Vector Calculus – Motion in 3-Space

Using what we have learned in the previous sections we now study more closely the motion of objects that are traveling along a path, $\boldsymbol{r}(t)$. We'll see how the familiar terms associated with motion, i.e. position, velocity, and acceleration, are related the more formal terms, e.g. unit tangent and unit normal vector, that we have seen in previous lessons.

Position, Velocity, and Acceleration

The position of an object in motion can be represented by the vector function, $\boldsymbol{r}(t)$. The rate of change of the position with respect to time is called the velocity, $\boldsymbol{v}(t) = \boldsymbol{r}'(t)$. This vector points in the direction of motion and its magnitude, $\|\boldsymbol{v}(t)\| = v(t)$, is the particles speed. Furthermore, the acceleration vector, the rate of change of the velocity with respect to time is denoted as $\boldsymbol{a}(t) = \boldsymbol{v}(t) = \boldsymbol{r}''(t)$.

Motion Describing Quantities				
$\boldsymbol{r}(t)$	:	Position Vector – Represents the Position of an Object	:	$\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$
$\boldsymbol{v}(t)$	:	Velocity Vector – Rate of change of Position	:	$\boldsymbol{v}(t) = \boldsymbol{r}'(t)$.
$v(t)$	:	Speed – Magnitude of Velocity	:	$v(t) = \|\boldsymbol{v}(t)\|$
$\boldsymbol{a}(t)$	:	Acceleration Vector - Rate of change of Velocity	:	$\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \boldsymbol{r}''(t)$

The next example illustrates how, given the position function, we can compute all other motion describing quantities.

Example 1: Compute the velocity vector, acceleration vector, and speed at $t = 1$ for an object whose position is described by the following vector function.

$$\boldsymbol{r}(t) = \langle sin(2t)\,, -\,cos(2t)\,, \sqrt{t+1}\rangle$$

Solution: The general functions are as follows:

$$\boldsymbol{v}(t) = \boldsymbol{r}'(t) = \langle 2\,cos(2t)\,, 2sin(2t), -\frac{1}{2\sqrt{t+1}}\rangle$$

$$\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \langle -4\,sin(2t)\,, 4\,cos(2t)\,, -\frac{1}{4(t+1)^{3/2}}\rangle$$

$$v(t) = \|\boldsymbol{v}(t)\| = \sqrt{4\,cos(2t) + 4\,sin(2t) + \frac{1}{4t+4}} = \sqrt{4 + \frac{1}{4t+4}} = \sqrt{\frac{16t+17}{4t+4}}$$

The functions evaluated at $t = 1$ are

$$\boldsymbol{v}(1) = \langle 2\,cos(2)\,, 2sin(2), -\frac{1}{2\sqrt{1+1}} \rangle \cong \langle -0.83, 1.82, -0.354 \rangle$$

$$\boldsymbol{a}(1) = \langle -4\,sin(2)\,, 4\,cos(2)\,, \frac{1}{4(1+1)^{3/2}} \rangle \cong \langle -3.64, -1.66, 0.088 \rangle$$

$$v(1) = \sqrt{\frac{16+17}{4+4}} \cong 2.03$$

We can also find the other motion describing quantities if only the acceleration is known. However, in this case we also need initial conditions to obtain a complete description. The next example illustrates this.

Example 2: Given the following acceleration vector and initial conditions, compute the velocity vector, position vector, and speed.

$$\boldsymbol{a}(t) = \langle 4, 2t, t+1 \rangle \qquad \boldsymbol{v}(0) = \langle 2, 1, 0 \rangle \qquad \boldsymbol{r}(0) = \langle 2, 3, 4 \rangle$$

Solution: By the Fundamental Theorem of Calculus, since $\boldsymbol{a}(t) = \boldsymbol{v}'(t)$ and $\boldsymbol{v}(t) = \boldsymbol{r}'(t)$, we know that

$$\boldsymbol{v}(t) = \int \boldsymbol{a}(t)dt \qquad \boldsymbol{r}(t) = \int \boldsymbol{v}(t)dt$$

Therefore, we have

$$\begin{aligned} \boldsymbol{v}(t) &= \int \boldsymbol{a}(t)dt \\ &= \int \langle 4, 2t, t+1 \rangle dt \\ &= \langle 4t, t^2, \frac{1}{2}t^2 + t \rangle + \boldsymbol{v}(0) \\ &= \langle 4t, t^2, \frac{1}{2}t^2 + t \rangle + \langle 2, 1, 0 \rangle \\ &= \langle 4t+2, t^2+1, \frac{1}{2}t^2 + t \rangle \end{aligned}$$

$$\begin{aligned} \boldsymbol{r}(t) &= \int \boldsymbol{v}(t)dt \\ &= \int \langle 4t+2, t^2+1, \frac{1}{2}t^2 + t \rangle\, dt \\ &= \langle 2t^2 + 2t, \frac{1}{3}t^3 + t, \frac{1}{6}t^3 + \frac{1}{2}t^2 \rangle + \boldsymbol{r}(0) \\ &= \langle 2t^2 + 2t, \frac{1}{3}t^3 + t, \frac{1}{6}t^3 + \frac{1}{2}t^2 \rangle + \langle 2, 3, 4 \rangle \\ &= \langle 2t^2 + 2t + 2, \frac{1}{3}t^3 + t + 3, \frac{1}{6}t^3 + \frac{1}{2}t^2 + 4 \rangle \end{aligned}$$

Lastly, the speed is

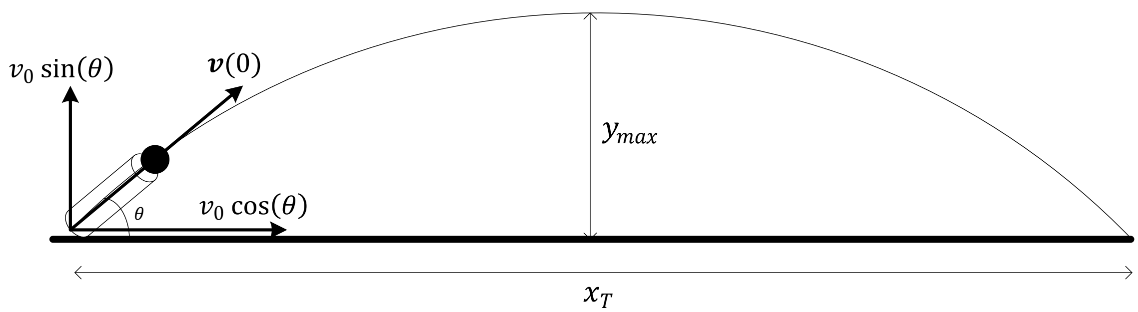
$$v(t) = \sqrt{(4t+2)^2 + \left(t^2+1\right)^2 + \left(\frac{1}{2}t^2 + t\right)^2}$$

Projectile motion

One interesting and useful application of the above concepts is projectile motion, i.e. motion of an object traveling in space that is subject to acceleration due to gravity only. A cannon fired from the ground is one example. Near the surface of the earth the gravitational force imparts an acceleration of $g = 9.8\ m/s^2$ in the negative z-direction irrespective of the mass of the object. Based on this knowledge we can predict the path of a projectile given only its initial conditions.

Example 3: A cannon is fired from the ground at an angle of 60° above the horizontal with an initial speed of $v(t) = 50\ m/s$. Find the following information related to the projectile.

1. The velocity and position vector functions for the cannon.
2. Find the location where the cannon lands.
3. Find the maximum height the cannon reaches.



Solution:

1. We assume the projectile stays in a plane. Therefore, the acceleration vector is given as

$$\boldsymbol{a}(t) = \langle 0, -g \rangle$$

To find the velocity and position vector we need initial conditions, which from the figure are

$$\boldsymbol{v}(0) = \langle v_0\, cos(\theta), v_0\, sin(\theta) \rangle \qquad \boldsymbol{r}(0) = \langle 0,0 \rangle$$

The velocity and position vector are then found as follows.

$$\begin{aligned} \boldsymbol{v}(t) &= \int \boldsymbol{a}(t)dt \\ &= \int \langle 0, -g \rangle dt \\ &= \langle 0, -gt \rangle + \langle v_0\, cos(\theta), v_0\, sin(\theta) \rangle \\ &= \langle v_0\, cos(\theta), v_0\, sin(\theta) - gt \rangle \end{aligned}$$

$$\begin{aligned} \boldsymbol{r}(t) &= \int \boldsymbol{v}(t)dt \\ &= \int \langle v_0\, cos(\theta), v_0\, sin(\theta) - gt \rangle dt \\ &= \langle v_0\, cos(\theta)\, t, v_0\, sin(\theta)\, t - \frac{1}{2}gt^2 \rangle + \langle 0,0 \rangle \\ &= \langle v_0\, cos(\theta)\, t, v_0 sin(\theta)\, t - \frac{1}{2}gt^2 \rangle \end{aligned}$$

Using the values given, the velocity and position function are

$$\boldsymbol{v}(t) = \langle 50\, cos(60^\circ), 50\, sin(60^\circ) - 9.8t\rangle \qquad \boldsymbol{r}(t) = \langle 50\, cos(60^\circ)\, t, 50\, sin(60^\circ)\, t - 4.9t^2\rangle$$

$$\boldsymbol{v}(t) = \langle 25, 43.3 - 9.8t\rangle \qquad \boldsymbol{r}(t) = \langle 25t, 43.3t - 4.9t^2\rangle$$

2. To find where the cannon lands we start by finding when the y-component of $\boldsymbol{r}(t)$ is zero. We'll use expressions before we substituted the time value for more generally solutions.

$$v_0\, sin(\theta)\, t - \frac{1}{2} g t^2 = 0$$

$$t\left(v_0\, sin(\theta) - \frac{1}{2} g t\right) = 0$$

Therefore, the cannon lands when

$$t = \frac{2v_0\, sin(\theta)}{g}$$

Substituting this into the x-component we find an expression for the total distance x_T.

$$x_T = v_0\, cos(\theta) \frac{2v_0\, sin(\theta)}{g}$$

$$x_T = \frac{{v_0}^2\, sin(2\theta)}{g}$$

Where, we used the trigonometric identity: $2sin(A)\, cos(A) = sin(2A)$.

Substituting values, we have

$$x_T = \frac{50^2\, sin(120^\circ)}{9.8} \cong 221\, m$$

3. To find the maximum height we find when the y-component of the velocity is equal to zero.

$$v_0\, sin(\theta) - gt = 0$$

$$t = \frac{v_0\, sin(\theta)}{g}$$

Substituting this into the y-component we find an expression for the maximum height y_{max}.

$$y_{max} = v_0 sin(\theta)\left(\frac{v_0\, sin(\theta)}{g}\right) - \frac{1}{2} g \left(\frac{v_0\, sin(\theta)}{g}\right)^2$$

$$y_{max} = \frac{{v_0}^2\, sin^2(\theta)}{2g}$$

Substituting values, we have

$$y_{max} = \frac{50^2\, sin^2(60^\circ)}{19.6} \cong 95.6\, m$$

Uniform Circular Motion and the Acceleration Vector

The acceleration is the rate of change of the velocity. So even if the speed, $\|\boldsymbol{v}(t)\|$, is constant, if the direction of an object is changing the acceleration is non-zero. An example of this is what is called *uniform circular motion*. It describes the scenario when an object travels in a circular path with a constant speed. The next example find the acceleration for this scenario.

Example 4: Find $\boldsymbol{a}(t)$ and $\|\boldsymbol{a}(t)\|$ for an object traveling around a circle with a radius R and a constant speed, v.

Solution: The angular speed, ω, and speed, v, are related as follows:

$$v = R\omega$$

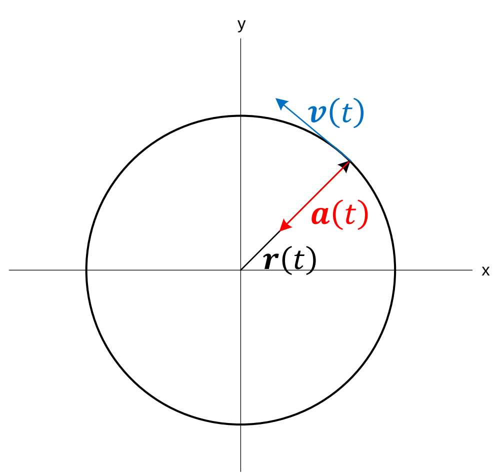
Therefore, the position of a particle traveling along a circle of radius R can be described as

$$\boldsymbol{r}(t) = R\langle cos(\omega t), sin(\omega t)\rangle = R\left\langle cos\left(\frac{v}{R}t\right), sin\left(\frac{v}{R}t\right)\right\rangle$$

The acceleration can then be found as

$$\begin{aligned}\boldsymbol{a}(t) = \boldsymbol{r}''(t) &= \frac{d}{dt}\left(R\left\langle -\frac{v}{R}sin\left(\frac{v}{R}t\right), \frac{v}{R}cos\left(\frac{v}{R}t\right)\right\rangle\right) \\ &= R\left(\left\langle -\frac{v^2}{R^2}cos\left(\frac{v}{R}t\right), -\frac{v^2}{R^2}sin\left(\frac{v}{R}t\right)\right\rangle\right) \\ &= -\frac{v^2}{R}\left(\left\langle cos\left(\frac{v}{R}t\right), sin\left(\frac{v}{R}t\right)\right\rangle\right) \\ &= -\frac{v^2}{R}\boldsymbol{r}(t)\end{aligned}$$

Note the acceleration vector is a scaled version of the negative of the position vector. Therefore, it points consistently toward the center of the circle as the figure above shows. The acceleration vector, $\boldsymbol{a}(t)$, is called the *centripetal acceleration*; meaning "center seeking".



Finally, we take a closer look at the acceleration vector and see how it relates to the unit tangent and unit normal vectors. Recall one of the definitions for the curvature was given as

$$\kappa = \frac{\left\|\boldsymbol{T}'(t)\right\|}{v(t)}$$

And since $\boldsymbol{N}(t) = \frac{\boldsymbol{T}'(t)}{\|\boldsymbol{T}'(t)\|}$, we can write

$$\boldsymbol{T}'(t) = \kappa v(t)\boldsymbol{N}(t)$$

Furthermore, the unit tangent vector is related to the velocity vector as follows.

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} = \frac{\boldsymbol{v}(t)}{\|\boldsymbol{v}(t)\|} = \frac{\boldsymbol{v}(t)}{v(t)}$$

Therefore, we can write

$$\boldsymbol{v}(t) = v(t)\boldsymbol{T}(t)$$

With this we can derive a general expression for the acceleration vector as shown below

$$\boldsymbol{a}(t) = \frac{d}{dt}\big(\boldsymbol{v}(t)\big) = \frac{d}{dt}\big(v(t)\boldsymbol{T}(t)\big) = v'(t)\boldsymbol{T}(t) + v(t)\boldsymbol{T}'(t) = \big(v'(t)\boldsymbol{T}(t) + \kappa v^2(t)\boldsymbol{N}(t)\big)$$

Now, since $\boldsymbol{T}(t)$ is tangential to the curve for all t, and $\boldsymbol{N}(t)$ is normal, (orthogonal), to the curve, we can express the acceleration vector as follows.

$$\boldsymbol{a}(t) = \underbrace{a_T\boldsymbol{T}(t)}_{tangential\ component} + \underbrace{a_N\boldsymbol{N}(t)}_{normal\ component}$$

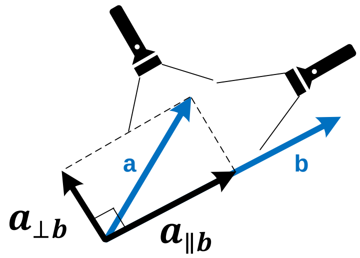
Where, $a_T = v'(t)$, *and* $a_N = \kappa v^2(t)$.

This expression provides insight in that each component describes a different aspect of the motion. Recall, we said that the acceleration is non-zero if *either* the speed *or* direction of motion changes. As we explain below, the tangential component describes the change in direction and the normal component describes the change in speed.

Acceleration Vector Components

The acceleration vector for an object traveling along a path is given as
 	
$$\boldsymbol{a}(t) = a_T\boldsymbol{T}(t) + a_N\boldsymbol{N}(t)$$
 	
Where, $a_T = v'(t)$, *and* $a_N = \kappa v^2(t)$
 	
• ***The Tangential Component "encodes" the change in the speed***
  ○ Since $a_T = v'(t)$ the tangential component is zero if the speed is constant.
• ***The Normal Component "encodes" the change in direction***
  ○ Since $a_N = \kappa v^2(t)$ the normal component is zero if $\kappa = 0$, which is the case when the path does not change direction.

In an earlier lesson we learned that any vector, $\boldsymbol{a}$, can be decomposed into two orthogonal components with respect to another vector $\boldsymbol{b}$; a parallel projection, $\boldsymbol{a}_{\parallel \boldsymbol{b}}$, and a perpendicular projection, $\boldsymbol{a}_{\perp \boldsymbol{b}}$.

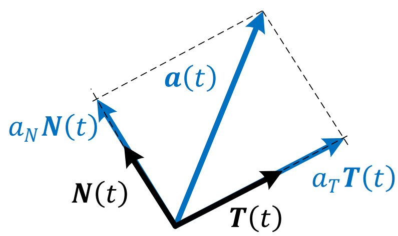


$$\boldsymbol{a} = \boldsymbol{a}_{\parallel \boldsymbol{b}} + \boldsymbol{a}_{\perp \boldsymbol{b}}$$

Where, $\boldsymbol{a}_{\parallel \boldsymbol{b}} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{b}\|} \hat{\boldsymbol{b}}$, and $\boldsymbol{a}_{\perp \boldsymbol{b}} = \boldsymbol{a} - \boldsymbol{a}_{\parallel \boldsymbol{b}}$.

Note the similarity to the acceleration vector we derived above.

$$\boldsymbol{a}(t) = a_T \boldsymbol{T}(t) + a_N \boldsymbol{N}(t)$$



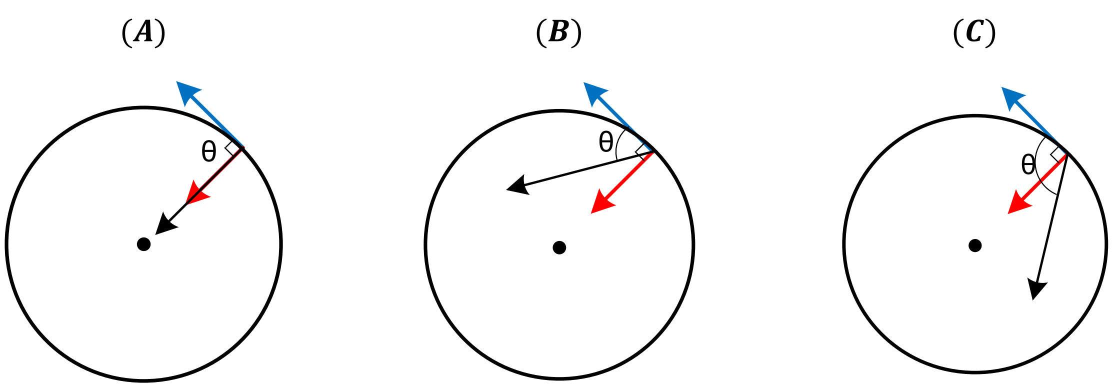
Since both $\boldsymbol{T}(t)$ and $\boldsymbol{N}(t)$ are unit vectors, the coefficients, a_T and a_N, are equal to the magnitude of the two components and can be computed using the dot product as follows

$$a_T = \boldsymbol{a}(t) \cdot \boldsymbol{T}(t) \qquad\qquad a_N = \boldsymbol{a}(t) \cdot \boldsymbol{N}(t)$$

Furthermore, using the fact that $a_T = v'(t)$ we can write

$$v'(t) = a_T = \boldsymbol{a}(t) \cdot \boldsymbol{T}(t) = \|\boldsymbol{a}(t)\| \|\boldsymbol{T}(t)\| \cos(\theta)$$

Which shows that the rate of change of speed depends on the angle between $\boldsymbol{a}(t)$ and $\boldsymbol{T}(t)$. We have three possible scenarios that can be best illustrated using circular motion as shown below.

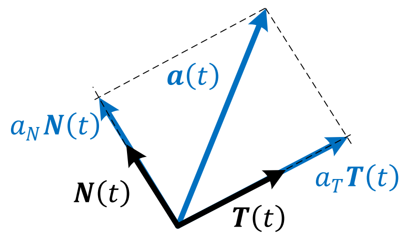


A. $\theta = 90^{\circ}$: Therefore, $cos(\theta) = 0$ and $v'(t) = 0$. The particles *speed is constant*, which results on uniform circular motion as shown in example 4.

B. $\theta < 90^{\circ}$: Therefore, $cos(\theta) > 0$ and $v'(t) > 0$. The particles *speed is increasing*.

C. $90^{\circ} < \theta < 180^{\circ}$: Therefore, $cos(\theta) < 0$ and $v'(t) < 0$. The particles *speed is decreasing*

Lastly, we derive additional formulas for decomposing the acceleration vector.



The magnitude of the tangential component is equal to the dot product of the acceleration vector with the unit tangent vector. The unit tangent vector, as you'll recall, is $\boldsymbol{T}(t) = \frac{\boldsymbol{v}(t)}{v(t)}$. Therefore, we can write

$$a_T\boldsymbol{T}(t) = \big(\boldsymbol{a}(t)\cdot\boldsymbol{T}(t)\big)\boldsymbol{T}(t) = \left(\frac{\boldsymbol{a}(t)\cdot\boldsymbol{v}(t)}{v(t)}\right)\frac{\boldsymbol{v}(t)}{v(t)} = \left(\frac{\boldsymbol{a}(t)\cdot\boldsymbol{v}(t)}{v^2(t)}\right)\boldsymbol{v}(t) = \left(\frac{\boldsymbol{a}(t)\cdot\boldsymbol{v}(t)}{\|\boldsymbol{v}(t)\|^2}\right)\boldsymbol{v}(t)$$

Next, we can use the Pythagorean theorom to write a relationship for the magnitude of the normal component as follows.

$$\|\boldsymbol{a}(t)\|^2 = \|a_T\boldsymbol{T}(t)\|^2 + \|a_N\boldsymbol{N}(t)\|^2 = |a_T|^2 + |a_N|^2$$

Therefore,

$$a_N = \sqrt{\|\boldsymbol{a}(t)\|^2 - |a_T|^2}$$

We can also express the normal vector as follows

$$a_N\boldsymbol{N}(t) = \boldsymbol{a}(t) - a_T\boldsymbol{T}(t) = \boldsymbol{a}(t) - \left(\frac{\boldsymbol{a}(t)\cdot\boldsymbol{v}(t)}{\|\boldsymbol{v}(t)\|^2}\right)\boldsymbol{v}(t)$$

Before summarizing this section let's look at a few more examples.

Example 5: Decompose the acceleration vector, $\boldsymbol{a}(t)$ of $\boldsymbol{r}(t) = \langle t^2, 2t, \ ln(t)\rangle$ into tangential and normal components at $t = \frac{1}{2}$.

Solution: We'll use the equations derived above and shown below.

$$\boldsymbol{a}_{\parallel}(t) = a_T\boldsymbol{T}(t) = \left(\frac{\boldsymbol{a}(t)\cdot\boldsymbol{v}(t)}{\|\boldsymbol{v}(t)\|^2}\right)\boldsymbol{v}(t) \qquad \boldsymbol{a}_{\perp}(t) = a_N\boldsymbol{N}(t) = \boldsymbol{a}(t) - \left(\frac{\boldsymbol{a}(t)\cdot\boldsymbol{v}(t)}{\|\boldsymbol{v}(t)\|^2}\right)\boldsymbol{v}(t)$$

We start by differentiating to find the velocity and acceleration vectors.

$$\boldsymbol{v}(t)|_{t={}^1/_2} = \boldsymbol{r}'(t) = \langle 2t, 2, \frac{1}{t}\rangle = \langle 1, 2{,}2\rangle \qquad \boldsymbol{a}(t)|_{t={}^1/_2} = \boldsymbol{v}'(t) = \langle 2, 0, -\frac{1}{t^2}\rangle = \langle 2, 0, -4\rangle$$

Therefore,

$$\begin{aligned}\boldsymbol{a}_{\parallel}({}^1/_2) &= \left(\frac{\boldsymbol{a}({}^1/_2)\cdot\boldsymbol{v}({}^1/_2)}{\|\boldsymbol{v}({}^1/_2)\|^2}\right)\boldsymbol{v}({}^1/_2) \\ &= \left(\frac{\langle 2, 0, -4\rangle\cdot\langle 1, 2{,}2\rangle}{\langle 1, 2{,}2\rangle\cdot\langle 1, 2{,}2\rangle}\right)\langle 1, 2{,}2\rangle \\ &= \left(\frac{-6}{9}\right)\langle 1, 2{,}2\rangle = \langle\frac{-2}{3}, \frac{-4}{3}, \frac{-4}{3}\rangle\end{aligned}$$

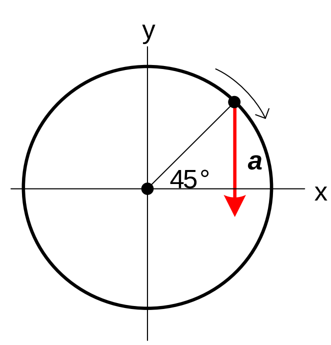
$$\begin{aligned}\boldsymbol{a}_{\perp}({}^1/_2) &= \boldsymbol{a}({}^1/_2) - \boldsymbol{a}_{\parallel}({}^1/_2) \\ &= \langle 2, 0, -4\rangle - \langle\frac{-2}{3}, \frac{-4}{3}, \frac{-4}{3}\rangle \\ &= \langle 2, 0, -4\rangle - \langle\frac{-2}{3}, \frac{-4}{3}, \frac{-4}{3}\rangle \\ &= \langle\frac{8}{3}, \frac{4}{3}, \frac{-8}{3}\rangle\end{aligned}$$

Lastly, we show that the vectors found do indeed add to the acceleration vector.

$$\boldsymbol{a} = \boldsymbol{a}_{\parallel} + \boldsymbol{a}_{\perp}$$

$$\boldsymbol{a} = \langle\frac{-2}{3}, \frac{-4}{3}, \frac{-4}{3}\rangle + \langle 8, \frac{4}{3}, \frac{-8}{3}\rangle = \langle 2, 0, -4\rangle$$

Example 6: Suppose a particle is moving on a circular path of $R = 30\ m$ with an acceleration of $\langle 0, -50\rangle\ {}^{m}/_{s^2}$ at the point shown in the figure below. Determine the speed and the tangential acceleration of the particle.

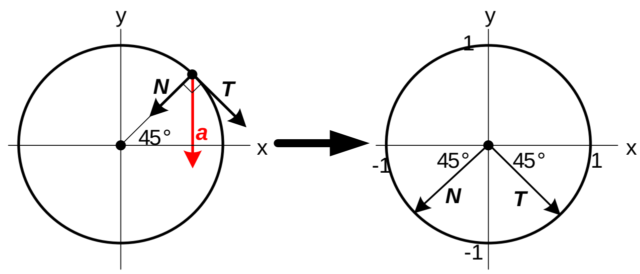


Solution: The acceleration vector is given by

$$\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$$

Where, $a_T = v'(t)$, *and* $a_N = \kappa v^2(t)$

The unit tangent and unit normal vector at the point given can be easily found by considering a unit circle as shown below. The first figure show the location of the unit tangent and unit normal vector on the particles path, while the second figure shows these vectors translated to a unit circle.



Therefore, we can write the following.

$$\boldsymbol{N} = \frac{\sqrt{2}}{2}\langle 1, -1\rangle \qquad\qquad \boldsymbol{T} = \frac{\sqrt{2}}{2}\langle -1, -1\rangle$$

With this we can write the following vector equation

$$\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}$$

$$\langle 0, -50\rangle = a_T\frac{\sqrt{2}}{2}\langle -1, -1\rangle + a_N\frac{\sqrt{2}}{2}\langle 1, -1\rangle$$

For which we can extract the following two equations

$$0 = -a_T\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}a_N \qquad\qquad -50 = -a_T\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}a_N$$

Solving we get

$$a_T = a_N \quad \rightarrow \quad \begin{aligned} a_T &= \frac{50}{\sqrt{2}} \\ v'(t) &= \frac{50}{\sqrt{2}} \cong 35.35\, {}^{m}/_{s^2} \end{aligned} \quad \rightarrow \quad \begin{aligned} \kappa v^2(t) &= a_N \\ v(t) &= \sqrt{\frac{a_T}{\kappa}} \\ v(t) &= \sqrt{\frac{\frac{50}{\sqrt{2}}}{\frac{1}{30}}} \\ v(t) &\cong 32.57\, {}^{m}/_{s} \end{aligned}$$

Where, in the last equation we used the fact that the curvature for a circle is ${}^{1}/_{R}$.

Final Summary for Vector Calculus – Motion in 3-Space

Motion Describing Quantities

$\boldsymbol{r}(t)$	:	Position Vector – Represents the Position of an Object	:	$\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$
$\boldsymbol{v}(t)$	:	Velocity Vector – Rate of change of Position	:	$\boldsymbol{v}(t) = \boldsymbol{r}'(t)$.
$v(t)$	:	Speed – Magnitude of Velocity	:	$v(t) = \lVert \boldsymbol{v}(t) \rVert$
$\boldsymbol{a}(t)$	:	Acceleration Vector - Rate of change of Velocity	:	$\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \boldsymbol{r}''(t)$

Acceleration Vector Decomposition

The acceleration vector for an object traveling along a path is given as

$$\boldsymbol{a}(t) = a_T\boldsymbol{T}(t) + a_N\boldsymbol{N}(t)$$

Where, $a_T = v'(t)$, *and* $a_N = \kappa v^2(t)$

- ***The Tangential Component "encodes" the change in the speed***
  - Since $a_T = v'(t)$ the tangential component is zero if the speed is constant.
- ***The Normal Component "encodes" the change in direction***
  - Since $a_N = \kappa v^2(t)$ the normal component is zero if $\kappa = 0$, which is the case when the path does not change direction.

The decomposition vectors can also be evaluated using the following formulas.

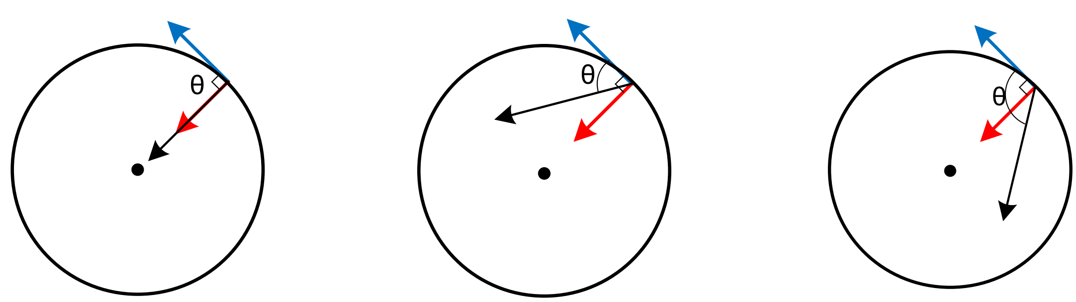
$$a_T\boldsymbol{T}(t) = \left(\frac{\boldsymbol{a}(t)\cdot\boldsymbol{v}(t)}{\lVert\boldsymbol{v}(t)\rVert^2}\right)\boldsymbol{v}(t)$$

$$\begin{aligned} a_N\boldsymbol{N}(t) &= \boldsymbol{a}(t) - a_T\boldsymbol{T}(t) \\ &= \boldsymbol{a}(t) - \left(\frac{\boldsymbol{a}(t)\cdot\boldsymbol{v}(t)}{\lVert\boldsymbol{v}(t)\rVert^2}\right)\boldsymbol{v}(t) \end{aligned}$$

Non-Uniform Circular Motion

$$v'(t) = a_T = \boldsymbol{a}(t)\cdot\boldsymbol{T}(t) = \lVert\boldsymbol{a}(t)\rVert\lVert\boldsymbol{T}(t)\rVert\, cos(\theta)$$

$(\boldsymbol{A})$ $(\boldsymbol{B})$ $(\boldsymbol{C})$



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B. $\theta < 90^\circ$: Therefore, $cos(\theta) > 0$ and $v'(t) > 0$. The particles *speed is increasing*.

C. $90^\circ < \theta < 180^\circ$: Therefore, $cos(\theta) < 0$ and $v'(t) < 0$. The particles *speed is decreasing*

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