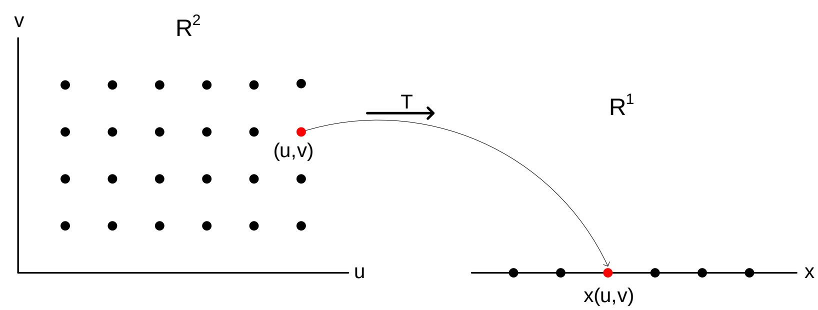
Multiple Integration – Change of Variables

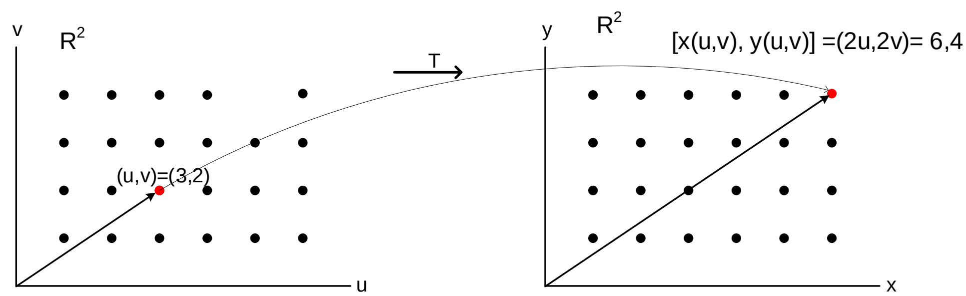
We have seen that multiple integration, in some cases, become much simpler if we change our coordinate system, e.g. rectangular to polar: $(x, y) \rightarrow (r, \theta)$. The process can be generalized with the *Change of Variables Formula*. In this lesson we formally introduce this formula. We attempt to develop the formula in way that provides an intuitive understanding, starting with some preliminary material.

Preliminary Material

A function, T, can be interpreted as a *map* or *transformation* in the sense that it maps elements from a set A in R^n to another set B in R^m. For example, a two variable function $x = T(u, v)$, which can be denoted as $T: R^2 \rightarrow R^1$, maps elements in R^2 to elements in R^1.



Changing coordinate variables, e.g. $(x, y) \rightarrow (r, \theta)$ can be interpreted as a transformation, $T: R^2 \rightarrow R^2$. Furthermore, we can treat each coordinate point as a vector. Once such transformation is a simple scaling of each vector shown below.



General mappings can become quite complicated. However, using only linear equations, i.e. linear mapping, simplifies the process. A linear transformation has the form

$$T(u, v) = (Au + Bv, Cu + Dv)$$

Where, A, B, C, and D are constants.

Treating the input and output variables as vectors, the linear transformation, T, from above can be represented more compactly using matrix notation as follows.

$$\begin{bmatrix} x \\ y \end{bmatrix} = T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} Au + Bv \\ Cu + Dv \end{bmatrix}$$

Where, the matrix, $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, defines the linear mapping from $\begin{bmatrix} x \\ y \end{bmatrix}$ vectors to $\begin{bmatrix} u \\ v \end{bmatrix}$ vectors.

Next, we can ask, "How does a coordinate system change when a linear transformation is applied to each coordinate, i.e. vector?". Let's explore this with an example below.

We start with the fact that a coordinate system can be defined by its standard basis vectors, which for R^2 are given by the following two vectors.

$$\hat{\boldsymbol{e}}_1 = \langle 1,0\rangle \quad \hat{\boldsymbol{e}}_2 = \langle 0,1\rangle$$

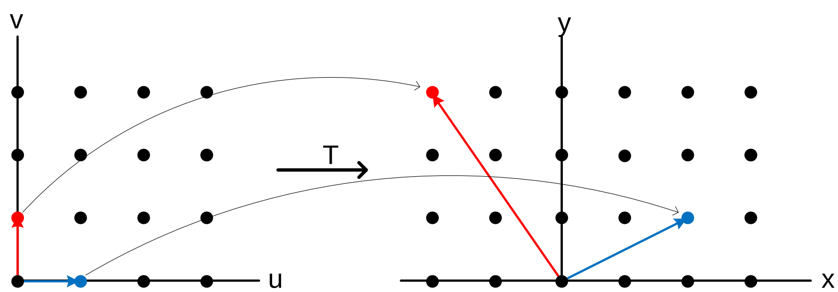
Next, we define an arbitrary linear transformation as

$$T = \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$$

The basis vectors are then mapped as follows:

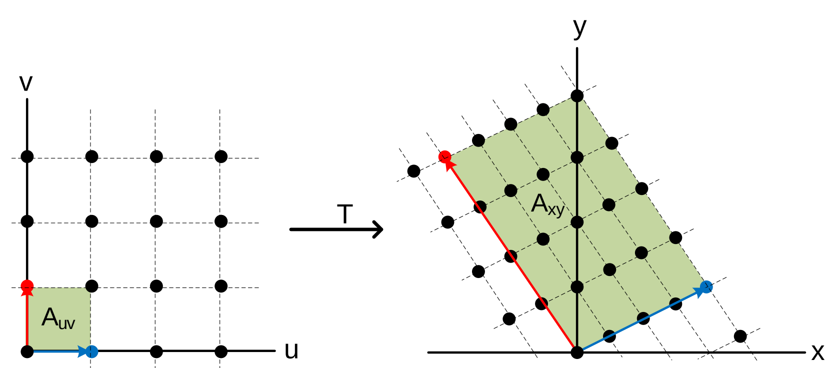
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = T(\hat{\boldsymbol{e}}_1) = \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = T(\hat{\boldsymbol{e}}_2) = \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

The process is illustrated geometrically below.



As the figure illustrates the basis vectors from u-v space are rotated and scaled on x-y space.

The next figure shows that if we continue this transformation process, the entire u-v coordinate system is rotated and scaled. This example answers our question from above, at least for the particular T chosen. The important fact for us is that as long as T is linear, *lines* in u-v space will map to *lines* in x-y space. This is not necessarily the case when T is nonlinear.



Next, we'll look how the area contained in the square created by the vectors in u-v compares to the area contained in the parallelogram created by the mapped vectors in x-y space. This process will ultimately lead us the general *Change of Variables Formula*.

The area, $A_{uv,1}$, is equal to one. For $A_{xy,1}$ we can use what we learned in an earlier lesson, that is the area of a parallelogram formed by two vectors is equal to the magnitude of the cross product. Note we add a $\widehat{\boldsymbol{k}}$ component of zero to each vector since the cross product is defined in R^3.

$$A_{xy,1} = \|\langle 2,1,0\rangle \times \langle -2,3,0\rangle\| = \left\| \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \widehat{\boldsymbol{k}} \\ 2 & 1 & 0 \\ -2 & 3 & 0 \end{vmatrix} \right\| = \|\langle 0, 0, (2 \cdot 3) - (1 \cdot -2)\rangle\| = 8$$

As you can see the cross product if equal to the determination of the linear map, T. Furthermore, from this we can deduce that each unit area in u-v space will map to $|T|$, 8 in this case, times the area in x-y space.

$$A_{xy} = |T| A_{uv}$$

Our next observation may seem arbitrary, but as we'll soon find out turns out to be a fundamental result that is the key to the Change of Variable Formula. Before that however, we define the so-called *Jacobian* in R^2, for which will utilize in our observation.

The Jacobian Determinant
Given the transformation $T: R^2 \to R^2$, where T is defined as $$T(u,v) = \big(x(u,v), y(u,v)\big)$$ The Jacobian of T, $Jac(T)$, is given as $$Jac(T) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$ The Jacobian of T is also denoted as $\frac{\partial(x,y)}{\partial(u,v)}$

The observation relates to a general linear mapping and the Jacobian of such mapping.

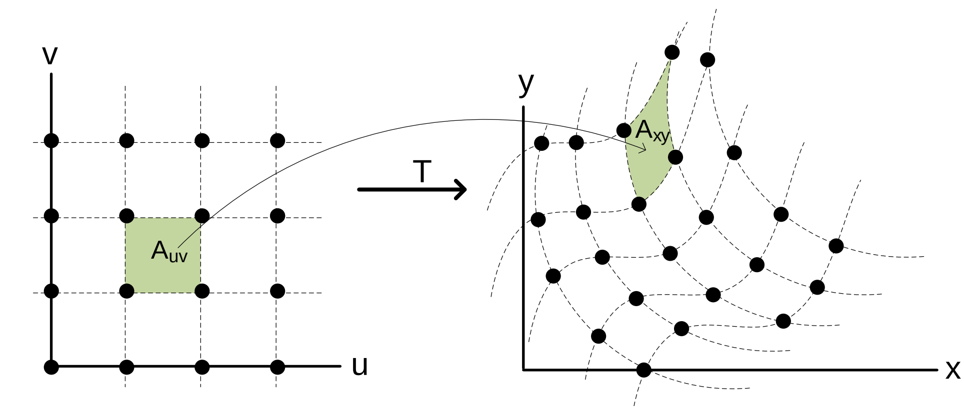
General Linear Mapping	***Jacobian of a Linear Map***
$T(u,v) = \big(x(u,v), y(u,v)\big)$ Where, $x(u,v) = Au + Bv$ $y(u,v) = Cu + Dv$	$Jac(T) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = \lvert T \rvert$

We see in the case of a linear map the Jacobian is equal to the determinant of the map. This is a fundamental result showing how area changes under a general linear map, T. It states that the Jacobian is precisely the scale factor for the area change when mapping from u-v to x-y. Note we take the absolute value of the Jacobian since we are measuring area.

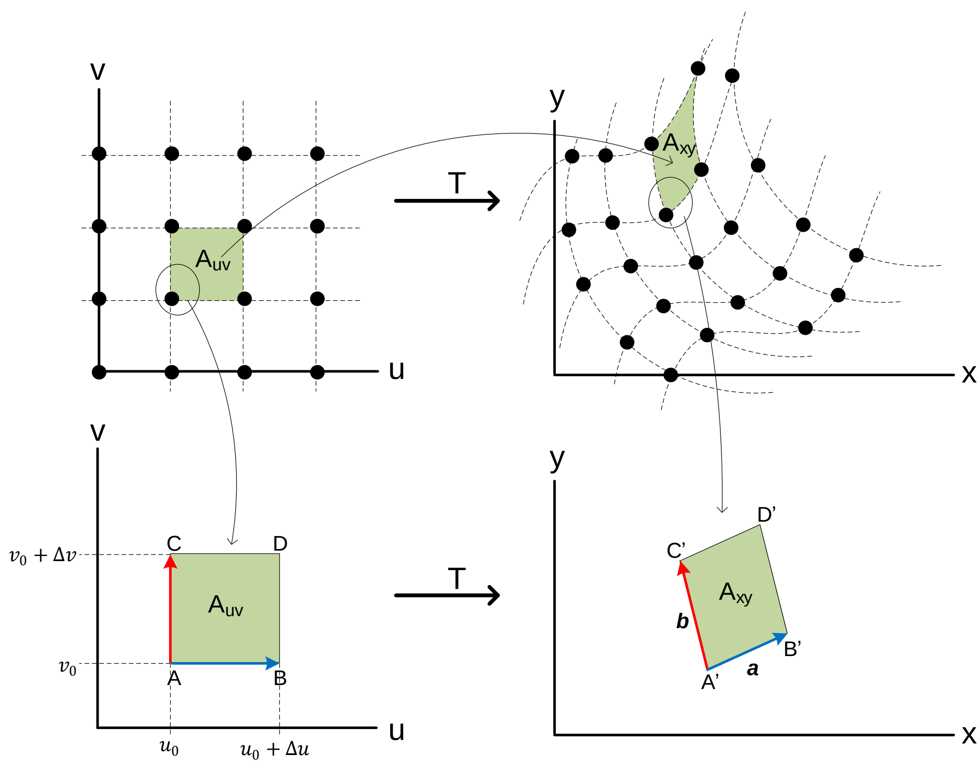
$$A_{xy} = |Jac(T)| A_{uv}$$

Change of Variable Formula

In the previous section we saw that changing variables of a coordinate system via a linear transformation, T, results in areas being scaled by the Jacobian of T. To show this we used the fact that lines remain lines under a linear transformation, i.e. parallelograms map to parallelograms. Unfortunately, this is true for linear mappings only. Moreover, most useful mappings are nonlinear. For example, polar mapping, $[x(r,\theta), y(r,\theta)] = [r\,cos(\theta)\,, r\,sin(\theta)]$, which we found so useful in previous lessons, is a nonlinear mapping. The concern with nonlinear mappings is that lines are not necessarily mapped to lines. The figure below, which illustrates a general transformation, $T(u,v) = \big(x(u,v), y(u,v)\big)$, shows how nonlinear mapping is not always straightforward.



In attempt to circumvent this problem we use a familiar calculus concept, an *infinitesimal*. As you can see below when we zoom in close enough the curves in x-y space become lines.



With this we can use the technique from the previous section to find a small area, ΔA_{xy}.

$$\Delta A_{xy} = \|\boldsymbol{a} \times \boldsymbol{b}\| \;= \left\|\overrightarrow{A'B'} \times \overrightarrow{A'C'}\right\|$$

The four corners in u-v space and the corresponding points in x-y space are shown below.

***u-v* Points**	***x-y* Points via: $\boldsymbol{T(u,v) = \big(x(u,v), y(u,v)\big)}$**
$A = (u_0, v_0)$ $B = (u_0 + \Delta u, v_0)$ $C = (u_0, v_0 + \Delta v)$ $D = (u_0 + \Delta u, v_0 + \Delta v)$	$A' = \big(x(u_0, v_0), y(u_0, v_0)\big)$ $B' = \big(x(u_0 + \Delta u, v_0), y(u_0 + \Delta u, v_0)\big)$ $C' = \big(x(u_0, v_0 + \Delta v), y(u_0, v_0 + \Delta v)\big)$ $D' = \big(x(u_0 + \Delta u, v_0 + \Delta v), y(u_0 + \Delta u, v_0 + \Delta v)\big)$

With this we can define the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as follows.

$$\boldsymbol{a} = \overrightarrow{A'B'} = \langle x(u_0 + \Delta u, v_0) - x(u_0, v_0), y(u_0 + \Delta u, v_0) - y(u_0, v_0)\rangle$$
$$\boldsymbol{b} = \overrightarrow{A'C'} = \langle x(u_0, v_0 + \Delta v) - x(u_0, v_0), y(u_0, v_0 + \Delta v) - y(u_0, v_0)\rangle$$

We call to mind the definition of the partial derivative, which is shown below as an approximation without the limit.

$$\boxed{\frac{\partial f(x_0, y)}{\partial x} \cong \frac{f(x_0 + \Delta x, y) - f(x_0, y)}{\Delta x}}$$

Multiplying this approximation through by Δx we can use it to rewrite the vectors using approximations as follows.

$$\boldsymbol{a} = \overrightarrow{A'B'} \cong \langle \frac{\partial x(u_0, v_0)}{\partial u} \Delta u, \frac{\partial y(u_0, v_0)}{\partial u} \Delta u\rangle$$
$$\boldsymbol{b} = \overrightarrow{A'C'} \cong \langle \frac{\partial x(u_0, v_0)}{\partial v} \Delta v, \frac{\partial y(u_0, v_0)}{\partial v} \Delta v\rangle$$

Finally, we can use these results to approximate the area, ΔA_{xy}. Note, in the cross product treat $\boldsymbol{a}$ and $\boldsymbol{b}$ as column vectors instead of row vectors.

$$\Delta A_{xy} \cong \|\boldsymbol{a} \times \boldsymbol{b}\|$$
$$\Delta A_{xy} \cong \left\| \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \Delta u \dfrac{\partial x(u_0, v_0)}{\partial u} & \Delta v \dfrac{\partial x(u_0, v_0)}{\partial v} & 0 \\ \Delta u \dfrac{\partial y(u_0, v_0)}{\partial u} & \Delta v \dfrac{\partial y(u_0, v_0)}{\partial v} & 0 \end{vmatrix} \right\|$$
$$\Delta A_{xy} \cong \underbrace{\left(\frac{\partial x(u_0, v_0)}{\partial u} \cdot \frac{\partial y(u_0, v_0)}{\partial v} - \frac{\partial x(u_0, v_0)}{\partial v} \cdot \frac{\partial y(u_0, v_0)}{\partial u} \right)}_{Jac(T_0)} \underbrace{\Delta u \Delta v}_{\Delta A_{uv}}$$

Which is the fundamental result we postulated at the end of the previous section. Namely, that the Jacobian is the scale factor for the area change when mapping from u-v to x-y. Note, we again use the absolute value for the Jacobian.

$$\Delta A_{xy} \cong \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta A_{uv}$$

The goal of this exercise was to develop a *Change of Variables Formula* that can be used to aid in the evaluation of multiple integrals. The relation above can be used to develop this formula. It gives us a way to find the area of a small region in x-y space using an equivalent area in u-v space. Instead of providing a formal proof we'll use a heuristic argument to arrive at the desired formula beginning with the relation above.

Starting with the left hand side we can imagine taking the small area, ΔA_{xy}, and dividing it up into even smaller sub-regions, dA_{xy}. If the original region is small enough, we can make the following approximation.

$$\iint_R f(x,y)dA_{xy} \cong f(x_0,y_0)\Delta A_{xy}$$

Using the same argument for the right hand side we have

$$\iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dA_{uv} \cong f(x(u_0,v_0),y(u_0,v_0))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta A_{uv}$$

Equating the left-hand sides of these results we come to the desired *Change of Variable Formula*.

$$\iint_R f(x,y)dA_{xy} = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dA_{uv}$$

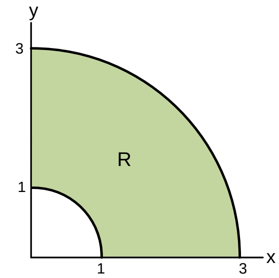
The formula is stated formally stated below.

Change of Variable Formula in R^2
Let $T:(u,v)\to(x,y)$ be a mapping from u-v space to x-y space that is one-to-one. If $f(x,y)$ is continuous, then $$\iint_D f(x,y)dxdy = \iint_S f(x(u,v),y(u,v))\left\|\frac{\partial(x,y)}{\partial(u,v)}\right\|dudv$$ Where, D is some region in x-y space and S is the corresponding region in u-v space.

Let's see how this formula is used, starting with an illustrative example using polar coordinates. Before we do keep in mind that the Change of Variables Formula as stated above turns an xy integral into a uv integral, but the map, T, goes from the uv domain to the xy domain, i.e. $T(u,v) = \big(x(u,v),y(u,v)\big)$

Example 1: Evaluate the following integral over the region shown

$$\iint_R (xy)dxdy$$



Solution: We would like to convert the xy integral into an integral in the $r\theta$ domain. According to our note from above this requires a transformation of the form

$$T(r,\theta) = \big(x(r,\theta), y(r,\theta)\big)$$

Step 1: Define the polar map

$$x = r\,cos(\theta) \qquad\qquad y = r\,sin(\theta)$$

Step 2: Compute the Jacobian for the map.

$$Jac(T) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos(\theta) & -r\,sin(\theta) \\ sin(\theta) & r\,cos(\theta) \end{vmatrix} = r\,cos^2(\theta) + r\,sin^2(\theta) = r$$

Step 3: Apply the map to $\boldsymbol{f(x(r,\theta), y(r,\theta))}$.

$$f\big(x(r,\theta), y(r,\theta)\big) = x(r,\theta)y(r,\theta) = r\,cos(\theta)\cdot r\,sin(\theta) = \frac{1}{2}r^2\,sin(2\theta)$$

Step 4: Apply the Change of Variable Formula.

$$\iint_R (xy)dxdy = \iint_S f\big(x(r,\theta), y(r,\theta)\big)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta$$

$$= \iint_S \left(\frac{1}{2}r^2\,sin(2\theta)\right) rdrd\theta$$

Note that the area element, $dA = rdrd\theta$, is exactly the polar area element we found in a previous lesson. The mapped region is obvious based on our a-prior knowledge of the mapping function.

$$S = \{(r,\theta)|\; 1 \le r \le 3, 0 \le \theta \le \pi/2\}$$

Returning to the integration we have

$$\iint_R (xy)dxdy = \frac{1}{2}\int_0^{\pi/2}\int_1^3 r^3\, sin(2\theta)\, drd\theta$$

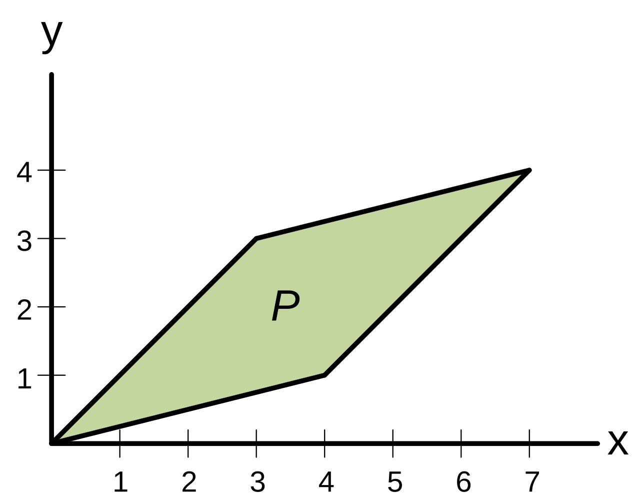
$$= \frac{1}{2}\int_0^{\pi/2} sin(2\theta)\left(\int_1^3 r^3\, dr\right) d\theta$$

$$= \frac{1}{2}\cdot 20\int_0^{\pi/2} sin(2\theta)\, d\theta$$

$$= 10\cdot\frac{1}{2}(-cos(\pi) + cos(0))$$

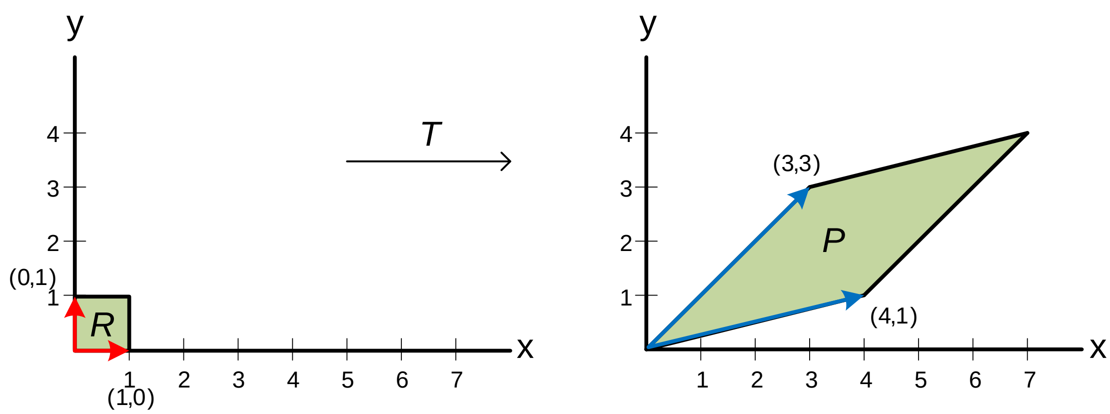
$$= 10$$

Example 2: Use the Change of Variables Formula to compute $\iint_P (e^{4x-y})dxdy$, by mapping the region below to a unit square at the origin.



Solution: Since we have a parallelogram, we can map to a square using a linear transformation of the form

$$T(u,v) = (Au + Bv, Cu + Dv)$$



The transformation is found by mapping the two vectors as shown

$$T(u,v) = (Au + Bv, Cu + Dv)$$
$$T(1,0) = (A \cdot 1 + B \cdot 0, C \cdot 1 + D \cdot 0)$$
$$(4,1) = (A, C)$$

$$T(u,v) = (Au + Bv, Cu + Dv)$$
$$T(0,1) = (A \cdot 1 + B \cdot 0, C \cdot 1 + D \cdot 0)$$
$$(3,3) = (B, D)$$

Therefore,

$$T(u,v) = (4u + 3v, 1u + 3v)$$

$$\iint_P (xy)dxdy = \iint_R f\big(x(u,v), y(u,v)\big)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

$$= \int_0^1 \int_0^1 e^{4(4u+3v)-(1u+3v)} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} dudv$$

$$= \int_0^1 \int_0^1 e^{15u+9v} \begin{vmatrix} 4 & 3 \\ 1 & 3 \end{vmatrix} dudv$$

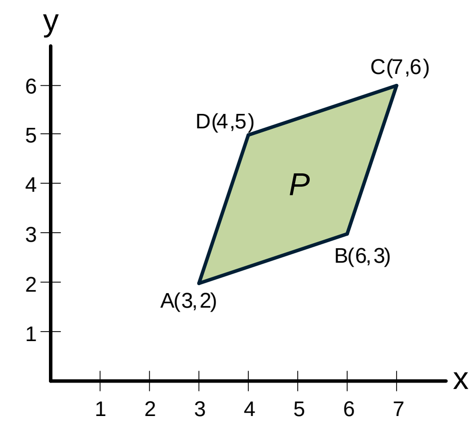
$$= 9\left(\int_0^1 e^{15u} du\right)\left(\int_0^1 e^{9v} dv\right)$$

$$= 9\left(\frac{1}{15}(e^{15} - 1)\right)\left(\frac{1}{9}(e^9 - 1)\right)$$

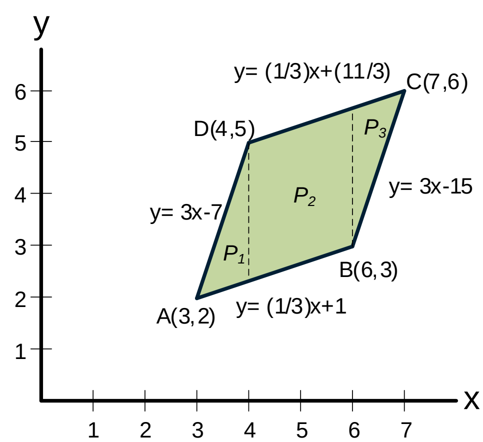
$$= \frac{1}{15}(e^{15} - 1)(e^9 - 1)$$

Example 3: Compute $\iint_P (x)dydx$ using the following methods.

1. Do not use the Change of Variables Formula.
2. Use the Change of Variables Formula such that the region is mapped to a unit square at the origin, where A is mapped to $(0,0)$.
3. Use the Change of Variables Formula such that the region is mapped to a 2x2 square at the origin, where A is mapped to $(0,0)$.



1. In this case we decompose the region into three horizontally simple regions as shown.



$$P_1 = \left\{(x, y)|\ 3 \le x \le 4, \quad \left(\frac{1}{3}x + 1\right) \le y \le (3x - 7)\right\}$$

$$P_2 = \left\{(x, y)|\ 4 \le x \le 6, \quad \left(\frac{1}{3}x + 1\right) \le y \le \left(\frac{1}{3}x + \frac{11}{3}\right)\right\}$$

$$P_2 = \left\{(x, y)|\ 6 \le x \le 7, \quad (3x - 15) \le y \le \left(\frac{1}{3}x + \frac{11}{3}\right)\right\}$$

Therefore, we have

$$\iint_P (x)dydx = \iint_{P_1} (x)dydx + \iint_{P_2} (x)dydx + \iint_{P_3} (x)dydx$$

The first integral is evaluated as shown.

$$\iint_{P_1} (x)dxdy = \int_3^4 x\left(\int_{\frac{1}{3}x+1}^{3x-7} dy\right)dx$$
$$= \int_3^4 x\left((3x-7) - \left(\frac{1}{3}x+1\right)\right)dx$$
$$= \int_3^4 \left(\frac{8}{3}x^2 - 8x\right)dx$$
$$= \frac{8}{9}x^3 - 4x^2\bigg|_3^4 = \frac{44}{9}$$

Next, we evaluate the second integral.

$$\iint_{P_2} (x)dxdy = \int_4^6 x\left(\int_{\frac{1}{3}x+1}^{\frac{1}{3}x+\frac{11}{3}} dy\right)dx$$
$$= \int_4^6 x\left(\left(\frac{1}{3}x+\frac{11}{3}\right) - \left(\frac{1}{3}x+1\right)\right)dx$$
$$= \int_4^6 \left(\frac{8}{3}x\right)dx$$
$$= \frac{4}{3}x^2\bigg|_4^6 = \frac{240}{9}$$

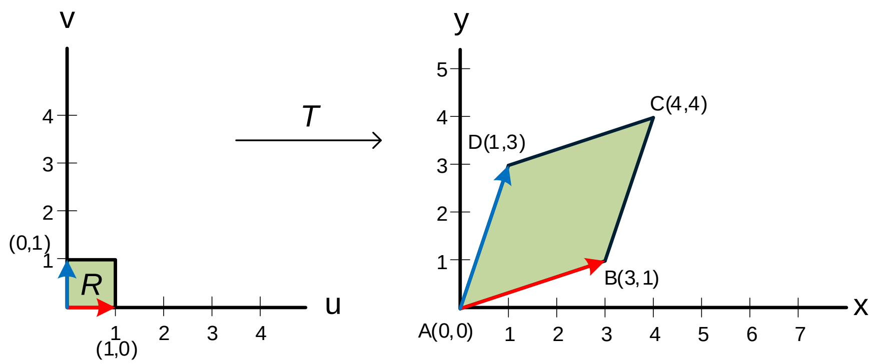
Finally, the third integral is

$$\iint_{P_3} (x)dxdy = \int_6^7 x\left(\int_{3x-15}^{\frac{1}{3}x+\frac{11}{3}} dy\right)dx$$
$$= \int_6^7 x\left(\left(\frac{1}{3}x+\frac{11}{3}\right) - (3x-15)\right)dx$$
$$= \int_6^7 \left(-\frac{8}{3}x^2 + \frac{56}{3}x\right)dx$$
$$= -\frac{8}{9}x^3 + \frac{28}{3}x^2\bigg|_6^7 = \frac{76}{9}$$

Summing we find

$$\iint_{P} (x)dxdy = \frac{44}{9} + \frac{240}{9} + \frac{76}{9} = 40$$

2. In this case, we can start by shifting the region 2 units down and 3 units to the left, for which we will compensate in the linear mapping below.



The transformation is found by mapping the vectors as shown

$$T(u,v) = (Au + Bv, Cu + Dv)$$
$$T(1,0) = (A \cdot 1 + B \cdot 0, C \cdot 1 + D \cdot 0)$$
$$(3,1) = (A, C)$$

$$T(u,v) = (Au + Bv, Cu + Dv)$$
$$T(0,1) = (A \cdot 0 + B \cdot 1, C \cdot 0 + D \cdot 1)$$
$$(1,3) = (B, D)$$

After accounting for the shift, we can write the transformation as

$$T(u,v) = (3u + v + 3, u + 3v + 2)$$

$$\iint_P (x)dxdy = \iint_R f\big(x(u,v), y(u,v)\big) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

$$= \int_0^1 \int_0^1 (3u + v + 3) \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} dudv$$

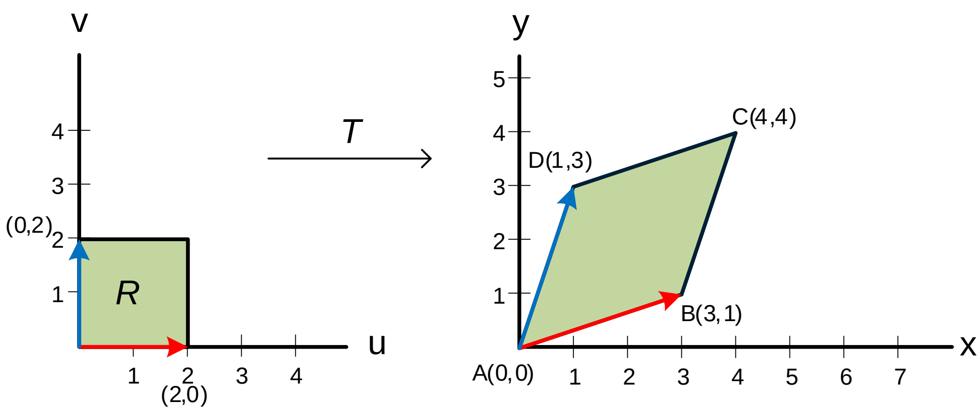
$$= \int_0^1 \int_0^1 (3u + v + 3) \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} dudv$$

$$= 8 \int_0^1 \left(\int_0^1 (3u + v + 3) du \right) dv$$

$$= 8 \int_0^1 \left(\frac{9}{2} + v \right) dv$$

$$= 8 \left(\frac{9}{2} + \frac{1}{2} \right) = 40$$

3. This case is similar to the previous except we map to the larger square as shown.



The transformation is found by mapping the vectors as shown

$$\begin{aligned} T(u,v) &= (Au + Bv, Cu + Dv) \\ T(2,0) &= (A\cdot 2 + B\cdot 0, C\cdot 2 + D\cdot 0) \\ (3,1) &= (2A, 2C) \end{aligned}$$

$$\begin{aligned} T(u,v) &= (Au + Bv, Cu + Dv) \\ T(0,2) &= (A\cdot 0 + B\cdot 2, C\cdot 0 + D\cdot 2) \\ (1,3) &= (2B, 2D) \end{aligned}$$

After accounting for the shift, we can write the transformation as

$$T(u,v) = \left(\frac{3}{2}u + \frac{1}{2}v + 3, \frac{1}{2}u + \frac{3}{2}v + 2\right)$$

$$\iint_P (x)dxdy = \iint_R f\big(x(u,v), y(u,v)\big)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

$$= \int_0^2\int_0^2 \left(\frac{3}{2}u + \frac{1}{2}v + 3\right)\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} dudv$$

$$= \int_0^2\int_0^2 \left(\frac{3}{2}u + \frac{1}{2}v + 3\right)\begin{vmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{vmatrix} dudv$$

$$= 2\int_0^2\left(\int_0^2 \left(\frac{3}{2}u + \frac{1}{2}v + 3\right)du\right)dv$$

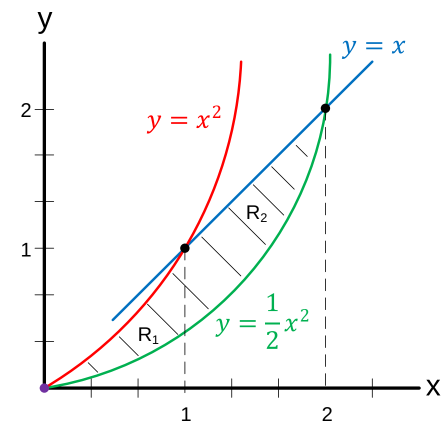
$$= 2\int_0^2 (9 + v)\,dv$$

$$= 2(18 + 2) = 40$$

Example 4: Sketch the region bounded by $y = x^2, y = \frac{1}{2}x^2$, and $y = x$. Then use the Change of Variables with the map $x = uv,\ y = u^2$ to compute

$$\iint_R \left(\frac{1}{y}\right) dxdy$$

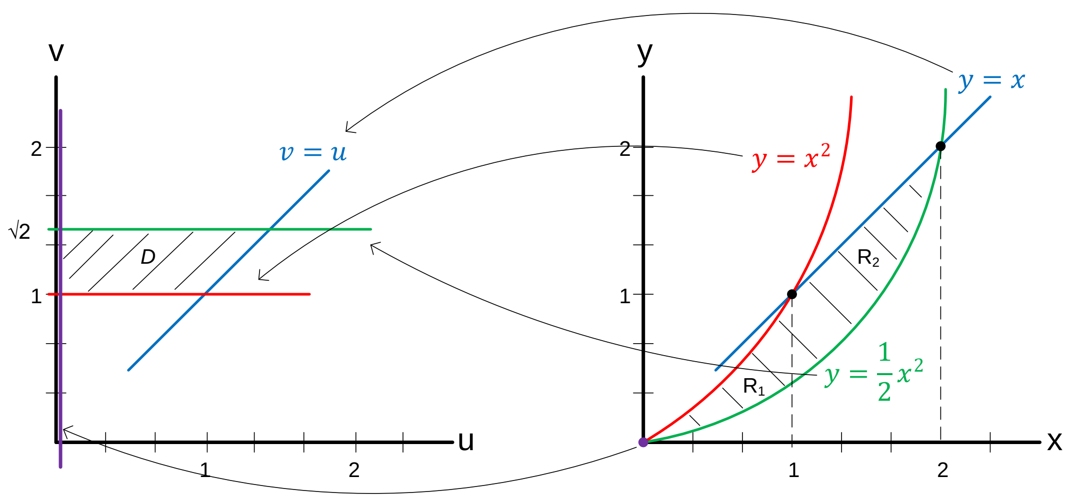
Solution: The region is sketched below.



To find the region in uv space we start by mapping the three xy curves.

$y = x$	$y = \frac{1}{2}x^2$	$y = x^2$
$u^2 = uv$	$u^2 = \frac{1}{2}u^2v^2$	$u^2 = u^2v^2$
$u = v$	$\sqrt{2} = v$	$1 = v$

To end the region on the left we map the point $(x, y) = (0,0)$. Since $y = u^2, u = 0$. Furthermore, since $x = uv$, v can be any value so that $(0,0)$ maps to a vertical line at $u = 0$.



$$\iint_R \left(\frac{1}{y}\right) dxdy = \int_1^{\sqrt{2}} \int_0^v \frac{1}{u^2} \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right| dudv$$

$$= \int_1^{\sqrt{2}} \int_0^v \frac{1}{u^2} \left| \begin{matrix} v & u \\ 2u & 0 \end{matrix} \right| dudv$$

$$= \int_1^{\sqrt{2}} \int_0^v \frac{1}{u^2} |2u^2| dudv$$

$$= 2 \int_1^{\sqrt{2}} \left(\int_0^v du \right) dv$$

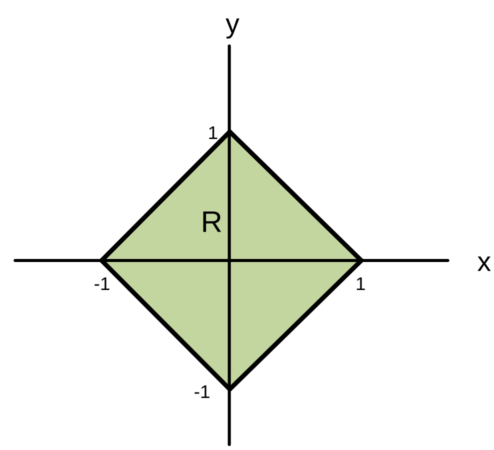
$$= 2 \int_1^{\sqrt{2}} v\, dv$$

$$= 2 \cdot \left(\frac{1}{2}(2-1) \right)$$

$$= 1$$

Example 5: Find an appropriate change of variable formula to evaluate the following

$$\iint_R (x+y)^2 e^{x^2-y^2} dxdy$$



Solution: In this case, we would like to find a change of variable formula to simplify the integration. First, note the integrand can be rewritten as follows

$$(x+y)^2 e^{x^2-y^2} = (x+y)^2 e^{(x+y)(x-y)}$$

Using the following change of variables: $u = (x+y),\ v = (x-y)$, the integrand can be written as $u^2 e^{uv}$.

Next, we solve these equations for x and y and find the following transformation.

$$T = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$$

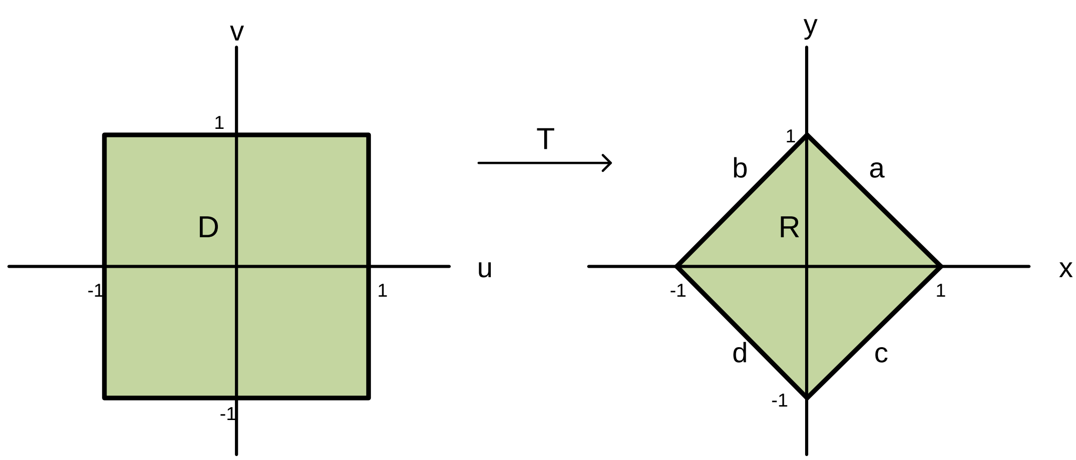
To map the region, we map the four lines as shown.

$$a: (y = x+1) \rightarrow \left(\frac{u-v}{2} = \frac{u+v}{2} + 1\right) \rightarrow (u - v = u + v + 2) \ \rightarrow v = -1$$

$$b: (\ y = 1 - x) \rightarrow \left(\frac{u-v}{2} = 1 - \frac{u+v}{2}\right) \rightarrow (u - v = 2 - u - v) \ \rightarrow u = 1$$

$$c: (y = x - 1) \rightarrow \left(\frac{u-v}{2} = \frac{u+v}{2} - 1\right) \rightarrow (u - v = u + v - 2\ ) \rightarrow v = 1$$

$$d: (y = -1 - x) \rightarrow \left(\frac{u-v}{2} = -1 - \frac{u+v}{2}\right) \rightarrow (u - v = -1 - u - v) \ \rightarrow u = -1$$



Finally, we evaluate using the Change of Variables Formula as follows:

$$\iint_R (x+y)^2 e^{x^2-y^2}\,dxdy = \int_{-1}^{1}\int_{-1}^{1} u^2 e^{uv} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} dvdu$$

$$= \int_{-1}^{1}\int_{-1}^{1} u^2 e^{uv} \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} dvdu$$

$$= \frac{1}{2}\int_{-1}^{1} u^2 \left(\int_{-1}^{1} e^{uv} dv\right) du$$

$$= \frac{1}{2}\int_{-1}^{1} u^2 \left(\frac{1}{u}(e^u - e^{-u})\right) du$$

$$= \frac{1}{2}\left(\int_{-1}^{1} ue^u\,du - \int_{-1}^{1} ue^{-u} du\right)$$

$$= \frac{1}{2}\left((ue^u - e^u) - (-ue^{-u} - e^{-u})\right)\Big|_{-1}^{1}$$

$$= \frac{1}{2}(ue^u - e^u + ue^{-u} + e^{-u})\Big|_{-1}^{1}$$

$$= \frac{1}{2}\left((e^1 - e^1 + e^{-1} + e^{-1}) - (-1e^{-1} - e^{-1} + -1e^1 + e^1)\right)\Big|_{-1}^{1}$$

$$= \frac{1}{2}(\cancel{e^1 - e^1} + e^{-1} + e^{-1} + e^{-1} + e^{-1} + \cancel{e^1 - e^1}) = \frac{2}{e}$$

The Change of Variables Formula has the same form in three, (or more), variables as in two variables. In three variables we have the following transformation equations.

$$x = x(u, v, w) \qquad y = y(u, v, w) \qquad z = z(u, v, w)$$

The Jacobian is the 3x3 determinant

$$Jac(T) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

The Change of Variables Formula is then

$$\iiint_R f(x, y, z)dxdydz = \iiint_S f\big(x(u, v, w), y(u, v, w), z(u, v, w)\big) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| dudvdw$$

To illustrate we derive the formula for integration in spherical coordinates. The spherical mapping equations are as shown.

$$x = \rho\, sin(\phi)\, cos(\theta) \qquad y = \rho\, sin(\phi)\, sin(\theta) \qquad z = \rho\, cos(\phi)$$

The Jacobian is then given as

$$Jac(T) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} sin(\phi)\, cos(\theta) & -\rho\, sin(\phi)\, sin(\theta) & \rho\, cos(\theta)\, cos(\phi) \\ sin(\phi)\, sin(\theta) & \rho\, sin(\phi)\, cos(\theta) & \rho\, cos(\phi)\, sin(\theta) \\ cos(\phi) & 0 & -\rho\, sin(\phi) \end{vmatrix}$$

Solving this is tedious but simplifies as expected. We illustrate below.

$$Jac(T) = sin(\phi)\,cos(\theta)\begin{vmatrix} \rho\,sin(\phi)\,cos(\theta) & \rho\,cos(\phi)\,sin(\theta) \\ 0 & -\rho\,sin(\phi) \end{vmatrix}$$

$$+\,\rho\,sin(\phi)\,sin(\theta)\begin{vmatrix} sin(\phi)\,sin(\theta) & \rho\,cos(\phi)\,sin(\theta) \\ cos(\phi) & -\rho\,sin(\phi) \end{vmatrix}$$

$$+\,\rho\,cos(\theta)\,cos(\phi)\begin{vmatrix} sin(\phi)\,sin(\theta) & \rho\,sin(\phi)\,cos(\theta) \\ cos(\phi) & 0 \end{vmatrix}$$

$$= \ sin(\phi)\,cos(\theta)\,(-\rho^2\,sin^2(\phi)\,cos(\theta))$$

$$+\,\rho\,sin(\phi)\,sin(\theta)\,(-\rho\,sin^2(\phi)\,sin(\theta) - \rho\,cos^2(\phi)\,sin(\theta))$$

$$+\,\rho\,cos(\theta)\,cos(\phi)\,(-\rho\,sin(\phi)\,cos(\theta)\,cos(\phi))$$

$$= \ (-\rho^2\,sin^3(\phi)\,cos^2(\theta)) - \rho^2\,sin^3(\phi)\,sin^2(\theta) - \rho^2\,sin^2(\theta)\,cos^2(\phi)\,sin(\phi)$$

$$-\,\rho^2\,cos^2(\phi)\,cos^2(\theta)\,sin(\phi)$$

$$= \ \left(-\rho^2\,sin^3(\phi)\,(cos^2(\theta) + sin^2(\theta))\right) + \left(-\rho^2\,sin(\phi)\,cos^2(\phi)\,(sin^2(\theta) + cos^2(\theta))\right)$$

$$= \ -\rho^2\,sin^3(\phi) - \rho^2\,sin(\phi)\,cos^2(\phi)$$

$$= \ -\rho^2\,sin(\phi)\,(sin^2(\phi) + cos^2(\phi))$$

$$Jac(T) = \ -\rho^2\,sin(\phi)$$

Note, since both ρ and $sin(\phi)$ are always positive we can simply drop the negative sign. Therefore, the volume element for spherical coordinates is as expected.

$$dxdydz \rightarrow |Jac(T)|d\rho d\phi d\theta = \rho^2\,sin(\phi)d\rho d\phi d\theta$$

Final Summary for Multiple Integration – Change of Variables

The Jacobian Determinant

Given the transformation $T: R^2 \to R^2$, where T is defined as

$$T(u, v) = \big(x(u, v), y(u, v)\big)$$

The Jacobian of T, $Jac(T)$, is given as

$$Jac(T) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

The Jacobian generalizes to n dimensions. For example, with three variables we have $T: R^3 \to R^3$, where T is defined as

$$T(u, v, w) = \big(x(u, v, w), y(u, v, w), z(u, v, w)\big)$$

$$Jac(T) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

- The Jacobian of T is also denoted as $\frac{\partial(x,y)}{\partial(u,v)}, \frac{\partial(x,y,z)}{\partial(u,v,w)}$
- The Jacobian is sometimes meant to express the matrix only and not its determinant. In these cases, we refer to the above as the *Jacobian Determinant*.

Change of Variable Formula in

Let $T: (u, v) \to (x, y)$ be a mapping from u-v space to x-y space that is one-to-one. If $f(x, y)$ is continuous, then

$$\iint_D f(x, y) dxdy = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv$$

Where, D is some region in x-y space and S is the corresponding region in u-v space.

Note: The Change of Variables Formula as stated above turns an xy integral into a uv integral, but the map, T, goes from the uv domain to the xy domain, i.e. $T(u, v) = \big(x(u, v), y(u, v)\big)$

In R^3 we have:

$$\iiint_D f(x, y, z) dxdydz = \iiint_S f\big(x(u, v, w), y(u, v, w), z(u, v, w)\big) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dudvdw$$

By: ferrantetutoring