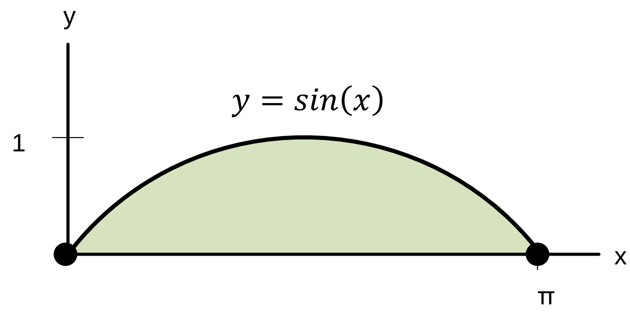
$$= \int_0^2 e^{y^2} \left(\int_0^{2y} 1dx\right) dy$$

$$= \int_0^2 2ye^{y^2} dy$$

In this case the first integral was simple since it was with respect to x. The second integral now has a factor of $2y$ and therefore can be solved with the substitution $u = y^2$.

$$\iint_D f(x,y)dA = \int_0^2 2ye^{y^2} dy = \int_0^4 e^u du = e^4 - 1$$

Example 5: Calculate the double integral of $f(x,y) = x$ over the region shown below.



Solution: The region is vertically simple. Therefore,

$$\iint_D f(x,y)dA = \int_0^{\pi}\int_0^{sin(x)} (x)dydx$$

$$= \int_0^{\pi} x\left(\int_0^{sin(x)} 1dy\right)dx$$

$$= \int_0^{\pi} x\, sin(x)\, dx$$

This integral can be solved using integration by parts as follows

$$u = x \qquad dv = sin(x)\,dx$$

$$du = dx \qquad v = -\,cos(x)$$

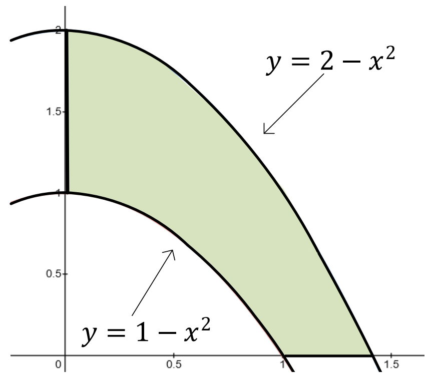
$$\iint_D f(x,y)dA = -x\,cos(x) + \int_0^{\pi} cos(x)\,dx$$

$$= -(-x\,cos(x) + sin(x))\Big|_0^{\pi}$$

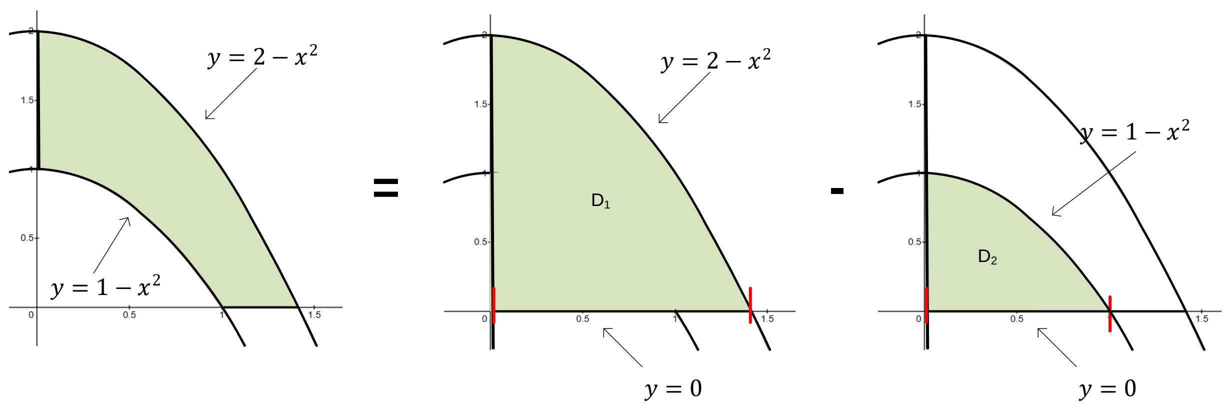
$$= (-\pi\,cos(\pi) + sin(\pi)) - (-0\,cos(0) + sin(0))$$

$$= \pi$$

Example 5: Calculate the double integral of $f(x, y) = x$ over the region shown below.



Solution: The region is neither vertically or horizontally simple. We can decompose the region with a vertical line at $x = 1$ or a horizontal line at $y = 1$. However, we can also create the desires region as shown below.



$$\iint\limits_{D} f(x,y)dA = \iint\limits_{D_1} f(x,y)dA - \iint\limits_{D_2} f(x,y)dA$$

$$= \int_0^{\sqrt{2}} \int_0^{2-x^2} (x)dydx - \int_0^1 \int_0^{1-x^2} (x)dydx$$

$$= \int_0^{\sqrt{2}} x(2 - x^2)dx - \int_0^1 x(1 - x^2)dx$$

$$= \int_0^{\sqrt{2}} (2x - x^3)dx - \int_0^1 (x - x^3)dx$$

$$= \left(x^2 - \frac{1}{4}x^4 \Big|_0^{\sqrt{2}} \right) - \left(\frac{1}{2}x^2 - \frac{1}{4}x^4 \Big|_0^1 \right)$$

$$= (1) - \left(\frac{1}{4}\right) = \frac{3}{4}$$

As we have seen, the double integral over a region represents the volume of the solid formed between the given surface, $f(x, y)$, and the x-y plane. This is completely analogous to the area between $f(x)$ and the x-axis from single variable calculus. Recall we were also able to compute the area *between* two curves, $f_1(x)$ and $f_2(x)$. Assuming $f_1(x) \geq f_2(x)$, the area between the curves is given as

$$A = \int_a^b (f_1(x) - f_2(x))dx$$

Similarly, we can compute the volume between two surfaces. Assuming $f_1(x, y) \geq f_2(x, y)$, the *volume* between the *surfaces* is given as

$$V = \iint_D (f_1(x, y) - f_2(x, y))dA$$

Before ending this lesson, we'll illustrate with two examples.

Example 6: Find the volume of the solid bounded by the paraboloids $f_1(x, y) = 8 - x^2 - y^2$ and $f_2(x, y) = x^2 + y^2$ lying over the region, D.

$$D = \{(x, y)| -1 \leq x \leq 1, -1 \leq y \leq 1\}$$

Solution: Based on the discussion above the volume can be computed as follows

$$V = \iint_D \big(f_1(x, y) - f_2(x, y)\big)dA$$

$$= \int_{-1}^{1}\int_{1}^{1} \big((8 - x^2 - y^2) - (x^2 + y^2)\big)dydx$$

$$= \int_{-1}^{1}\int_{1}^{1} (8 - 2x^2 - 2y^2)dydx$$

$$= \int_{-1}^{1} \left(8y - 2yx^2 - \frac{2}{3}y^3\Big|_{-1}^{1}\right) dx$$

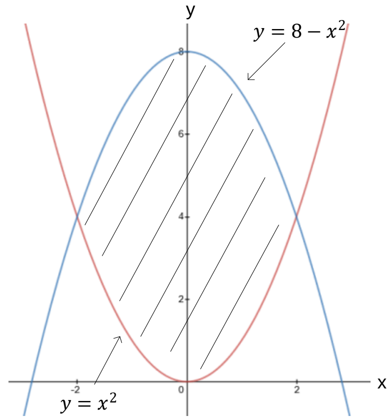
$$= \int_{-1}^{1} \left(8 - 2x^2 - \frac{2}{3}\right) - \left(-8 + 2x^2 + \frac{2}{3}\right) dx$$

$$= \int_{-1}^{1} \left(16 - 4x^2 - \frac{4}{3}\right) dx$$

$$= \left(16x - \frac{4}{3}x^3 - \frac{4}{3}x\right)\Big|_{-1}^{1} = \frac{80}{3}$$

Example 7: Find the volume of the region bounded by $z_1 = 16 - y,\ z_2 = y,\ y = x^2$, and $y = 8 - x^2$.

Solution: The region in the x-y plane is shown below.



The surfaces are planes that intersect at $y = 8$.

$$y = 16 - y \to 2y = 16 \to y = 8$$

Furthermore, between $y = 0$ and $y = 8$ he plane $z_1 \geq z_2$, and therefore the volume can be computed as

$$V = \iint_D \big(f_1(x,y) - f_2(x,y)\big)dA$$

$$= \int_{-2}^{2}\int_{x^2}^{8-x^2} (16 - 2y)dydx$$

$$= \int_{-2}^{2}\left(-(16y - y^2)\Big|_{x^2}^{8-x^2}\right)dx$$

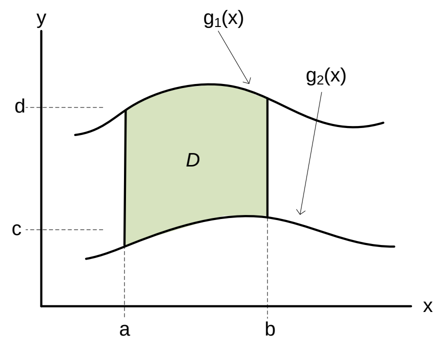
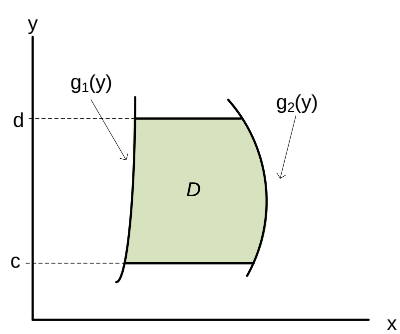
$$= \int_{-2}^{2}(16(8 - x^2) - (8 - x^2)^2) - (16(x^2) - (x^2)^2)dx$$

$$= \int_{-2}^{2}(128 - 16x^2 - 64 + 16x^2 - x^4 - 16x^2 + x^4)dx$$

$$= \int_{-2}^{2}(64 - 16x^2)dx$$

$$= 64x - \frac{16}{3}x^3\Big|_{-2}^{2} = \frac{512}{3}$$

Final Summary for Multiple Integration – Double Integrals over Arbitrary Regions

Double Integral over Vertically Simple Regions
A vertically simple region is defined as: $$D = (x,y)\vert\ a \le x \le b, \qquad g_1(x) \le y \le g_2(x)$$ And the double integral of $f(x,y)$ over D is $$\iint_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$ 
Double Integral over Horizontally Simple Regions
A horizontally simple region is defined as: $$D = \{(x,y)\vert\ g_1(y) \le x \le g_2(y), \qquad c \le y \le d\}$$ And the double integral of $f(x,y)$ over D is $$\iint_D f(x,y)dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)dx\,dy$$ 
Volume Between Two Surfaces
Assuming the integrable functions, $f_1(x,y) \ge f_2(x,y)$, for all points in D, then the *volume* between the *surfaces* is given as $$V = \iint_D (\ f_1(x,y) -\ f_2(x,y))dA$$

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