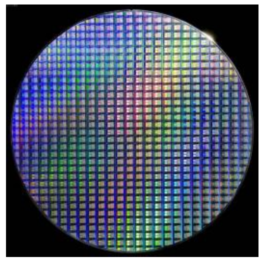
Multiple Integration – Applications

In this lesson we discuss a few applications of double and triple integrals. There are, of course, innumerable applications, however we will focus on a small subset.

Total Amount and Density

In some applications it is of interest to find the *total amount* of some quantity when we know the amount per unit measure. The amount per unit measure is referred to as *density*. We can illustrate this with an example of a *wafer*, which is a thin circular disk of a semiconductor material, typically silicon, used in the manufacturing of computer chips. An illustration of a wafer with computer chips already etched into the silicon substrate is shown below.



Assume the radius of the wafer is 15 cm with a thickness of $0.0775\ cm$. Given the mass density of silicon is known to be 2.33 g/cm^3 we can find the *total mass* of the wafer by multiplying the mass density of silicon with the volume of the wafer.

$$M_S = \delta_S V_W = (2.33) \cdot (\pi 15^2 \cdot 0.0775) \cong 127\ g$$

Note that integration was not required since the density was a constant, i.e. the silicon was uniformly distributed. Integration is required when the density is not constant. To illustrate this, let's use the example of a wafer made with a mix of both silicon and germanium.

Example 1: A silicon/germanium wafer is fabricated in such a way that the density of the wafer varies linearly as a function of the height starting with the density of silicon at the bottom and ending with the density of germanium, which is 5.232 g/cm^3, at the top. Find the total mass of the wafer. The dimension of the wafer are the same as given above.

Solution: The density equation can be found using the known density values at $z_1 = 0$ and $z_2 = 0.0775$, i.e. $P_1 = (2.33, 0)$, $P_2 = (5.232, 0.0775)$.

$$\delta(z) = \left(\frac{5.232 - 2.33}{0.0775}\right) z + 2.33 = \left(\frac{2.902}{0.0775}\right) z + 2.33$$

Next, we create an expression for an infinitesimal mass element, dM, which is given by the product of an infinitesimal volume element and the density at that point.

$$dM = \delta(x, y, z)dV$$

The total mass is equal to an infinite sum of these mass elements, i.e. the integral over the region of the wafer.

$$M = \iiint_R \delta(x, y, z)dV$$

Since the wafer is in the shape of a cylinder the integral is most easily evaluated using cylindrical coordinates. The integration region is defined as

$$R = \{(r, \theta, z) | 0 \le r \le 15, 0 \le \theta \le 2\pi, 0 \le z \le 0.0775\}$$

Therefore, we can evaluate the integral as shown.

$$\iiint_R \delta(x, y, z)dV = \int_{\theta=0}^{2\pi} \int_{r=0}^{15} \int_{z=0}^{0.0775} \delta(r\,cos(\theta)\,, r\,sin(\theta)\,, z)\, rdzdrd\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{15} r\left(\int_{z=0}^{0.0775} \left(\left(\frac{2.902}{0.0775}\right) z + 2.33\right) dz\right) drd\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{15} r\left(\left(\frac{2.902}{2 \cdot 0.0775}\right) 0.0775^2 + 2.33 \cdot 0.0775\right) drd\theta$$

$$= 0.0775 \cdot 3.781 \int_{\theta=0}^{2\pi} \left(\int_{r=0}^{15} r\,dr\right) d\theta$$

$$= \frac{0.0775 \cdot 3.781}{2} (15^2) \int_{\theta=0}^{2\pi} 1 d\theta$$

$$= 3.781(\pi 15^2 \cdot 0.0775)$$

$$M \cong 207.13\ g$$

Note the expression for the total mass for the pure silicon wafer was

$$M_S = (2.33) \cdot (\pi 15^2 \cdot 0.0775)$$

Which is the same exact form as the last expression above except for the first term, 3.781, which as you may have guessed, is the average density of silicon and germanium.

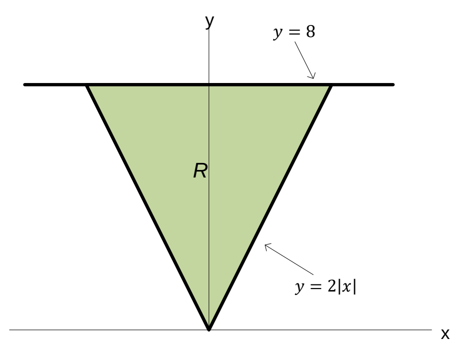
$$\bar{\delta} = \frac{2.33 + 5.232}{2} = 3.781$$

Total Amount Using Density
Given the amount per unit measure, i.e. density, of a quantity, the total amount is given by the following integrals.

One Dimension
$$Total\ Amount\ = \int_R \delta(x)dx$$ Where, $\delta(x)$ is the amount per unit length and R is the interval of integration
Two Dimensions
$$Total\ Amount\ = \iint_R \delta(x,y)dA$$ Where, $\delta(x,y)$ is the amount per unit area and R is the region of integration.
Three Dimensions
$$Total\ Amount = \iiint_R \delta(x,y,z)dV$$ Where, $\delta(x,y,z)$ is the amount per unit volume and R is the region of integration.

Example 2: Find the total population within the sector, $2|x| \leq y \leq 8$, assuming a population density of $\delta(x,y) = 100e^{-0.1y}$ people per square kilometer.

Solution: The region is sketched below.



The region is defined as

$$R = \left\{(x,y) | -\frac{1}{2}y \leq x \leq \frac{1}{2}y, 0 \leq y \leq 8, \right\}$$

The total population is then given by the following double integral.

$$Total\ Population\ = \int_{y=0}^{8} \int_{x=-\frac{1}{2}y}^{\frac{1}{2}y} 100e^{-0.1y}\, dxdy$$

Which we solve as follows

$$= \int_0^8 100e^{-0.1y} \left(\int_{-\frac{1}{2}y}^{\frac{1}{2}y} 1\, dx \right) dy$$

$$= \int_0^8 100e^{-0.1y} \left(\frac{1}{2}y + \frac{1}{2}y \right) dy$$

$$= 100 \int_0^8 ye^{-0.1y} dy$$

To solve this final integral, we use integration by parts as follows

$u = y$ $\quad du = dy$ $\quad dv = e^{-0.1y} dy$ $\quad v = -10e^{-0.1y}$

Therefore,

$$100 \int_{y=0}^{8} ye^{-0.1y} dy = 100\left(-10ye^{-0.1y} + \int_0^8 10e^{-0.1y} dy \right)$$

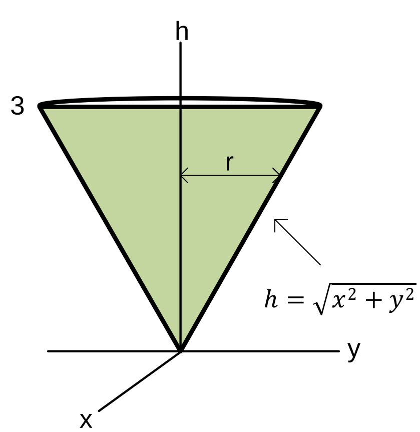
$$= 100\left(-10ye^{-0.1y} - \frac{10}{0.1} e^{-0.1y} \Big|_0^8 \right)$$

$$= 100\big((-80e^{-0.8} - 100e^{-0.8}) - (0 - 100e^{0}) \big)$$

$$= (-18000e^{-0.8} + 10000)$$

$$Total\ Population \cong 1912\ total\ people$$

Example 3: Assume the density of the atmosphere is $\delta(h) = ae^{-bh}\ \ kg/km^3$, where h is the height above sea level in km, $a = 1.225E^9$, and $b = 0.13$. Calculate the total mass of the atmosphere contained in a cone-shaped region $\sqrt{x^2 + y^2} \leq h \leq 3$.



Solution: As we have seen in the previous lesson a cone is best described using cylindrical coordinates. We can describe the cone using cylindrical coordinates as $h(r) = r$. With this the region can be defined as follows.

$$R = \{(r, \theta, h) | 0 \le r \le h, 0 \le \theta \le 2\pi, 0 \le h \le 3\}$$

The total mass is then given by

$$\iiint_R \delta(x, y, z) dV = \int_{h=0}^{3} \int_{\theta=0}^{2\pi} \int_{r=0}^{h} ae^{-bh}\, r dr d\theta dh$$

$$= a \int_{h=0}^{3} e^{-bh} \int_{\theta=0}^{2\pi} \left(\int_{r=0}^{h} r\, dr \right) d\theta dh$$

$$= \frac{a}{2} \int_{h=0}^{3} h^2 e^{-bh} \left(\int_{\theta=0}^{2\pi} 1 d\theta \right) dh$$

$$= \pi a \int_{h=0}^{3} (h^2 e^{-bh}) dh$$

Where, the final integral requires us to use integration by parts twice. We can also use a table of integrals where we can find the following formula.

$$\boxed{\int x^2 e^{ax} dx = \frac{e^{ax}}{a} \left(x^2 - \frac{2x}{a} + \frac{2}{a^2} \right)}$$

Applying this formula, we have

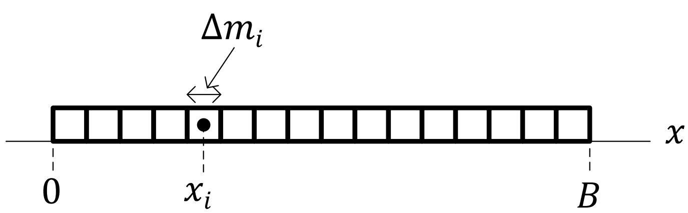
$$\pi a \int_{h=0}^{3} (h^2 e^{-bh}) dh = \pi a \left(\frac{e^{-bh}}{-b} \left(h^2 - \frac{2h}{-b} + \frac{2}{b^2} \right) \right) \Bigg|_0^3$$

$$= \pi a \left[\left(\frac{e^{-3b}}{-b} \left(9 - \frac{6}{-b} + \frac{2}{b^2} \right) \right) - \left(-\frac{2}{b^3} \right) \right]$$

$$= \pi a \left[\left(-\frac{e^{-3b}}{b} \left(9 + \frac{6}{b} + \frac{2}{b^2} \right) \right) + \left(\frac{2}{b^3} \right) \right] \cong 2.593E^{10}\ kg$$

Center of Mass

The center of mass of an object is defined as the position, with respect to some reference point, where the object is balanced. It can be computed using the concept of a *weighted average*. We can derive a general formula starting with a hypothetical rod that exist in one dimensional space along the x-axis.



The '*weights*' in the weighted average are equal to the percent mass contribution of each of the mass elements and the average is with respect to the position of these mass elements.

$$x_{com} = x_1\left(\frac{\Delta m_1}{M}\right) + x_2\left(\frac{\Delta m_2}{M}\right) + \cdots + x_N\left(\frac{\Delta m_N}{M}\right)$$

$$x_{com} = \frac{1}{M}(x_1\Delta m_1 + x_2\Delta m_2 + \ldots + x_N\Delta m_N)$$

$$x_{com} = \frac{1}{M}\sum_{i=1}^{N} x_i\Delta m_i$$

Where, M is the total mass of the rod, Δm_i is the mass of a small element located a distance x_i from a reference point, i.e. $x = 0$.

Furthermore, if the mass density of the rod at each point, x_i, is $\delta(x_i)$, then the mass of each element is $\Delta m_i = \delta(x_i)\Delta x$. Therefore, we can write.

$$x_{com} = \frac{1}{M}\sum_{i=1}^{N} x_i\delta(x_i)\Delta x$$

Next, we let Δx get infinitesimally small, i.e. $\Delta x \to dx$, resulting in an infinite sum, i.e. an integral.

$$x_{com} = \frac{1}{M}\int_R x\delta(x)dx$$

Where, R is the interval of integration and $\delta(x)$ is the mass density in $mass/length$.

Lastly, from the previous section the total mass, M, is given as

$$M = \int_R \delta(x)dx$$

Analysis for two and three dimensional objects is completely analogous. A summary is given below.

Center of Mass

The center of mass of an object is defined as the position, with respect to some reference point, where the object is balanced.

One Dimension

$$x_{com} = \frac{\int_R x\delta(x)dx}{\int_R \delta(x)dx}$$

Where, $\delta(x)$ is mass density per unit length and R is the interval of integration

Two Dimensions

$$x_{com} = \frac{\iint_R x\delta(x,y)dA}{\iint_R \delta(x,y)dA} \qquad y_{com} = \frac{\iint_R y\delta(x,y)dA}{\iint_R \delta(x,y)dA}$$

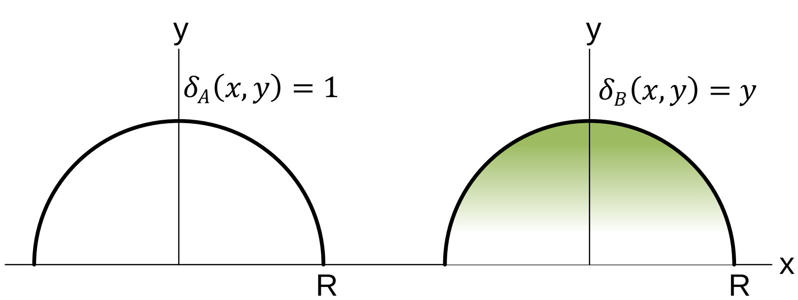
Where, $\delta(x,y)$ is the mass density per unit area and R is the region of integration.

Three Dimensions

$$x_{com} = \frac{\iiint_R x\delta(x,y)dV}{\iiint_R \delta(x,y)dV} \qquad y_{com} = \frac{\iiint_R y\delta(x,y)dV}{\iiint_R \delta(x,y)dV} \qquad z_{com} = \frac{\iiint_R z\delta(x,y)dV}{\iiint_R \delta(x,y)dV}$$

Where, $\delta(x,y,z)$ is the mass density per unit volume and R is the region of integration.

Example 4: Compare the center of mass for the two half circles shown below.



Solution: The first step should always be to look for symmetry, which can be used to simplify the center of mass computations. In both cases the object displays symmetry about the y-axis. Additionally, since the mass density is constant with respect to x, the center of mass with respect to x-axis is zero, i.e. $x_{com} = 0$. The center of mass in the y-direction is given as

$$y_{com} = \frac{\iint_D \ y\delta(x,y)dA}{\iint_D \ \delta(x,y)dA}$$

Next, we choose a coordinate system that best fits the problem. In this case, since the object displays a radial symmetry, its likely easier to use polar coordinates. The domain of integration in polar coordinates is

$$D = \{(r,\theta)|\ 0 \le r \le R, 0 \le \theta \le \pi\}$$

Therefore, the y_{com} for each object is

$$y_{A,com} = \frac{\int_{\theta=0}^{\pi}\int_{r=0}^{R}(r\,sin(\theta))\delta_A(x,y)rdrd\theta}{\int_{\theta=0}^{\pi}\int_{r=0}^{R}\delta_A(x,y)rdrd\theta} \qquad y_{B,com} = \frac{\int_{\theta=0}^{\pi}\int_{r=0}^{R}(r\,sin(\theta))\delta_B(x,y)rdrd\theta}{\int_{\theta=0}^{\pi}\int_{r=0}^{R}\delta_B(x,y)rdrd\theta}$$

$$y_{A,com} = \frac{\int_{\theta=0}^{\pi}\int_{r=0}^{R} r^2\,sin(\theta)\,drd\theta}{\int_{\theta=0}^{\pi}\int_{r=0}^{R} rdrd\theta} \qquad y_{B,com} = \frac{\int_{\theta=0}^{\pi}\int_{r=0}^{R} r^3\,sin^2(\theta)\,drd\theta}{\int_{\theta=0}^{\pi}\int_{r=0}^{R} r^2\,sin(\theta)\,drd\theta}$$

Note the numerator of $y_{A,com}$ is the same integral as the denominator of $y_{B,com}$. Furthermore, the denominator of $y_{A,com}$ is simply the area of the half circle, i.e. $\pi R^2/2$. Therefore, we need only compute two integrals. We start with the numerator of $y_{A,com}$, (the denominator of $y_{B,com}$).

$$\int_{\theta=0}^{\pi}\int_{r=0}^{R} r^2\,sin(\theta)\,drd\theta = \frac{R^3}{3}\int_{\theta=0}^{\pi} sin(\theta)\,d\theta$$

$$= \frac{R^3}{3}(-\,cos(\pi) + cos(0))$$

$$= \frac{2R^3}{3}$$

Next, we evaluate the numerator of $y_{B,com}$.

$$\int_{\theta=0}^{\pi}\int_{r=0}^{R} r^3 \, sin^2(\theta)\, drd\theta = \frac{R^4}{4}\int_{\theta=0}^{\pi} sin^2(\theta)\, d\theta$$

$$= \frac{R^4}{4}\cdot\frac{1}{2}\int_{\theta=0}^{\pi}(1-cos(2\theta))d\theta$$

$$= \frac{R^4}{8}\left(\theta - \frac{1}{2}sin(2\theta)\Big|_0^{\pi}\right)$$

$$= \frac{R^4}{8}\left(\left(\pi - \frac{1}{2}sin(2\pi)\right) - \left(0 - \frac{1}{2}sin(0)\right)\right)$$

$$= \frac{\pi R^4}{8}$$

With these values we can calculate the center of mass for each object.

$$y_{A,com} = \frac{\int_{\theta=0}^{\pi}\int_{r=0}^{R} r^2 \, sin(\theta)\, drd\theta}{\int_{\theta=0}^{\pi}\int_{r=0}^{R} rdrd\theta} \qquad y_{B,com} = \frac{\int_{\theta=0}^{\pi}\int_{r=0}^{R} r^3 \, sin^2(\theta)\, drd\theta}{\int_{\theta=0}^{\pi}\int_{r=0}^{R} r^2 \, sin(\theta)\, drd\theta}$$

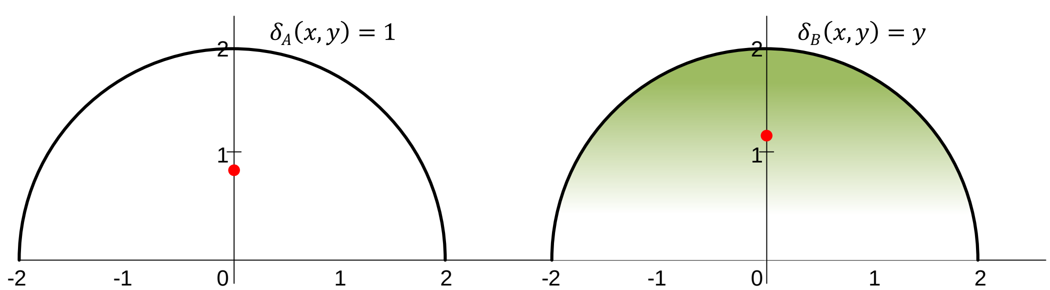
$$y_{A,com} = \frac{\frac{2R^3}{3}}{\frac{\pi R^2}{2}} = \frac{4R}{3\pi} \qquad y_{B,com} = \frac{\frac{\pi R^4}{8}}{\frac{2R^3}{3}} = \frac{3\pi R}{16}$$

Therefore,

$$\left(x_{A,com}, y_{A,com}\right) = \left(0, \frac{4}{3\pi}R\right) \qquad \left(x_{B,com}, y_{B,com}\right) = \left(0, \frac{3\pi}{16}R\right)$$

The figure below shows an illustration for the center of mass with $R = 2$.

When $\delta_A(x, y) = 1$ the center of mass is slightly below the halfway point since the shape contains more mass below the halfway point. However, for $\delta_B(x, y) = y$, the mass is concentrated in the top half of the shape and therefore the center of mass moves above the halfway point.



Example 5: Find the center of mass for the solid hemisphere, $x^2 + y^2 + z^2 \leq R^2, z \geq 0$. Assume $\delta(x, y, z) = 1$

Solution: The hemisphere is symmetrical with respect to the z-axis with a constant density. Therefore, the center of mass in the x-y plane is located at $(0,0)$, and we need only find z_{com}.

$$z_{com} = \frac{\iiint_D z\delta(x,y,z)dV}{\iiint_D \delta(x,y,z)dV}$$

The region lends itself to description in spherical coordinates. A sphere of radius R in spherical coordinates if $\rho = R$, and the region of integration is

$$D = \{(\rho, \phi, \theta) |\ 0 \leq \rho \leq R, 0 \leq \phi \leq \pi/2\,, 0 \leq \theta \leq 2\pi\}$$

Therefore, we have

$$z_{com} = \frac{\int_0^{2\pi} \int_0^{\pi/2} \int_0^R (\rho\, cos(\phi))\, \rho^2\, sin(\phi)\, d\rho d\phi d\theta}{\int_0^{2\pi} \int_0^{\pi/2} \int_0^R \rho^2\, sin(\phi)\, d\rho d\phi d\theta}$$

$$z_{com} = \frac{\int_0^{2\pi} \int_0^{\pi/2} \int_0^R (\rho^3 cos(\phi))\, sin(\phi)\, d\rho d\phi d\theta}{\int_0^{2\pi} \int_0^{\pi/2} \int_0^R \rho^2\, sin(\phi)\, d\rho d\phi d\theta}$$

The denominator is simply the volume of a half of a sphere, i.e. $\frac{2}{3}\pi R^3$, and the numerator is evaluated as follows.

$$\int_0^{2\pi} \int_0^{\pi/2} cos(\phi)\, sin(\phi) \left(\int_0^R \rho^3\, d\rho \right) d\phi d\theta = \frac{R^4}{4} \int_0^{2\pi} \left(\int_0^{\pi/2} \frac{sin(2\phi)}{2} d\phi \right) d\theta$$

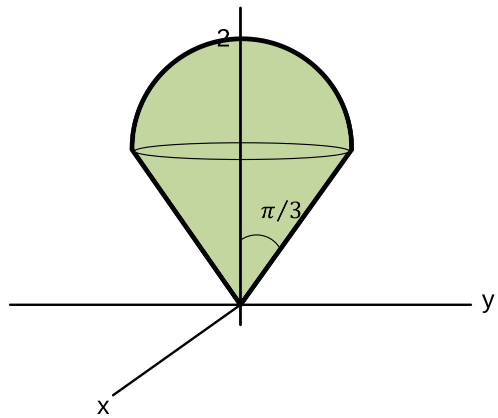
$$= \frac{R^4}{4} \cdot \frac{1}{4} \int_0^{2\pi} (-\,cos(\pi) + cos(0))\ d\theta$$

$$= \frac{R^4}{4} \cdot \frac{2}{4} \int_0^{2\pi} d\theta = \frac{R^4}{8} 2\pi = \frac{\pi R^4}{4}$$

Therefore,

$$z_{com} = \frac{\frac{\pi R^4}{4}}{\frac{2\pi R^3}{3}} = \frac{3}{8} R \qquad \rightarrow \qquad (x_{com}, y_{com}, z_{com}) = \left(0{,}0, \frac{3}{8} R\right)$$

Example 6: Find the center of mass of the "ice cream cone" region bounded, in spherical coordinates, by the cone $\phi = \pi/3$ and the sphere, $\rho = R$. Assume $\delta(x, y, z) = 1$



Solution: Again, we have symmetry with respect to the z-axis with a constant density. Therefore, the center of mass in the x-y plane is located at $(0,0)$, and we need only find z_{com}.

$$z_{com} = \frac{\iiint_D zdV}{\iiint_D dV}$$

The region can be defined as

$$D = \{(\rho, \phi, \theta)|\ 0 \le \rho \le R, 0 \le \phi \le \pi/3\,, 0 \le \theta \le 2\pi\}$$

We start with the numerator substituting $z = \rho\, cos(\phi)$.

$$\iiint_D zdV = \int_0^{2\pi} \int_0^{\pi/3} \int_0^{R} (\rho\, cos(\phi))\, \rho^2\, sin(\phi)\, d\rho d\phi d\theta$$

$$= \int_0^{2\pi} \int_{-\pi/3}^{\pi/3} cos(\phi)\, sin(\phi) \left(\int_0^2 \rho^3\, d\rho\right) d\phi d\theta$$

$$= \frac{R^4}{8} \int_0^{2\pi} \left(\int_0^{\pi/3} sin(2\phi)\, d\phi\right) d\theta$$

$$= \frac{R^4}{16} \int_0^{2\pi} (-\,cos(2\pi/3) + cos(0)) d\theta$$

$$= \frac{3R^4}{32} \int_0^{2\pi} d\theta = \frac{3\pi R^4}{16}$$

Next, we evaluate the denominator.

$$\iiint_D dV = \int_0^{2\pi} \int_0^{\pi/3} \left(\int_0^R \rho^2 \, sin(\phi) \, d\rho \right) d\phi d\theta$$

$$= \frac{R^3}{3} \int_0^{2\pi} \left(\int_0^{\pi/3} sin(\phi) \, d\phi \right) d\theta$$

$$= \frac{R^3}{3} \int_0^{2\pi} (-\,cos(\pi/3) + cos(0)) d\theta$$

$$= \frac{R^3}{6} \int_0^{2\pi} d\theta = \frac{\pi R^3}{3}$$

Therefore,

$$z_{com} = \frac{\frac{3\pi R^4}{16}}{\frac{\pi R^3}{3}} = \frac{9}{16} R \qquad \rightarrow \qquad (x_{com}, y_{com}, z_{com}) = \left(0,0, \frac{9}{16} R\right)$$

Rotational Inertia

We start this section by providing a generalization of the types of integrals we have been using in this lesson. Since $x^0 = 1$ we can write the one dimensional center of mass as

$$x_{com} = \frac{\int_R \; x^1 \delta(x) dx}{\int_R \; x^0 \delta(x) dx}$$

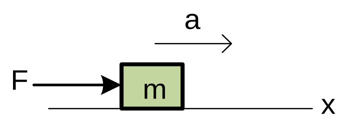
The integrals are identical up to the exponent of the variable x. These types of integrals are called *moments* and are classified by the value of the exponent as shown below.

$$n^{th} \, moment = \int_R x^n \delta(x) dx$$

The interpretation depends on what is represented by the function, $\delta(x)$. For example, when $\delta(x)$ represents the mass density then the 0th moment is the total mass and the 1st moment divided by the 0th moment is the center of mass. We may now ask, "Does the 2nd moment have a physical meaning?". In this section we aim to show, somewhat heuristically, that using mass density the 2nd moment is the rotational inertia.

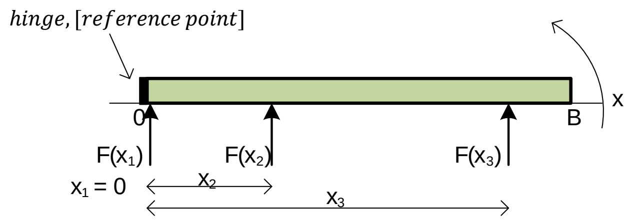
Let's start with Newton's second law.

$$F = ma$$



Where, F represents the net force on the object, a is the acceleration, and m is the mass of the object. The mass of an object is equivalent to what we refer to as the *linear inertia,* defined as a measure of an objects resistance to a change in its *linear*, i.e. straight line, velocity. As a direct analog we can define the *rotational inertia* as a measure of an objects resistance to a change in its *rotational velocity*.

We illustrate with a hypothetical rod in one dimension as shown below. The rod is hinged at $x = 0$, so that a force directed upward will tend to rotate the rod about this reference point.



Intuitively, we can see that $F(x_1)$ will not cause any rotational acceleration, whereas $F(x_3)$ will cause a greater rotational acceleration than $F(x_2)$. In other words, the magnitude of the rotational acceleration depends on both the force *and* the distance from the reference point to the point where the force is applied. Now, we take Newton's 2nd law and multiply both sides by the distance from the reference point, x.

$$Fx = mxa$$

Next, we replace the linear acceleration, a, with the rotational acceleration, α, using $a = x\alpha$.

$$Fx = mx(x\alpha) = mx^2\alpha$$

The value, Fx, is referred to as torque, $\tau = Fx$, giving us a version of Newton's second law applied to rotational motion.

$$\tau = (mx^2)\alpha$$

Where, the quantity $I = mx^2$, is the *rotational inertia* and is directly analogous to linear inertia. In other words, the rotational inertia is an intrinsic property of an object that that measures its resistance to a change in its *rotational velocity*.

$$I = mx^2$$

Finally, as expected, if the mass is not uniformly distributed throughout the object, we instead use a mass element as $dm = \delta(x)dx$ and define the rotational inertia of the entire object as

$$I = \int_R x^2\delta(x)dx$$

Therefore, if $\delta(x)$ represents the mass density of an object, the 2nd moment is the rotational inertia of the object.

The rotational inertia can be defined in a similar fashion for two and three dimensions. A summary is given below.

Rotational Inertia (2nd Moment)

The rotational inertia of an object is a measure of an objects resistance to a change in its change in its *rotational velocity* relative to a reference point, e.g. $x = 0$.

One Dimension

$$I = \int_R x^2 \delta(x) dx$$

Where, $\delta(x)$ is mass density per unit length and R is the interval of integration

Two Dimensions

$$I_x = \iint_R x^2 \delta(x, y) dA \qquad I_y = \iint_R y^2 \delta(x, y) dA \qquad I_z = \iint_R (x^2 + y^2) \delta(x, y) dA$$

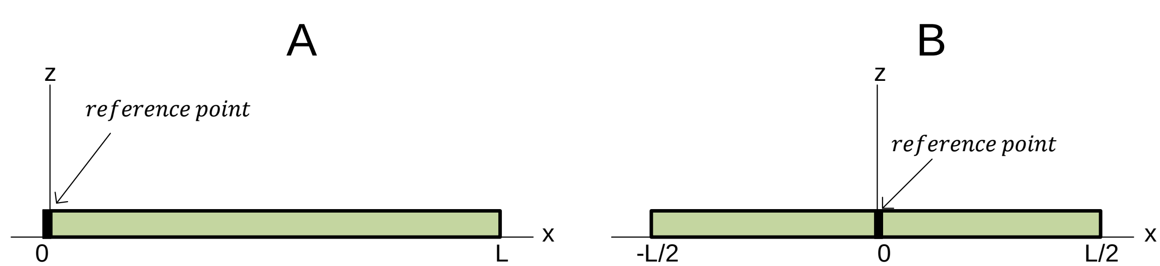
Where, $\delta(x, y)$ is the mass density per unit area, R is the region of integration, and $I_{x,y,z}$ is the rotational inertia with respect to the x, y, z-axis respectively.

Three Dimensions

$$I_x = \iiint_R (y^2 + z^2) \delta(x, y, z) dV \qquad I_y = \iiint_R (x^2 + z^2) \delta(x, y, z) dV \qquad I_z = \iiint_R (x^2 + y^2) \delta(x, y, z) dV$$

Where, $\delta(x, y, z)$ is the mass density per unit volume, R is the region of integration, and $I_{x,y,z}$ is the rotational inertia with respect to the x, y, z-axis respectively.

Example 7: Calculate I_z for a thin rod of length, L, for two different refence point shown below. In both cases assume $\delta(x) = M/L$, i.e. the mass is uniformly distributed.



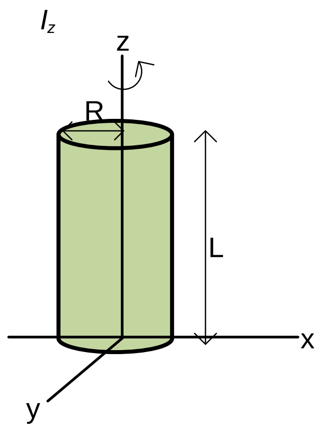
Solution: For rod A we have

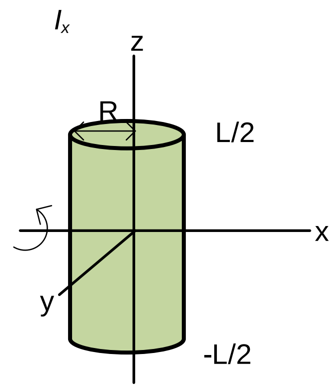
$$I_A = \int\limits_R x^2\delta(x)dx = \int\limits_0^L \left(\frac{M}{L}\right)x^2 dx = \frac{M}{L}\cdot\frac{1}{3}x^3\Big|_0^L = \frac{1}{3}ML^2$$

And for rod B we have

$$I_B = \int\limits_R x^2\delta(x)dx = \int\limits_{-L/2}^{L/2} \left(\frac{M}{L}\right)x^2 dx = \frac{M}{L}\cdot\frac{1}{3}x^3\Big|_{-L/2}^{L/2} = \frac{M}{3L}\left(L^3\left(\frac{1}{8}+\frac{1}{8}\right)\right) = \frac{1}{12}ML^2$$

Example 8: Calculate the rotational inertia of a solid cylinder with respect to the axes shown below. In both cases assume $\delta = M/\pi R^2 L$.





Solution: Based on the shape of the object we'll use cylindrical coordinates. We'll find the rotational inertia with respect to the z-axis first.

$$\begin{aligned}
I_z &= \iiint\limits_R (x^2+y^2)\delta(x,y,z)dV \\
&= \int_0^L\int_0^{2\pi}\int_0^R (r^2)\frac{M}{\pi R^2 L} r\,dr\,d\theta\,dz \\
&= \frac{M}{\pi R^2 L}\int_0^L\int_0^{2\pi}\left(\int_0^R r^3 dr\right)d\theta\,dz \\
&= \frac{MR^4}{4\pi R^2 L}\int_0^L\left(\int_0^{2\pi} d\theta\right)dz \\
&= \frac{MR^2 2\pi}{4\pi L}\int_0^L dz = \frac{MR^2 L}{2L} = \frac{1}{2}MR^2
\end{aligned}$$

With respect to the x-axis we have

$$I_x = \iiint_R (y^2 + z^2)\delta(x,y,z)dV$$

$$= \int_0^L \int_0^{2\pi} \int_0^R (r^2 \sin^2(\theta) + z^2)\frac{M}{\pi R^2 L} r dr d\theta dz$$

$$= \frac{M}{\pi R^2 L}\int_0^L \int_0^{2\pi} \left(\int_0^R (r^3 \sin^2(\theta) + rz^2)dr\right) d\theta dz$$

$$= \frac{M}{\pi R^2 L}\int_0^{2\pi} \left(\int_0^L \left(\frac{R^4}{4}\sin^2(\theta) + \frac{R^2}{2}z^2\right) dz\right) d\theta$$

$$= \frac{M}{\pi R^2 L}\int_0^{2\pi} \left(\frac{R^4}{4}\sin^2(\theta)\, z + \frac{R^2}{6}z^3 \Bigg|_{-L/2}^{L/2}\right) d\theta$$

$$= \frac{M}{\pi R^2 L}\int_0^{2\pi} \left(\left(\frac{LR^4}{8}\sin^2(\theta) + \frac{R^2L^3}{48}\right) - \left(\frac{-LR^4}{8}\sin^2(\theta) - \frac{R^2L^3}{48}\right)\right) d\theta$$

$$= \frac{M}{\pi R^2 L}\int_0^{2\pi} \left(\frac{LR^4}{4}\sin^2(\theta) + \frac{R^2L^3}{24}\right) d\theta$$

$$= \frac{M}{\pi R^2 L}\cdot\left(\frac{LR^4}{4}\pi + \frac{R^2L^3}{24}2\pi\right)$$

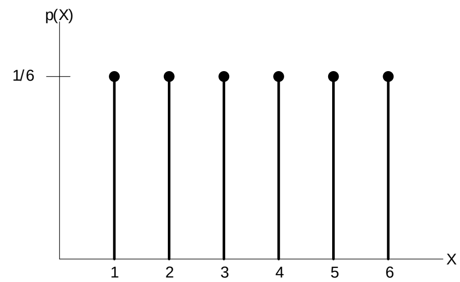
$$= \frac{M}{\cancel{\pi R^2 L}}\cdot\frac{\cancel{\pi L}R^{2\cancel{4}}}{4} + \frac{M}{\cancel{\pi R^2 L}}\cdot\frac{\cancel{\pi R^2}L^{2\cancel{3}}}{12}$$

$$= \frac{1}{4}MR^2 + \frac{1}{12}ML^2$$

Where, we used $\int_0^{2\pi} \sin^2(\theta)\, d\theta = \pi$ from an earlier example. Note I_z is independent from L, but I_x depnds on both L and R.

Probability Theory

A different type of density function is known as a *probability density function*. In probability theory we define a random variable, X, as the outcome of some experiment or measurement where the result is not predetermined. As an example, let's define an "experiment" as rolling a die. The random variable is the set of all possible outcomes, i.e. $\mathrm{X} = \{1,2,3,4,5,6\}$. The probability density function for this experiment, denoted as $p(X)$, is the discrete function shown below. Using a fair die, all outcomes are equally probable and equal to $1/6$.

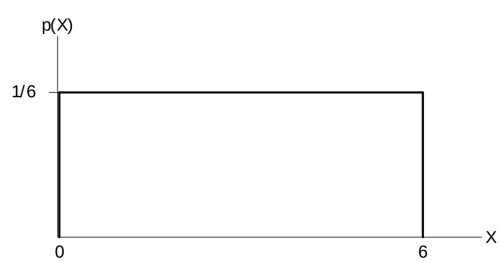


The probability that X will take on a value between, for example, 2 and 4 is denoted as $P(2 \le X \le 4)$ and is found by summing the individual probability density values.

$$
\begin{aligned}
P(2 \le X \le 4) &= P(X = 2) + P(X = 3) + P(X = 4) \\
&= p(2) + p(3) + p(4) \\
&= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}
\end{aligned}
$$

Note that a fundamental property of probability density functions is that the sum of all outcomes must equal one.

The above concepts can be extended to continuous probability density functions, i.e. the random variable, X, takes on a continuum of values. For example, assume we throw a dart randomly into a box that measures 6 feet in length. We let the random variable, X, represent the distance into the box where the dart lands. Assuming all darts land inside the box, the random variable, X, can take on any value between zero and six, $X = [0,6]$. Assuming again that all outcomes are equally probable the probability density function is as shown below.



We can again compute the probability that X will take on a certain range of values, e.g. between 0 and 3, i.e. $P(0 \le X \le 3)$. However, in this case the discrete summation turns into an integration.

$$
P(0 \le X \le 3) = \int_0^3 p(x)dx = \int_0^3 \frac{1}{6} dx \frac{1}{6}(3-0) = \frac{1}{2}
$$

We can also perform two experiments, with outcomes denoted as X and Y. In this case the probability that X and Y will take on a certain range values is called the joint probability and is computed using a *joint probability density function*, $p(x, y)$.

$$P(a \leq X \leq b;\ c \leq Y \leq d) = \int_{y=c}^{d} \int_{x=a}^{b} p(x, y)dxdy$$

The results can be extended to any number of random variables and is summarized below up to three variables.

Probability Density Functions
A random variable, X, is defined as the outcome of some experiment or measurement where the result is not predetermined. If X is a continuous random variable, then the probability that the outcome lies in a certain interval is equal to the integral of the probability density function evaluated over the interval.
One Random Variable
$$P(a \leq X \leq b) = \int_{x=a}^{b} p(x)dx$$ Where, $p(x, y)$ is the probability density function that must satisfy $$\int_{x=-\infty}^{\infty} p(x)dx = 1$$
Two Random Variables
$$P(a \leq X \leq b;\ c \leq Y \leq d) = \int_{y=c}^{d} \int_{x=a}^{b} p(x, y)dxdy$$ Where, $p(x, y)$ is the probability density function that must satisfy $$\int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} p(x, y)dxdy = 1$$
Three Random Variables
$$P(a \leq X \leq b;\ c \leq Y \leq d;\ e \leq Z \leq f) = \int_{z=e}^{f} \int_{y=c}^{d} \int_{x=a}^{b} p(x, y, z)dxdydz$$ Where, $p(x, y, z)$ is the probability density function that must satisfy $$\int_{z=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} p(x, y, z)dxdydz = 1$$

Example 9: The time to failure, (in months), of two critical components in an aircraft sensor system is modeled with two random variables, X and Y. The joint probability density function is

$$p(x,y) = \begin{cases} \frac{1}{864} e^{-\frac{x}{24}-\frac{y}{36}}, & x \geq 0, y \geq 0 \\ 0, & otherwise \end{cases}$$

What is the probability that neither component functions after 2 years?

Solution: The probability that the time to failure for both components is less than 2 years, i.e. 24 months is given as

$$\begin{aligned} P(0 \leq X \leq 24;\ 0 \leq Y \leq 24) &= \int_{y=0}^{24} \int_{x=0}^{24} \left(\frac{1}{864} e^{-\frac{x}{24}-\frac{y}{36}} \right) dxdy \\ &= \frac{1}{864} \int_{y=0}^{24} e^{-\frac{y}{36}} \left(\int_{x=0}^{24} e^{-\frac{x}{24}} dx \right) dy \\ &= \frac{1}{864} \int_{y=0}^{24} e^{-\frac{y}{36}} \left(-24\, e^{-\frac{x}{24}} \Big|_0^{24} \right) dy \\ &= \frac{1}{864} \int_{y=0}^{24} e^{-\frac{y}{36}} (24(1-e^{-1})) dy \\ &= \frac{24(1-e^{-1})}{864} \left(-36\, e^{-\frac{y}{36}} \Big|_0^{24} \right) \\ &= \frac{24(1-e^{-1})36\left(1-e^{-\frac{2}{3}}\right)}{864} \\ &= (1-e^{-1})\left(1-e^{-\frac{2}{3}}\right) \\ &\cong 0.31 \end{aligned}$$

Example 10: In quantum mechanics the state of a particle is described by its so-called *wave function*. Considering the state as being the distance, ρ, from the center of the atom in spherical coordinates, the wave function for an electron in the 1s state of a hydrogen atom is

$$\psi(\rho) = \frac{1}{\sqrt{\pi a^3}} e^{-\rho/a}$$

Where, $a = 5.3E^{-11}\ m$, is the Bohr radius.

The magnitude squared of the wave function can be interpreted as the probability density function for the particle.

$$p(\rho, \phi, \theta) = |\psi(\rho)|^2$$

Find the probability that the electron is found to be outside of the Bohr radius.

Solution: The probability density function is

$$p(\rho,\phi,\theta) = |\psi(\rho)|^2 = \left|\frac{1}{\sqrt{\pi a^3}} e^{-\rho/a}\right|^2 = \frac{1}{\pi a^3} e^{-2\rho/a}$$

The probability that the electron is found in the range, $a \le \rho \le \infty$, is then found as follows

$$P(a \le \rho \le \infty, 0 \le \phi \le \pi, -\pi \le \theta \le \pi) = \int_{-\pi}^{\pi}\int_{0}^{\pi}\int_{a}^{\infty}\left(\frac{1}{\pi a^3} e^{-2\rho/a}\right)\rho^2 \sin(\phi)\, d\rho d\phi d\theta$$

$$= \frac{1}{\pi a^3}\int_{-\pi}^{\pi} d\theta \int_{0}^{\pi} \sin(\phi)\, d\phi \int_{a}^{\infty}(\rho^2 e^{-2\rho/a} d\rho)$$

$$= \frac{1}{\pi a^3}\left(\int_{-\pi}^{\pi} d\theta\right)\left(\int_{0}^{\pi} \sin(\phi)\, d\phi\right)\left(\int_{a}^{\infty} \rho^2 e^{-2\rho/a} d\rho\right)$$

$$= \frac{1}{\pi a^3}(2\pi)(2)\left(\int_{a}^{\infty} \rho^2 e^{-2\rho/a} d\rho\right)$$

$$= \frac{4}{a^3}\left(\int_{a}^{\infty} \rho^2 e^{-2\rho/a} d\rho\right)$$

The last integral can be solved using integration by parts twice, However, we will instead use the following formula from a table of integrals.

$$\boxed{\int x^2 e^{bx} dx = \frac{e^{bx}}{b}\left(x^2 - \frac{2x}{b} + \frac{2}{b^2}\right)}$$

Therefore,

$$\int_{a}^{\infty}(\rho^2 e^{\rho(-2/a)} d\rho) = \frac{-ae^{-2\rho/a}}{2}\left(\rho^2 + \frac{a\rho}{1} + \frac{a^2}{2}\right)\Bigg|_{a}^{\infty}$$

$$= \left[\frac{-ae^{-\infty}}{2}\left(\infty^2 + \frac{\infty}{1} + \frac{a^2}{2}\right)\right] - \left[\frac{-ae^{-2}}{2}\left(a^2 + \frac{a^2}{1} + \frac{a^2}{2}\right)\right]$$

$$= \left[\left(\frac{a\infty^2}{2e^\infty} - \frac{a\infty}{2e^\infty} - \frac{a^3}{4e^\infty}\right)\right] + \left[\left(\frac{a^3}{2e^2} + \frac{a^3}{2e^2} + \frac{a^3}{4e^2}\right)\right]$$

$$= 0 + \frac{5a^3}{4e^2}$$

Substituting we have

$$P(a \le \rho \le \infty, 0 \le \phi \le \pi, -\pi \le \theta \le \pi) = \frac{4}{a^3}\left(\int_{a}^{\infty} \rho^2 e^{-2\rho/a} d\rho\right)$$

$$= \frac{4}{a^3}\left(\frac{5a^3}{4e^2}\right) = \frac{5}{e^2} \cong 0.677$$

Final Summary for Multiple Integration – Applications

Total Amount Using Density

One Dimension

$$Total\ Amount\ = \int_R \delta(x)dx$$

Where, $\delta(x)$ is the amount per unit length and R is the interval of integration

Two Dimensions

$$Total\ Amount\ = \iint_R \delta(x,y)dA$$

Where, $\delta(x,y)$ is the amount per unit area and R is the region of integration.

Three Dimensions

$$Total\ Amount = \iiint_R \delta(x,y,z)dV$$

Where, $\delta(x,y,z)$ is the amount per unit volume and R is the region of integration.

Center of Mass

One Dimension

$$x_{com} = \frac{\int_R x\delta(x)dx}{\int_R \delta(x)dx}$$

Where, $\delta(x)$ is mass density per unit length and R is the interval of integration

Two Dimensions

$$x_{com} = \frac{\iint_R x\delta(x,y)dA}{\iint_R \delta(x,y)dA} \qquad y_{com} = \frac{\iint_R y\delta(x,y)dA}{\iint_R \delta(x,y)dA}$$

Where, $\delta(x,y)$ is the mass density per unit area and R is the region of integration.

Three Dimensions

$$x_{com} = \frac{\iiint_R x\delta(x,y)dV}{\iiint_R \delta(x,y)dV} \qquad y_{com} = \frac{\iiint_R y\delta(x,y)dV}{\iiint_R \delta(x,y)dV} \qquad z_{com} = \frac{\iiint_R z\delta(x,y)dV}{\iiint_R \delta(x,y)dV}$$

Where, $\delta(x,y,z)$ is the mass density per unit volume and R is the region of integration.

Rotational Inertia (2nd Moment)

One Dimension

$$I = \int_R x^2 \delta(x) dx$$

Where, $\delta(x)$ is mass density per unit length and R is the interval of integration

Two Dimensions

$$I_x = \iint_R x^2 \delta(x,y) dA \qquad I_y = \iint_R y^2 \delta(x,y) dA \qquad I_z = \iint_R (x^2 + y^2) \delta(x,y) dA$$

Where, $\delta(x,y)$ is the mass density per unit area, R is the region of integration, and $I_{x,y,z}$ is the rotational inertia with respect to the x, y, z-axis respectively.

Three Dimensions

$$I_x = \iiint_R (y^2 + z^2) \delta(x,y,z) dV \qquad I_y = \iiint_R (x^2 + z^2) \delta(x,y,z) dV \qquad I_z = \iiint_R (x^2 + y^2) \delta(x,y,z) dV$$

Where, $\delta(x,y,z)$ is the mass density per unit volume, R is the region of integration, and $I_{x,y,z}$ is the rotational inertia with respect to the x, y, z-axis respectively.

Probability Density Functions

One Random Variable

$$P(a \le X \le b) = \int_{x=a}^{b} p(x) dx$$

Two Random Variables

$$P(a \le X \le b;\ c \le Y \le d) = \int_{y=c}^{d} \int_{x=a}^{b} p(x,y) dx dy$$

Three Random Variables

$$P(a \le X \le b;\ c \le Y \le d;\ e \le Z \le f) = \int_{z=e}^{f} \int_{y=c}^{d} \int_{x=a}^{b} p(x,y,z) dx dy dz$$

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