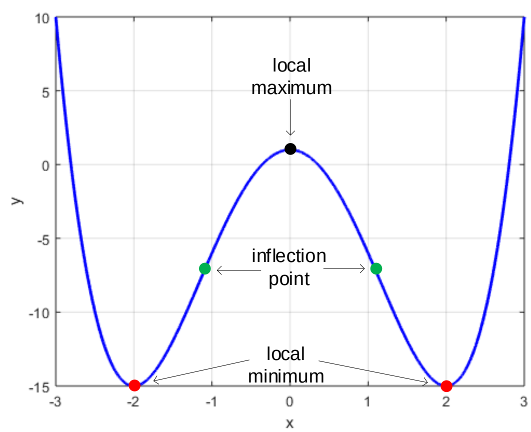
Multivariable Differentiation – Optimization

In our calculus 1 lesson we developed techniques to find the maximum and minimum values for single variable functions, $f(x)$. In this lesson we extend these techniques to multivariable functions. Then main concepts are completely analogous, however additional computation is generally required compared to single variable optimization. We'll limit our lesson to studying optimization of two variable functions.

Single Variable Optimization Review

The figure below shows an example single variable function containing two local minimum points, one local maximum, and two inflection points. Furthermore, if we consider the interval $[-3,3]$, the endpoints of this interval are considered absolute maximums.



Conceptually, optimization for two variable functions is completely analogous to optimization for single variable functions. As such we start with a brief review of some of the most important concepts used in single variable optimization.

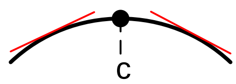
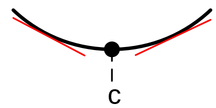
Critical points are locations where the derivative is either equal to zero or does not exist.

Critical Points Definition
A number c in the domain of $f(x)$ is called a **critical point** if either of the following are true: • $f'(c) = 0$ • $f'(c)$ does not exist.

Next, we come to Fermat's theorem, which states that if $f(c)$ is a local extreme value, i.e. a local minimum or maximum value, then c is a critical point and thus the tangent line (if it exists), is horizontal at $x = c$.

Fermat's Theorem of Local Extrema
If $f(c)$ is a local minimum or maximum, then c is a critical point of f. Note: This theorem does not claim that all critical points are local extreme values, but rather that all local extreme values are critical points.

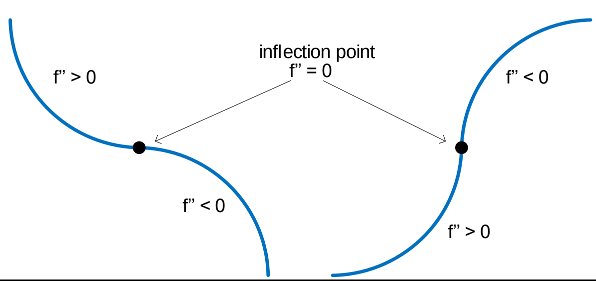
Once critical points are located where the tangent line is horizontal, we need to determine if the points are local minimums or local maximums. One of the tests we used was called the first derivative test, which is shown below.

First Derivative Test for Critical Points
Let c be a critical point of the function, $f(x)$. Then:
1. If $f'(x)$ changes from $+$ to – at c, $\implies$ $f(c)$ is a local maximum. 
2. If $f'(x)$ changes from – to $+$at c, $\implies$ $f(c)$ is a local minimum. 

Next, we introduced a more straightforward method called the second derivative test.

Second Derivative Test for Critical Points
Let c be a critical point of the function, $f(x)$. Then:
1. If $f''(c) > 0$ $\implies$ $f(c)$ is a local maximum. 2. If $f''(c) < 0$ $\implies$ $f(c)$ is a local minimum. 3. If $f''(c) = 0$ $\implies$ The test is inconclusive. a. $f(c)$ may be a local maximum, a local minimum, an inflection point, or none of the above. b. Use the first derivative test and/or the test for inflection points.

In the third case, when the test is inconclusive, one possibility is that we have an inflection point. An inflection point is identified by the fact that the second derivative changes sign at c.

Test for Inflection Points
A number c in the domain of $f(x)$ is called an **inflection point** if either of the following are true: • $f''(c) = 0$ and $f''(x)$ changes sign at $x = c$ • $f''(c)$ does not exist and $f''(x)$ changes sign at $x = c$


Multivariable Optimization

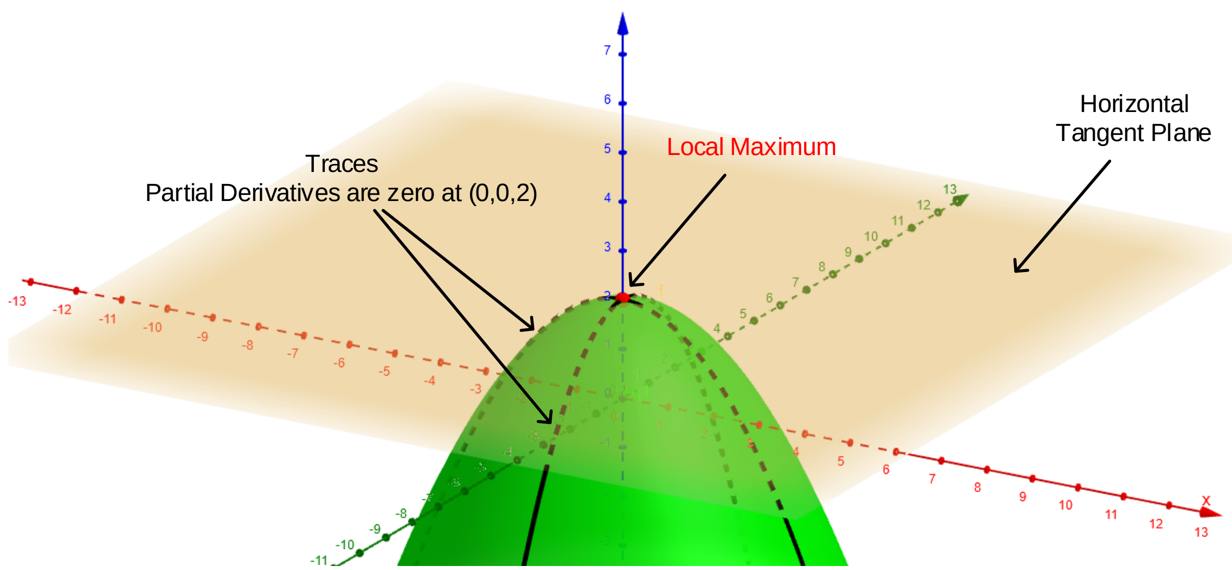
Using the quick review from above on single variable optimization we start by giving the analogous definition of critical points for two variable functions.

Critical Points Definition for Two Variable Functions
A point $P = (a, b)$ in the domain of $f(x, y)$ is called a **critical point** if: • $f_x(a, b) = 0$ or $f_x(a, b)$ does not exists, AND • $f_y(a, b) = 0$ or $f_y(a, b)$ does not exists.

Similar to the singular variable case we also have Fermat's theorem for two variable functions.

Fermat's Theorem of Local Extrema for Two Variable Functions
If $f(a, b)$ is a local minimum or maximum, then $P = (a, b)$ is a critical point of $f(x, y)$. Note: This theorem does not claim that all critical points are local extreme values, but rather that all local extreme values are critical points.

The figure below illustrates a critical point that is also a local maximum. Recall for single variable functions the point where the derivative is zero corresponds a horizontal *tangent line*. For two variable functions, the point at which **both** partial derivatives evaluate to zero correspond to the location where the *tangent plane* is horizontal. In the example below the point correspond to a local maximum.



Once the critical points are identified we need to determine their nature. Just as in the single variable case the possibilities are local minimums, local maximums, or saddle points, which are analogous to inflection points. For two variable functions we use a test that is analogous to the second derivative test for single variable functions. The proof will be provided in an appendix section at a later date.

Second Derivative Test for Two Variable Functions
Let $P = (a, b)$ be a critical point of the function, $f(x, y)$ and assume f_{xx}, f_{yy} and f_{xy} are continuous near P. Then: 1. If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum. 2. If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum. 3. If $D < 0$ then $f(a, b)$ is a saddle point. 4. If $D = 0$ then the test is inconclusive. Where D is called the discriminate $$D = f_{xx}(a, b)f_{yy}(a, b) - {f_{xy}}^2(a, b)$$

Now let's look at some examples.

Example 1: Find all critical points of $f(x, y) = 11x^2 - 2xy + 2y^2 + 3y$ and then apply the second derivative test to classify each.

Solution: We check for critical points by setting the partial derivatives to zero.

$$f_x = 22x - 2y = 0 \qquad\qquad f_y = -2x + 4y + 3 = 0$$

Which results in a set of simultaneous linear equations which are solved below.

From the first equation we find $y = 11x$. Substituting this into the second equation we find

$$-2x + 4(11x) + 3 = 0$$
$$x = -\frac{1}{14} \qquad \rightarrow \qquad y = -\frac{11}{14}$$

Therefore, the function has one critical point, $P = \left(-\frac{1}{14}, -\frac{11}{14}\right)$.

To classify this critical point, we use the second derivative test with the computations below.

$$f_{xx}|_P = 22 \qquad f_{yy}\big|_P = 4 \qquad f_{xy}\big|_P = -2$$

$$D = f_{xx}|_P \cdot f_{yy}\big|_P - \left(f_{xy}\big|_P\right)^2$$
$$D = 22 \cdot 4 - (-2)^2$$
$$D = 40$$

Finally, since $D > 0$ and $f_{xy}\big|_P > 0$, $P = \left(-\frac{1}{14}, -\frac{11}{14}\right)$ is a local minimum.

Example 2: Find all critical points of $f(x,y) = 4xy - x^4 - y^4$ and then apply the second derivative test to classify each.

Solution:

$$f_x = 4y - 4x^3 = 0 \qquad\qquad f_y = 4x - 4y^3 = 0$$
$$y = x^3 \qquad\qquad x = y^3$$

Substituting the first equation into the second we have

$$x = x^9$$

Which implies $x = 0, 1$ or -1, and since $y = x^3$ we have the following three critical points.

$$P_1 = (0,0) \qquad P_2 = (1,1) \qquad P_3 = (-1,-1)$$

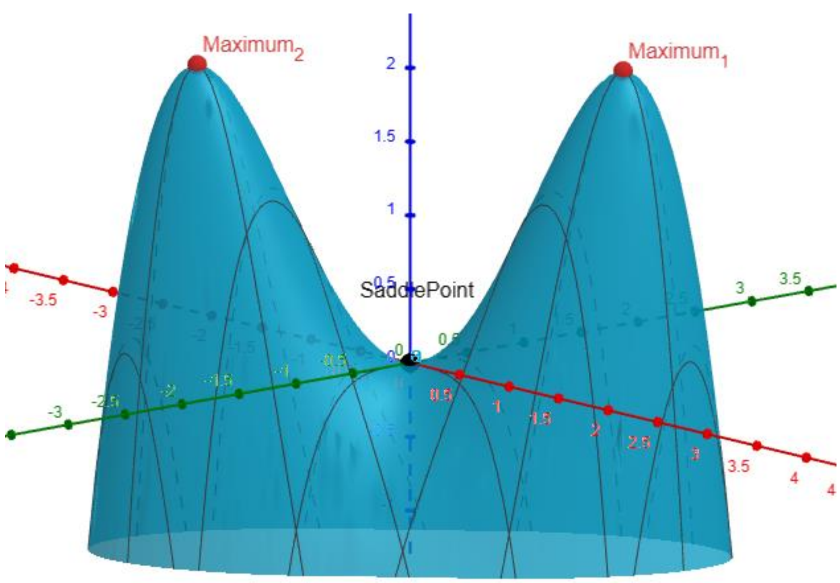
To classify these points, we use the second derivative test.

$$f_{xx}|_P = 22 \qquad f_{yy}\big|_P = 4 \qquad f_{xy}\big|_P = -2$$

$$D = f_{xx}|_P \cdot f_{yy}\big|_P - \left(f_{xy}\big|_P\right)^2$$
$$D = 22 \cdot 4 - (-2)^2$$
$$D = 40$$

P	$f_{xx}\|_P$	$f_{yy}\|_P$	$f_{xy}\|_P$	D	***Classification***
$(0,0)$	0	0	4	-16	Saddle Point
$(1,1)$	-12	-12	4	128	Maximum
$(-1,-1)$	-12	-12	4	128	Maximum

The graph is shown below for illustration.



Absolute Extrema

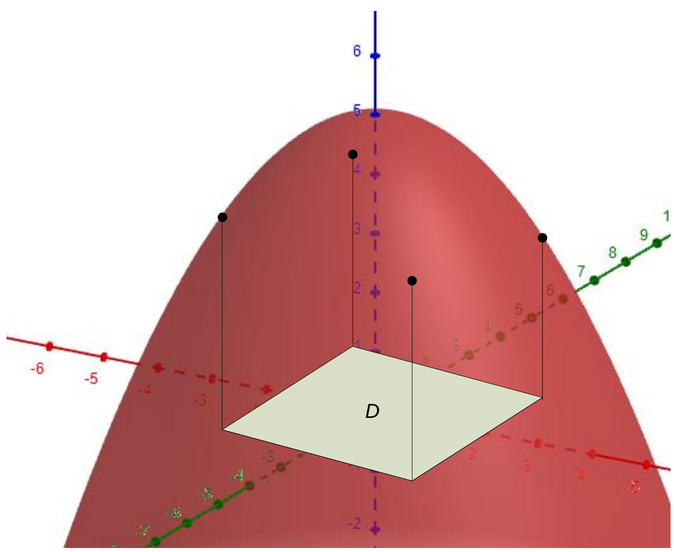
As mentioned in the single variable optimization review section, local extrema are not necessarily the absolute extreme values of the function. The figure in that section shows a function with a *local* maximum value as well as two *absolute* maximum values over the interval $[-3,3]$. For single variable calculus we can state the following.

Existence and Location of Absolute Extrema
Let $f(x)$ be a continuous function on a closed interval, I. Then: 1. $f(x)$ takes on both a minimum and maximum value on I. 2. The extreme values occur either at critical points in the interior of I or at the endpoints.

This theorem also applies for two variable functions as stated below.

Existence and Location of Absolute Extrema
Let $f(x,y)$ be a continuous function on a closed domain D in R^2. Then: 1. $f(x,y)$ takes on both a minimum and maximum value on D. 2. The extreme values occur either at critical points in the interior of D or at points on the boundary of D.

The figure below illustrates a function that is to be considered on the domain, D, only.



Let’s look at a specific example.

Example 3: Find the maximum and minimum values of $f(x, y) = 2x + y - 3xy$ on the unit square $D = \{(x, y) : 0 \le x, y \le 1\}$.

Solution: We'll start by finding the critical points within the domain given.

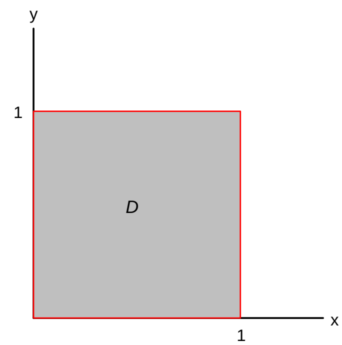
$$f_x = 2 - 3y = 0 \qquad\qquad f_y = 1 - 3x = 0$$

$$y = \frac{2}{3} \qquad\qquad x = \frac{1}{3}$$

Therefore, a critical point exists at $P = \left(\frac{1}{3}, \frac{2}{3}\right)$. The value of the function at this location is

$$f\left(\frac{1}{3}, \frac{2}{3}\right) = 2\left(\frac{1}{3}\right) + \frac{2}{3} - 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{2}{3}$$

Next, we check the value of the function on the boundaries of the domain, which we can divide into four edges of the square shown below.



Left Edge: $x = 0, 0 \le y \le 1$	***Right Edge:*** $x = 1, 0 \le y \le 1$
$f(0, y) = y$ Max: $f(0,1) = 1$ Min: $f(0,0) = 0$	$f(1, y) = 2 + y - 3y = 2 - 2y$ Max: $f(1,0) = 2$ Min: $f(1,1) = 0$

Bottom Edge: $0 \le x \le 1, y = 0$	***Top Edge:*** $0 \le x \le 1, y = 1$
$f(x, 0) = 2x$ Max: $f(1,0) = 2$ Min: $f(0,0) = 0$	$f(x, 1) = 2x + 1 - 3x = 1 - x$ Max: $f(0,1) = 1$ Min: $f(1,1) = 0$

Comparing these values with the critical point from above we find

- Absolute maximum on boundary: $f(1,0) = 2$.
- Absolute minimum on boundary: $f(0,0) = f(1,1) = 0$

Most optimization problems in practice will not be given with the equations already developed. Therefore, the next few examples will require us to develop the optimization equation in addition to solving for the applicable extreme value.

Example 4: Find the shortest distance from the point $P = (1,2,3)$ to the plane $x + y + z = 1$

Solution: We start by considering an arbitrary point on the plane, $Q = (x, y, z)$. Then the distance between P and Q is given by the following equation, i.e. a simple extension of the Pythagorean Theorem.

$$D = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$$

Where, $(a, b, c) = (1,2,3)$

We would like to minimize this function, which we refer to as the *objective function*. The minimization is however constrained by the fact that Q has to be on the plane. We account for this by solving the equation of the plane for one of the variables and substituting into the objective function.

$$z = (1 - x - y\ )$$

The objective function is now a two variable function given as

$$D(x, y) = \sqrt{(x-a)^2 + (y-b)^2 + (1 - x - y\ - c)^2}$$

Finally, to reduce computation complexity we notice that the distance will be minimum when the distance squared is minimized. Therefore, the final objective function is given as

$$D^2 = f(x, y) = (x-a)^2 + (y-b)^2 + (1 - x - y\ - c)^2$$

Setting the partials equal to zero we obtain.

$$f_x = 2(x-a) - 2(1 - x - y\ - c)$$
$$0 = x - a - 1 + x + y + c$$
$$2x + y = (1 + a - c)$$

$$f_y = 2(y-b) - 2(1 - x - y\ - c)$$
$$0 = y - b - 1 + x + y + c$$
$$2y + x = (1 + b - c)$$

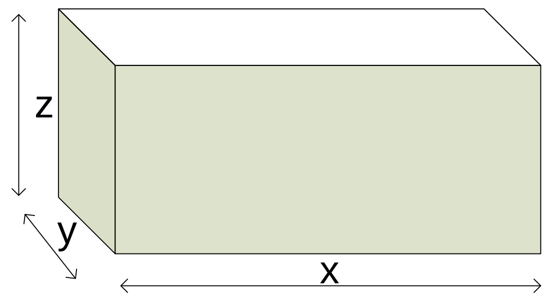
Substituting $(a, b, c) = (1,2,3)$ and solving, we find

$$x = -\frac{2}{3} \qquad y = \frac{1}{3} \qquad \rightarrow \qquad z = \left(1 + \frac{2}{3} - \frac{1}{3}\ \right) = \frac{4}{3}$$

Therefore, the point $Q = \left(-\frac{2}{3}, \frac{1}{3}, \frac{4}{3}\right)$, and the distance between the two points is

$$D = \sqrt{\left(-\frac{2}{3} - 1\right)^2 + \left(\frac{1}{3} - 2\right)^2 + \left(\frac{4}{3}\ - 3\right)^2} = \frac{5}{\sqrt{3}}$$

Example 5: An open top rectangular bin is designed to hold V cubic feet of garbage. The four walls are fabricated from material costing \$2 per square foot while the bottom is made from material costing \$4 per square foot. Find the dimensions of the bin to minimize the cost.



Solution: We start with the objective function being the cost of the material.

Cost of Bottom	***Cost of Sides***	***Total Cost***
$\$4(xy)$	$\$2(2yz + 2xz)$	$C = 4(xy + yz + xz)$

We are constrained by the volume of the box being V cubic feet.

$$V = xyz$$

Solving for z and substituting, we create the desired 2 variable objective function.

$$C = 4\left(xy + y\left(\frac{V}{xy}\right) + x\left(\frac{V}{xy}\right)\right)$$

$$C(x, y) = 4\left(xy + \frac{V}{x} + \frac{V}{y}\right)$$

Setting the partials equal to zero we obtain.

$$C_x = 4\left(y - \frac{V}{x^2}\right) = 0$$

$$y = \frac{V}{x^2}$$

$$C_y = 4\left(x - \frac{V}{y^2}\right) = 0$$

$$x = \frac{V}{y^2}$$

Simultaneously solving we find

$$x = \frac{V}{\left(\frac{V}{x^2}\right)^2}$$

$$1 = \frac{x^3}{V}$$

$$x = \sqrt[3]{V}$$

$$y = \frac{V}{\left(\sqrt[3]{V}\right)^2}$$

$$y = \sqrt[3]{V}$$

$$\rightarrow$$

$$z = \frac{V}{xy}$$

$$z = \frac{V}{\sqrt[3]{V} \cdot \sqrt[3]{V}}$$

$$z = \sqrt[3]{V}$$

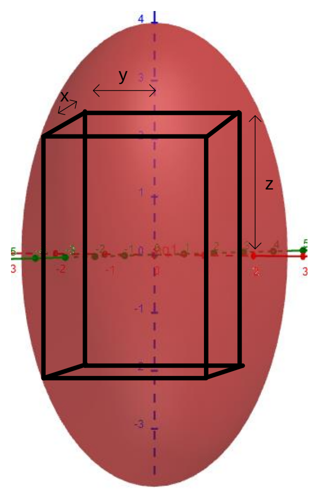
Therefore, the cost in minimized when the box is a cube! The cost can be written in terms of the volume as

$$C = 4\left(xy + y\left(\frac{V}{xy}\right) + x\left(\frac{V}{xy}\right)\right)$$
$$= 4\left(V^{2/3} + V^{1/3}\left(\frac{V}{V^{2/3}}\right) + V^{1/3}\left(\frac{V}{V^{2/3}}\right)\right)$$
$$= 4\left(3V^{2/3}\right)$$
$$= 12\left(\sqrt[3]{V^2}\right)$$

For example, if $V = 100\ ft^2$, the minimum cost is $12\left(\sqrt[3]{100^2}\right) \cong \$259$

Example 6: Find the dimensions of the largest rectangular box that can be inscribed in a general ellipsoid.

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$



Solution: In this case the objective function is the volume of the rectangular box given by

$$V = 2x \cdot 2y \cdot 2z = 8xyz$$

The constraint function is given by the ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

Solving the ellipsoid for z and substituting into the objective function we get

$$V(x,y) = 8xy\left(c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\right) = 8cxy\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

In this case computing the partials and solving is somewhat tedious. However, doing so will allow us to appreciate the next lesson on Lagrange Multipliers where we'll redo this problem and find it much less painful. The partial with respect to x is computed, set to zero, and simplified below.

$$
\begin{aligned}
V_x &= (8cy)\left(\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right)+(8cxy)\left(\frac{-\frac{x}{a^2}}{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\right) \\
&= \frac{8cy\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}{1}-\frac{8cyx^2}{a^2\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} \\
&= \frac{8cya^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)-8cyx^2}{a^2\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}
\end{aligned}
$$

Setting to zero allows us to work with the numerator only. Additionally, we divide through by $8cy$ and ignore the $y=0$ solution.

$$
\begin{aligned}
x^2 &= a^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right) \\
\frac{x^2}{a^2} &= 1-\frac{x^2}{a^2}-\frac{y^2}{b^2} \\
\frac{2x^2}{a^2} &= 1-\frac{y^2}{b^2} \\
x^2 &= \frac{a^2}{2}-\frac{a^2y^2}{2b^2}
\end{aligned}
$$

The computation of partial with respect to y is similarly and the result sis shown below.

$$
y^2 = \frac{b^2}{2}-\frac{b^2x^2}{2a^2}
$$

Next, we substitute for x^2 and solve for y.

$$
\begin{aligned}
y^2 &= \frac{b^2}{2}-\frac{b^2}{2a^2}\left(\frac{a^2}{2}-\frac{a^2y^2}{2b^2}\right) \\
y^2 &= \frac{b^2}{2}-\frac{b^2}{4}+\frac{y^2}{4} \\
\frac{3y^2}{4} &= \frac{b^2}{4} \\
y &= \frac{b}{\sqrt{3}}
\end{aligned}
$$

Where, we ignored the negative answer.

Substituting back to solve for x we find

$$x^2 = \frac{a^2}{2} - \frac{a^2}{2\cancel{b^2}}\left(\frac{\cancel{b^2}}{3}\right)$$
$$x^2 = \frac{a^2}{3}$$
$$x = \frac{a}{\sqrt{3}}$$

Finally, we can solve for z

$$z = c\sqrt{1 - \frac{\frac{\cancel{a^2}}{3}}{\cancel{a^2}} - \frac{\frac{\cancel{b^2}}{3}}{\cancel{b^2}}}$$
$$z = \frac{c}{\sqrt{3}}$$

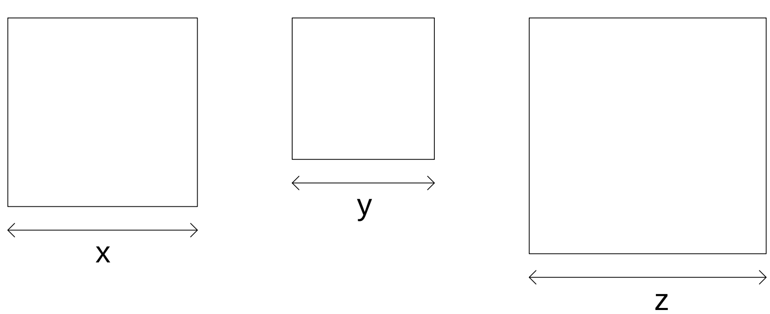
Therefore, the dimensions of the maximum box is $\frac{1}{\sqrt{3}}(a, b, c)$. The volume of the box is given as

$$V = 8xyz = 8\left(\frac{a}{\sqrt{3}} \cdot \frac{b}{\sqrt{3}} \cdot \frac{c}{\sqrt{3}}\right) = \frac{8abc}{3\sqrt{3}}$$

Note, if we let $a = b = c = R$, then we have a sphere of radius R. The dimensions of the box is then $\left(\frac{R}{\sqrt{3}}, \frac{R}{\sqrt{3}}, \frac{R}{\sqrt{3}}\right)$ with a volume of

$$V = \frac{8R^3}{3\sqrt{3}}$$

Example 7: A 120-m long fence is to be cut into pieces to make three enclosures, each of which is a square. How should the fence be cut up in order to minimize the total area enclosed by the fence?



Solution: In this case the objective function is the total area, A, which is subject to the available fencing, P.

Objective Function	***Constraint Function***
$A = x^2 + y^2 + z^2$	$P = 4x + 4y + 4z$

Solving the constraint function for z and substituting into the objective function we have

$$A(x, y) = x^2 + y^2 + \left(\frac{P}{4} - x - y\right)^2$$

With partials

$$A_x = 2x - 2\left(\frac{P}{4} - x - y\right) = 4x + 2y - \frac{P}{2}$$

$$A_y = 2y - 2\left(\frac{P}{4} - x - y\right) = 4y + 2x - \frac{P}{2}$$

Setting the partials to zero and solving we find

$$x = y = \frac{P}{12}$$

Therefore,

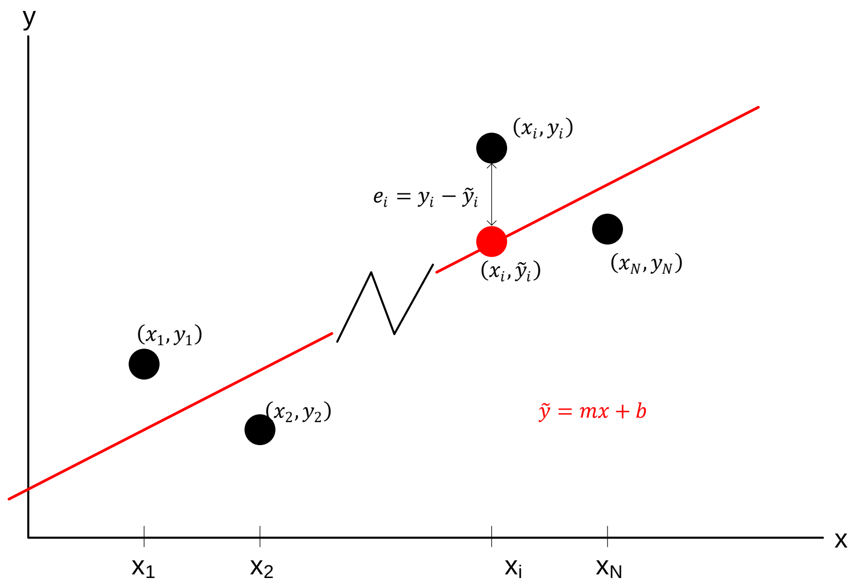
$$z = \frac{P}{4} - x - y = \frac{3P}{12} - \frac{P}{12} - \frac{P}{12} = \frac{P}{12}$$

To minimize the area all enclosures should be of the same size, $x = y = z = \frac{P}{12} = s$

$$s = \frac{120}{12} = 10\ m$$

The fence should be cut into 12 pieces, each $10\ m$ long, to create 3 - $10x10$ enclosures.

Example 8: In many engineering and science applications data is collected empirically that appears to fit to a straight line. The figure below illustrates data, (x, y), that is collected and plotted on an x-y plane. The red line, $\tilde{y} = mx + b$, illustrates a line that attempts to fit the data collected. How can we choose the 'best' line based on the data collected?



Solution: We can start by looking at the error for a particular point between the empirical data, y_i, and the estimate, $\tilde{y}_i$, as illustrated in the figure above.

$$e_i = \tilde{y}_i - y_i$$
$$e_i = (mx_i + b) - y_i$$

The total error, E, for N data points can then be written as

$$E = \sum_{i=1}^{N} (mx_i + b - y_i)$$

However, to avoid adding negative errors with positive errors we instead use the squared error for each data point. The final parameter used to measure the total error is then

$$E(m, b) = \sum_{i=1}^{N} (mx_i + b - y_i)^2$$

To choose the line of 'best' fit we would like to minimize this value, which is a function of two variables, m and b. Therefore, we can use the techniques learned in the lesson to find the best fit line!

Since the derivative is a linear operator, we can move it inside the summation. Therefore, the partials are

$$E_m = \sum_{i=1}^{N} 2x_i(mx_i + b - y_i) \qquad\qquad E_b = \sum_{i=1}^{N} 2(mx_i + b - y_i)$$

Setting the partials to zero we have

$$\sum_{i=1}^{N} x_i(mx_i + b - y_i) = 0 \qquad\qquad \sum_{i=1}^{N}(mx_i + b - y_i) = 0$$

$$m\sum_{i=1}^{N} x_i{}^2 + b\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} x_i y_i = 0 \qquad\qquad m\sum_{i=1}^{N} x_i + \sum_{i=1}^{N} b - \sum_{i=1}^{N} y_i = 0$$

We have found the two simultaneous equations that must be solved to find the best fit line for the empirical data. Next, recall that the average value of a set of values is the sum of the values divided by the number of values. With this we can write the equations in a more concise form by using the following notations.

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \qquad \bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i \qquad \overline{x^2} = \frac{1}{N}\sum_{i=1}^{N} x_i{}^2 \qquad \overline{xy} = \frac{1}{N}\sum_{i=1}^{N} x_i y_i$$

Substituting with the new notation the set of simultaneous linear equations becomes.

$$\overline{xy} = m\overline{x^2} + b\bar{x} \qquad\qquad \bar{y} = m\bar{x} + b$$

Finally, solving for m and b we have

$$m = \left(\frac{\overline{xy} - \bar{x}\bar{y}}{\left(\overline{x^2}\right) - (\bar{x})^2}\right) \qquad\qquad b = \left(\bar{y} - \bar{x}\left(\frac{\overline{xy} - \bar{x}\bar{y}}{\left(\overline{x^2}\right) - (\bar{x})^2}\right)\right)$$

Which can be used to find the best fit line for a particular set of data.

Final Summary for Multivariable Differentiation – Optimization

Critical Points Definition for Two Variable Functions
A point $P = (a, b)$ in the domain of $f(x, y)$ is called a **critical point** if: • $f_x(a, b) = 0$ or $f_x(a, b)$ does not exists, AND • $f_y(a, b) = 0$ or $f_y(a, b)$ does not exists.

Fermat's Theorem of Local Extrema for Two Variable Functions
If $f(a, b)$ is a local minimum or maximum, then $P = (a, b)$ is a critical point of $f(x, y)$. Note: This theorem does not claim that all critical points are local extreme values, but rather that all local extreme values are critical points.

Second Derivative Test for Two Variable Functions
Let $P = (a, b)$ be a critical point of the function, $f(x, y)$ and assume f_{xx}, f_{yy} and f_{xy} are continuous near P. Then: 5. If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum. 6. If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum. 7. If $D < 0$ then $f(a, b)$ is a saddle point. 8. If $D = 0$ then the test is inconclusive. Where D is called the discriminate $D = f_{xx}(a, b) f_{yy}(a, b) - {f_{xy}}^2(a, b)$

Existence and Location of Absolute Extrema
Let $f(x, y)$ be a continuous function on a closed domain D in R^2. Then: 3. $f(x, y)$ takes on both a minimum and maximum value on D. 4. The extreme values occur either at critical points in the interior of D or at points on the boundary of D.

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