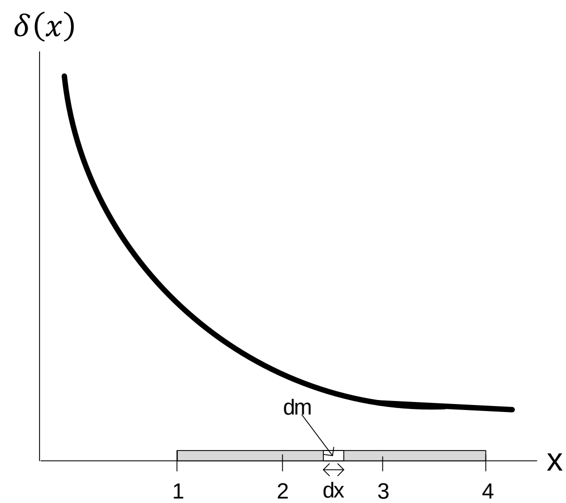
Line and Surface Integrals – Line Integrals

As mentioned, one of the goals of this series of lessons is to extend integration further than what we have done so far. After introducing the concept of a vector field in the previous lesson we are now in a position to begin this quest. In this lesson we start by introducing integrals over curves, i.e. *line integrals*. We'll learn two different types of line integrals; 1.) *scalar line integrals* – integrals of scalar functions along a curve, and 2.) *vector line integrals* – integrals of vector fields along a curve.

Scalar Line Integrals

In an earlier lesson we looked at some applications for multiple integrals. In that lesson, we introduced how we can use the integral of density to compute the "total amount". For example, assume we have a straight rod with a mass density of $\delta(x) = 1/x \;\; kg/m$. We can illustrate the scenario as shown, where the rod lies along the x-axis. The y-axis then represents the density along the length of the rod.

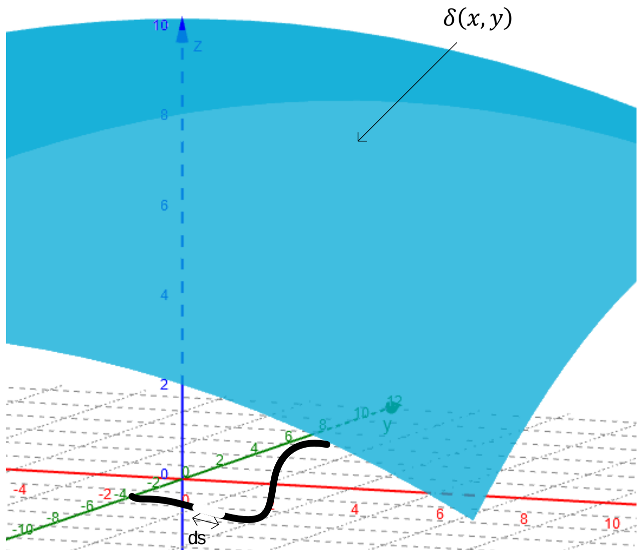


The total mass is calculated by summing the mass elements along the rod using integration.

$$M \;=\; \int_1^4 dm = \int_1^4 \delta(x)dx = \int_1^4 (1/x)dx = ln(4) \cong 1.4\;kg$$

The above integral is an example of a scalar line integral, i.e. we integrated a scalar function, $\delta(x)$, along the a 'curve'. In this case, the 'curve' was a straight line and therefore the integral could be evaluated in one dimension.

On the other hand, if the rod were bent into a curve in three dimensions the density would be function of three dimensions, i.e. $\delta(x, y, z)$ and the infinitesimal lengths along the rod could be indicated as ds. To illustrate we bend the rod in the x-y plane only. The density function, $\delta(x, y)$ can then be envisioned as a surface in three dimensions as shown on the figure below.



In this case, the total mass of the rod can be written as

$$M = \int_C \delta(x,y)ds$$

Where, C represents the path along the rod in the x-y plane, i.e. the curve, and ds is an infinitesimal section along the length of the rod. Next, we write a vector parameterization of the curve, $\boldsymbol{r}(t) = \langle x(t), y(t) \rangle$. In one of our earlier lessons, '*Arc Length and Speed*', we showed the arc length for a parametric curve is given as

$$ds = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2} dt = \|\boldsymbol{r}'(t)\| dt$$

With this we can rewrite the expression for the total mass from above as

$$M = \int_C \delta\left(x(t), y(t)\right) ds = \int_a^b \delta\left(\boldsymbol{r}(t)\right)\|\boldsymbol{r}'(t)\| dt$$

Generalizing this concept, we define the scalar line integral as shown.

Scalar Line Integral
The scalar line integral of the function $f(x,y,z)$ over the curve, C, is given as $$\int_C f(x,y,z)ds$$ Let $\boldsymbol{r}(t)$ be a parameterization of a curve, C, for $a \leq t \leq b$, the scalar line integral can then be written as $$\int_C f(x,y,z)ds = \int_a^b f\left(\boldsymbol{r}(t)\right)\|\boldsymbol{r}'(t)\| dt$$

Example 1: Find the total mass of a wire in the shape of a parabola $y = x^2$ for $1 \leq x \leq 4$ with the mass density given by $\delta(x, y) = y/x \;\; g/cm$.

Solution: The total mass is given by the following scalar line integral.

$$M = \int_C \delta(x, y, z) ds = \int_a^b \delta(\boldsymbol{r}(t)) \|\boldsymbol{r}'(t)\| dt$$

We start by parameterizing the curve, letting $t = x, \;\therefore\; y = t^2$.

$$\boldsymbol{r}(t) = \langle t, t^2 \rangle, \qquad 1 \leq t \leq 4$$

Therefore,

$$\|\boldsymbol{r}'(t)\| = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2} = \sqrt{1 + 4t^2}$$

Next, we evaluate the density function along the curve

$$\delta\left(x(t), y(t)\right) = \frac{t^2}{t} = t$$

Substituting into the line integral we have

$$\int_a^b \delta(\boldsymbol{r}(t)) \|\boldsymbol{r}'(t)\| dt = \int_1^4 t\sqrt{1 + 4t^2} dt$$

Which can be evaluated with the following substation

$$u = 1 + 4t^2 \qquad\qquad du = 8tdt$$

$$\int_1^4 t\sqrt{1 + 4t^2} dt = \frac{1}{8}\int_5^{65} u^{1/2} dt$$

$$= \frac{1}{8}\int_1^4 u^{1/2} dt$$

$$= \frac{1}{12}\left(65^{3/2} - 5^{3/2}\right)$$

$$M \cong 42.74 \; g$$

Example 2: The scalar electric potential, V, at a point, P, in space from a point change, Q, is given as

$$V(P) = k\frac{Q}{r}$$

Where, r is the distance between the charge and the point P, and $k = 8.99E^9$, is the Coulomb constant.

If the charge is instead distributed along a curve with a given change density, we can find the electric potential at a point P using a line integral as follows:

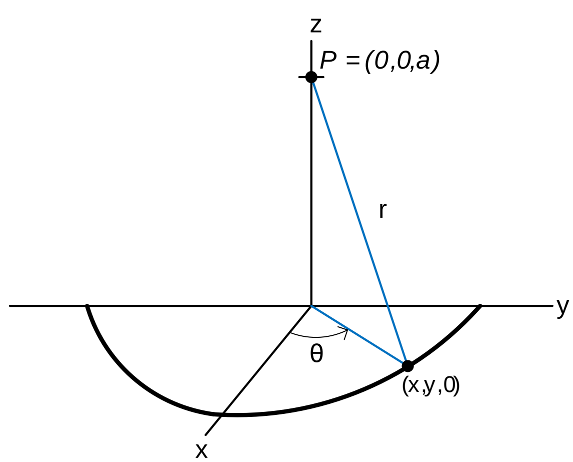
$$V(P) = \int_C k\frac{\delta(x,y,z)}{r}ds$$

Where, P is the point in space where the potential is measured, C represents the curve where the charge lies, $\delta(x,y,z)$ is the charge density in coulombs per length, r denotes the distance from the point (x,y,z) on the curve and the point P.

If a charged semicircle of radius $R = 0.1\ m$ centered at the origin in the x-y plane has a charge density

$$\delta(x,y,0) = 10^{-8}\left(2 - \frac{x}{R}\right)\ C/m$$

Find the electric potential at the point $P = (0,0,a)$.



Solution: In this case, we can parameterize the curve using the polar angle, θ.

$$r(\theta) = \langle R\,cos(\theta)\,, R\,sin(\theta)\rangle, \qquad -\pi/2 \le \theta \le \pi/2$$

Therefore,

$$\|\boldsymbol{r}'(\theta)\| = \sqrt{R^2\,sin^2(\theta) + R^2\,cos^2(\theta)} = R$$

Next, we write the charge density as a function of the parameter, θ.

$$\delta\big(r(\theta)\big) = 10^{-8}\left(2 - \frac{\cancel{R}\,cos(\theta)}{\cancel{R}}\right) = 10^{-8}(2 - cos(\theta))$$

The distance, r, from a point on the wire to the point P, is giving by the Pythagorean Theorem.

$$r = \sqrt{R^2 + a^2}$$

Substituting into the line integral we have

$$V(P) = \int_C k\frac{\delta(x,y,z)}{r}ds$$

$$= \int_{-\pi/2}^{\pi/2} k\frac{\delta\big(r(\theta)\big)}{r}\|\mathbf{r}'(\theta)\|d\theta$$

$$= \frac{k\cdot 10^{-8}\cdot R}{\sqrt{R^2+a^2}}\int_{-\pi/2}^{\pi/2}(2 - cos(\theta))d\theta$$

$$= \frac{k\cdot 10^{-8}\cdot R}{\sqrt{R^2+a^2}}\cdot(2\theta - sin(\theta))\Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{k\cdot 10^{-8}\cdot R\cdot(2\pi - 2)}{\sqrt{R^2+a^2}}$$

$$= \frac{8.9\cdot(2\pi - 2)}{\sqrt{0.01+a^2}} \cong \frac{38.5}{\sqrt{0.01+a^2}}\ volts$$

Example 3: Compute the scalar line integral, $\int_C xe^{z^2}ds$, where the curve is the piecewise linear path from $A = (1,0,1)$ to $B = (0,2,0)$ to $C = (1,1,1)$.

Solution: We can start by parameterizing the two lines.

$$\boldsymbol{r}_{AB}(t) = \langle 1,0,1\rangle + (\langle 0,2,0\rangle - \langle 1,0,1\rangle)t = \langle 1,0,1\rangle + \langle -1,2,-1\rangle t, \qquad 0 \le t \le 1$$

$$\boldsymbol{r}_{BC}(t) = \langle 0,2,0\rangle + (\langle 1,1,1\rangle - \langle 0,2,0\rangle)t = \langle 0,2,0\rangle + \langle 1,-1,1\rangle t, \qquad 0 \le t \le 1$$

Therefore,

$$\|\boldsymbol{r}_{AB}'(t)\| = \sqrt{-1^2 + 2^2 + -1^2} = \sqrt{6} = 2\sqrt{2}$$

$$\|\boldsymbol{r}_{BC}'(t)\| = \sqrt{1^2 + -1^2 + 1^2} = \sqrt{3}$$

$$\int_C xe^{z^2}ds = \int_0^1 f(\boldsymbol{r}_{AB}(t))\|\boldsymbol{r}_{AB}'(t)\|dt + \int_0^1 f(\boldsymbol{r}_{BC}(t))\|\boldsymbol{r}_{BC}'(t)\|dt$$

$$= 2\sqrt{2}\int_0^1 (1-t)e^{(1-t)^2}dt + \sqrt{3}\int_0^1 (t)e^{(t)^2}dt$$

The integrals can be evaluated using similar substitutions

First Integral Substitution

$$u = (1-t)^2$$

$$du = -2(1-t)dt$$

Second Integral Substitution

$$u = t^2$$

$$du = 2tdt$$

$$= 2\sqrt{2}\int_0^1 (1-t)e^{(1-t)^2}dt + \sqrt{3}\int_0^1 (t)e^{(t)^2}dt = -\frac{2\sqrt{2}}{2}\int_1^0 e^u du + \frac{\sqrt{3}}{2}\int_0^1 e^u du$$

$$= \frac{2\sqrt{2}}{2}\int_0^1 e^u du + \frac{\sqrt{3}}{2}\int_0^1 e^u du$$

$$= \left(\frac{2\sqrt{2}+\sqrt{3}}{2}\right)\int_0^1 e^u du$$

$$= \left(\frac{2\sqrt{2}+\sqrt{3}}{2}\right)(e^1 - 1) \cong 3.9$$

Example 4: Calculate the mass of a metal tube in the helical shape $\boldsymbol{r}(t) = \langle cos(t), sin(t), t^2 \rangle$ for $0 \leq t \leq 2\pi$ if the mass density is $\delta(x, y, z) = \sqrt{z}\ g/cm$.

Solution: The curve is given in parametrized form and therefore we can start by computing ds.

$$ds = \|\boldsymbol{r}'(t)\|dt = \left(\sqrt{sin^2(t) + cos^2(t) + 4t^2}\right)dt = \left(\sqrt{1 + 4t^2}\right)dt$$

The total mass is then given by the following scalar line integral.

$$M = \int_C \delta(x, y, z)ds = \int_0^{2\pi} \delta(\boldsymbol{r}(t))\|\boldsymbol{r}'(t)\|dt$$

$$= \int_0^{2\pi} t\sqrt{1 + 4t^2}dt$$

The integral can be evaluated using following substitution.

$$u = 1 + 4t^2 \qquad du = 8tdt$$

$$\int_0^{2\pi} t\sqrt{1 + 4t^2}dt = \frac{1}{8}\int_1^{1+16\pi^2} u^{1/2}du$$

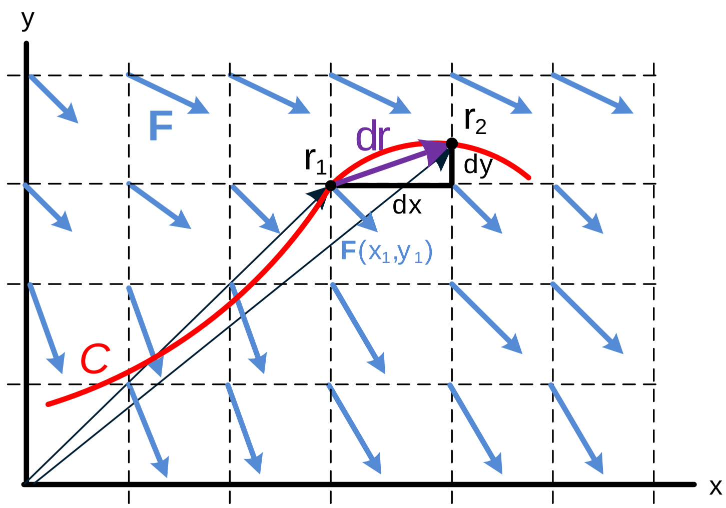
$$= \frac{1}{12}\left((1 + 16\pi^2)^{3/2} - 1\right)$$

$$M \cong 166.86\ g$$

Vector Line Integrals

As you may have imagined, the vector line integral is a similar concept to the scalar line integral. The difference being that for a scalar line integral we integrate a *scalar function* along a *curve* and for a vector line integral we integrate a *vector field* along a *path*. We emphasize the word *curve* versus *path* because with a vector line integral the direction we move along the curve it vitally important. We'll illustrate through an example application.

Recall that in physics the work done by a force is equal to the dot product of the force vector and the distance vector that points in the direction of movement, i.e. $W = \boldsymbol{F} \cdot \boldsymbol{r}$. As usual for cases where one or more of these quantities are not constant, we use infinitesimals. The figure below illustrates this concept in two dimensions.



We assume a particle is moving along the curve, C, in the presence of a vector force field, $\boldsymbol{F}$. The work done by the force as the particle moves an infinitesimal distance, $\|d\boldsymbol{r}\|$, is given as

$$dW = \boldsymbol{F}(x, y) \cdot d\boldsymbol{r}.$$

The vector, $d\boldsymbol{r}$, can be written component-wise as $d\boldsymbol{r} = \langle dx, dy \rangle$. Multiplying by dt/dt we can write $d\boldsymbol{r}$ as

$$d\boldsymbol{r} = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle \, dt = \boldsymbol{r}'(t)dt$$

Therefore, we can write the work as

$$dW = \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)dt$$

Finally, we integrate to find the total work.

$$W = \int_a^b \left(\boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) \right) dt$$

Furthermore, if we carry out the dot product symbolically, the integral can also be written as

$$W = \int_a^b \left(F_1(\boldsymbol{r}(t)) \frac{dx}{dt} + F_2(\boldsymbol{r}(t)) \frac{dy}{dt} \right) dt$$

Vector Line Integral
The vector line integral of the vector field $\boldsymbol{f}(x, y, z)$ over the curve, C, is given as $$\int_C \boldsymbol{f}(x, y, z) \cdot d\boldsymbol{r}$$ Let $\boldsymbol{r}(t)$ be a parameterization of a curve, C, for $a \leq t \leq b$, then the vector line integral is also given as by the two equivalent expressions $$\int_a^b \left(\boldsymbol{f}\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{r}'(t)\right) dt = \int_a^b \left(f_1\big(\boldsymbol{r}(t)\big)\frac{dx}{dt} + f_2\big(\boldsymbol{r}(t)\big)\frac{dy}{dt} + f_3\big(\boldsymbol{r}(t)\big)\frac{dz}{dt}\right) dt$$

Example 5: Evaluate the vector line integral of $\boldsymbol{f}(x, y, z) = \langle z, y^2, x\rangle$ over the curve, C, parameterized by $\boldsymbol{r}(t) = \langle t + 1, e^t, t^2\rangle, 0 \leq t \leq 2$**.**

Solution: We'll compute the line integral using the following formula.

$$\int_0^2 \left(\boldsymbol{f}\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{r}'(t)\right) dt$$

The two vectors are given as

$$\boldsymbol{f}\big(\boldsymbol{r}(t)\big) = \langle t^2, e^{2t}, t + 1\rangle \qquad \boldsymbol{r}'(t) = \langle 1, e^t, 2t\rangle$$

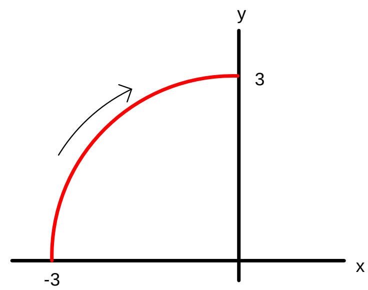
Therefore,

$$\begin{aligned}
\int_0^2 \left(\boldsymbol{f}\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{r}'(t)\right) dt &= \int_0^2 (\langle t^2, e^{2t}, t + 1\rangle \cdot \langle 1, e^t, 2t\rangle) dt \\
&= \int_0^2 (e^{3t} + 3t^2 + 2t) dt \\
&= \frac{1}{3}e^{3t} + t^3 + t^2 \Big|_0^2 \\
&= \left(\frac{1}{3}e^6 + 12\right) - \left(\frac{1}{3}\right) \\
&= \frac{e^6 + 35}{3}
\end{aligned}$$

Example 6: Evaluate the vector line integral of $\boldsymbol{f}(x,y) = \langle x^2, xy \rangle$ over part of a circle $x^2 + y^2 = 9$ with $x \leq 0, y \geq 0$, oriented clockwise.

Solution: The path is a quarter circle of radius 3 in the second quadrant as shown below. Note the direction of the curve is shown as clockwise as specified. The path is parameterized as

$$r(\theta) = \langle 3\,cos(\theta)\,, 3\,sin(\theta) \rangle, \;\; \theta\; goes\; from\; \pi\; to\; \pi/2$$



Therefore

$$\int_{\pi}^{\pi/2} \left(\boldsymbol{f}\big(\boldsymbol{r}(\theta)\big) \cdot \boldsymbol{r}'(\theta)\right) d\theta = \int_{\pi}^{\pi/2} (\langle 9\,cos^2(\theta)\,, 9\,sin(\theta)\,cos(\theta)\,\rangle \cdot \langle -3\,sin(\theta)\,, 3\,cos(\theta) \rangle) d\theta$$

$$= \int_{\pi}^{\pi/2} (-27\,sin(\theta)\,cos^2(\theta) + 27\,sin(\theta)\,cos^2(\theta)) d\theta$$

$$= 0$$

Example 7: Evaluate the vector line integral of $\boldsymbol{f}(x,y) = \langle x^2, xy \rangle$ over the line segment from $(0,0)$ to $(2,2)$.

Solution: The line segment is parametrized as

$$\boldsymbol{r}(t) = \langle 0,0 \rangle + (\langle 2,2 \rangle - \langle 0,0 \rangle)t = \langle 2,2 \rangle t, \qquad 0 \leq t \leq 1$$

Therefore,

$$\int_{0}^{1} \left(\boldsymbol{f}\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{r}'(t)\right) dt = \int_{0}^{1} (\langle 4t^2, 4t^2 \rangle \cdot \langle 2,2 \rangle) dt$$

$$= \int_{0}^{1} 16t^2 dt$$

$$= \frac{16}{3}$$

Example 8: Evaluate the vector line integral of $\boldsymbol{f}(x,y,z) = \langle e^z, e^{x-y}, e^y \rangle$ over a piecewise linear path from $A = (0,0,0)$ to $B = (0,0,1)$ to $C = (0,1,1)$ to $D = (-1,1,1)$.

Solution: We'll start by parameterizing the three lines.

$$\boldsymbol{r}_{AB}(t) = \langle 0,0,0\rangle + (\langle 0,0,1\rangle - \langle 0,0,0\rangle)t = \langle 0,0,1\rangle t \qquad 0 \le t \le 1$$

$$\boldsymbol{r}_{BC}(t) = \langle 0,0,1\rangle + (\langle 0,1,1\rangle - \langle 0,0,1\rangle)t = \langle 0,0,1\rangle + \langle 0,1,0\rangle t \qquad 0 \le t \le 1$$

$$\boldsymbol{r}_{CD}(t) = \langle 0,1,1\rangle + (\langle -1,1,1\rangle - \langle 0,1,1\rangle)t = \langle 0,1,1\rangle + \langle -1,0,0\rangle t \qquad 0 \le t \le 1$$

Let's take one integral at a time

1. $A \to B$

$$\int_0^1 \left(\boldsymbol{f}\left(\boldsymbol{r}_{AB}{}'(t)\right) \cdot \boldsymbol{r}_{AB}{}'(t)\right) dt = \int_0^1 (\langle e^t, 1, 1\rangle \cdot \langle 0,0,1\rangle) dt$$
$$= \int_0^1 1 dt = 1$$

2. $B \to C$

$$\int_0^1 \left(\boldsymbol{f}\left(\boldsymbol{r}_{BC}{}'(t)\right) \cdot \boldsymbol{r}_{BC}{}'(t)\right) dt = \int_0^1 (\langle e^1, e^{-t}, e^t\rangle \cdot \langle 0,1,0\rangle) dt$$
$$= \int_0^1 e^{-t} dt = \left(1 - \frac{1}{e^1}\right)$$

3. $C \to D$

$$\int_0^1 \left(\boldsymbol{f}\left(\boldsymbol{r}_{CD}{}'(t)\right) \cdot \boldsymbol{r}_{CD}{}'(t)\right) dt = \int_0^1 (\langle e^1, e^{1-t}, e^1\rangle \cdot \langle -1,0,0\rangle) dt$$
$$= \int_0^1 -e^1 dt = -e^1$$

Summing the results, we have

$$\int_C \boldsymbol{f}(x,y,z) \cdot d\boldsymbol{r} = 1 + \left(1 - \frac{1}{e^1}\right) + -e^1 = \left(2 - \frac{1}{e^1} - e^1\right)$$

Example 9: Calculate the work performed in moving a particle along the path, $\boldsymbol{r}(t) = \langle t^2, t^3, t\rangle \ meters, \ for\ 1 \leq t \leq 2$, in the presence of a force field

$$\boldsymbol{F} = \langle x^2, -z, -yz^{-1}\rangle\ Newtons$$

Solution: The work done against a force field is the negative of the vector line integral.

$$W = -\int_1^2 \Big(\boldsymbol{F}\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{r}'(t)\Big)\,dt$$

$$= -\int_1^2 (\langle t^4, -t, -t^2\rangle \cdot \langle 2t, 3t^2, 1\rangle)dt$$

$$= -\int_1^2 (2t^5 - 3t^3 \ \ - t^2)dt$$

$$= -\left(\frac{1}{3}t^6 - \frac{3}{4}t^4 \ \ - \frac{1}{3}t^3\Big|_1^2\right)$$

$$= \frac{89}{12}\,Joules$$

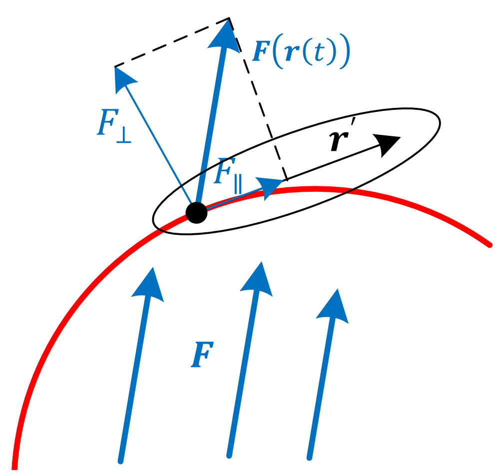
Flux Across a Curve

When a particle moves in the presence of a vector force field, the work done on the particle is computed by integrating the infinitesimal work elements, dW, over the path of the particle.

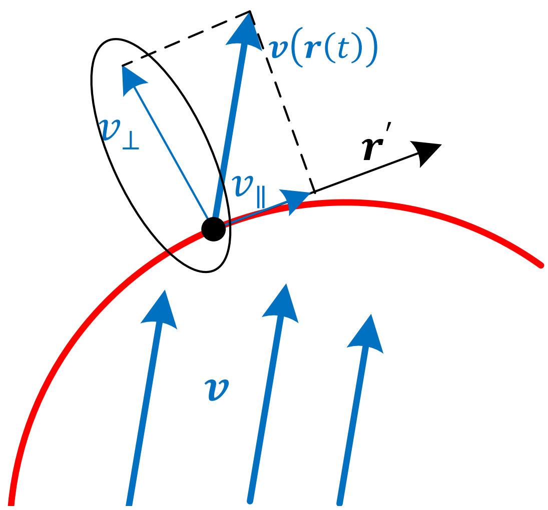
$$dW = \boldsymbol{F}\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{r}'(t)$$

In other words, we use the component of the force vector that is *tangential* to the path, $\boldsymbol{r}'(t)$.

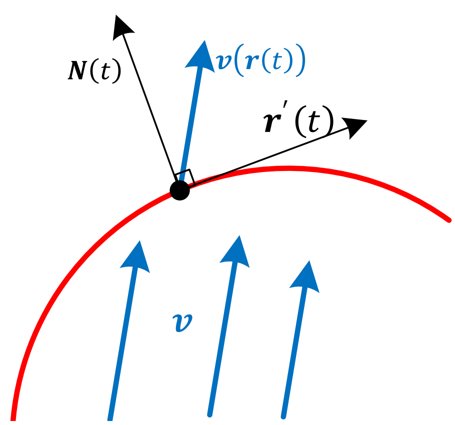
$$dW = \boldsymbol{F}\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{r}'(t) = \big\|\boldsymbol{F}\big(\boldsymbol{r}(t)\big)\big\|\|\boldsymbol{r}'(t)\|\cos(\theta) = F_{\parallel}\big(\boldsymbol{r}(t)\big)\|\boldsymbol{r}'(t)\|$$



Next, we can ask: "Is there any significance to using the *normal* component, $F_\perp\big(\boldsymbol{r}(t)\big)$ instead?". The answer is *yes*, and the quantity computed is referred to as the *flux across a curve*. The term flux is generally used to describe the flow of a certain substance through a surface, or in this case across a curve. For example, if the vector field represents a velocity of a liquid the vector line integral using the normal component is a measure of how much liquid is flowing across the curve.



Before we look at an example let's derive the expression used to compute this measure. Computationally, instead of using the normal component of the vector field we use a normal vector relative to the path, i.e. the normal vector that corresponds to $\boldsymbol{r}'(t)$.



Using the figure, the infinitesimal flux element, $d\Phi$, is computed as

$$d\Phi = \boldsymbol{v}\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{N}(t)$$

In two dimensions we can show that $\boldsymbol{N}(t)$ is related $\boldsymbol{r}'(t)$ as follows:

$$If\ \ \boldsymbol{r}'(t) = \langle x'(t), y'(t)\rangle, \qquad then, \qquad \boldsymbol{N}(t) = \langle y'(t), -x'(t)\rangle$$

We prove this by showing the dot product is equal to zero.

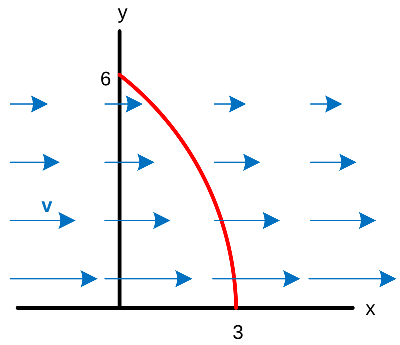
$$\boldsymbol{r}'(t)\cdot\boldsymbol{N}(t) = \langle x'(t), y'(t)\rangle\cdot\langle y'(t), -x'(t)\rangle = \big(x'(t)y'(t) - y'(t)x'(t)\big) = 0$$

Therefore, we can state the following two results.

Work done by a Vector Force Field
The work done on a particle moving along curve parameterized by $\boldsymbol{r}(t)$ in the presence of a vector force field, $\boldsymbol{F}$, is given as $$W = \int_a^b \Big(\boldsymbol{F}\big(\boldsymbol{r}(t)\big)\cdot\boldsymbol{r}'(t)\Big)\,dt$$

Flux Across a Plane Curve
The flux across a plane curve parameterized by $\boldsymbol{r}(t)$ in the presence of a vector field, $\boldsymbol{v}$, is given as $$\Phi = \int_a^b \Big(\boldsymbol{v}\big(\boldsymbol{r}(t)\big)\cdot\boldsymbol{N}(t)\Big)\,dt$$ Where, $\boldsymbol{N}(t) = \langle y'(t), -x'(t)\rangle$ and $\boldsymbol{r}'(t) = \langle x'(t), y'(t)\rangle$

Example 10: Calculate the flux of the velocity vector field, $\boldsymbol{v}(t) = \langle 3 + 2y - y^2/3 \, , 0 \rangle \ cm/s$ across the quarter of the ellipse $\boldsymbol{r}(t) = \langle 3\,cos(t)\,, 6\,sin(t) \rangle$ for $0 \leq t \leq \pi/2$.



Solution: We can start by computing $\boldsymbol{N}(t)$

$$\boldsymbol{r}'(t) = \langle -3\,sin(t)\,, 6\,cos(t) \rangle \rightarrow \boldsymbol{N}(t) = \langle 6\,cos(t)\,, 3\,sin(t) \rangle$$

Furthermore,

$$\boldsymbol{v}\big(\boldsymbol{r}(t)\big) = \langle 3 + 12\,sin(t) - 12\,sin^2(t)\,, 0 \rangle$$

The flux is then computed as follows

$$\Phi = \int_a^b \Big(\boldsymbol{v}\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{N}(t)\Big)\,dt$$

$$= \int_0^{\pi/2} (\langle 3 + 12\,sin(t) - 12\,sin^2(t)\,, 0 \rangle \cdot \langle 6\,cos(t)\,, 3\,sin(t) \rangle)dt$$

$$= \int_0^{\pi/2} (18\,cos(t) + 72\,sin(t)\,cos(t) - 72\,sin^2(t)\,cos(t))dt$$

$$= 18\int_0^{\pi/2} cos(t)\,dt + 36\int_0^{\pi/2} sin(2t)\,dt - 72\int_0^{1} u^2 du$$

$$= 18(sin(\pi/2) - sin(0)) \pm 18(cos(\pi) - cos(\pi)) - 12(1/3)$$

$$= 18 + 36 - 24$$

$$= 30\ cm^2/s$$

Final Summary for Line and Surface Integrals – Line Integrals

Scalar Line Integral

The scalar line integral of the function $f(x, y, z)$ over the curve, C, is given as

$$\int_C f(x, y, z)ds$$

Let $\boldsymbol{r}(t)$ be a parameterization of a curve, C, for $a \leq t \leq b$, then the scalar line integral is also given as

$$\int_C f(x, y, z)ds = \int_a^b f(\boldsymbol{r}(t))\|\boldsymbol{r}'(t)\|dt$$

Vector Line Integral

The vector line integral of the vector field $\boldsymbol{f}(x, y, z)$ over the curve, C, is given as

$$\int_C \boldsymbol{f}(x, y, z) \cdot d\boldsymbol{r}$$

Let $\boldsymbol{r}(t)$ be a parameterization of a curve, C, for $a \leq t \leq b$, then the vector line integral is also given as by the two equivalent expressions

$$\int_a^b \left(\boldsymbol{f}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\right) dt = \int_a^b \left(f_1(\boldsymbol{r}(t))\frac{dx}{dt} + f_2(\boldsymbol{r}(t))\frac{dy}{dt} + f_3(\boldsymbol{r}(t))\frac{dz}{dt}\right) dt$$

Work done by a Vector Force Field

The work done on a particle moving along curve parameterized by $\boldsymbol{r}(t)$ in the presence of a vector force field, $\boldsymbol{F}$, is given as

$$W = \int_a^b \left(\boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\right) dt$$

Flux Across a Plane Curve

The flux across a plane curve parameterized by $\boldsymbol{r}(t)$ in the presence of a vector field, $\boldsymbol{v}$, is given as

$$\Phi = \int_a^b \left(\boldsymbol{v}(\boldsymbol{r}(t)) \cdot \boldsymbol{N}(t)\right) dt$$

Where, $\boldsymbol{N}(t) = \langle y'(t), -x'(t)\rangle$ and $\boldsymbol{r}'(t) = \langle x'(t), y'(t)\rangle$

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