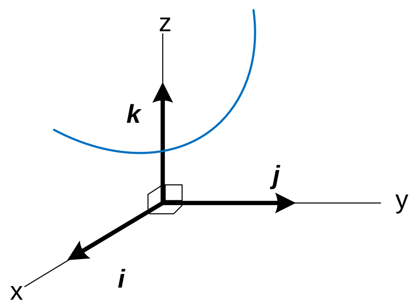
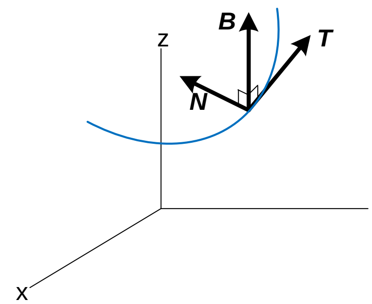
Vector Calculus – Frenet Frames

In the previous section we used curvature as a measure of how a curve bends. In its definition we used the unit tangent vector, $\boldsymbol{T}(t)$, which points in the direction of motion. In this section we derive two additional unit vectors related to the curve. These two vectors, along with the unit tangent vector are mutually orthogonal. A set of orthogonal unit vectors, *orthonormal vectors,* in R^3 can be used to establish a coordinate system just like the standard $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$, and $\widehat{\boldsymbol{k}}$ unit vectors. In this case however, the coordinate system is referenced to the curve itself. We call this a *Frenet frame*. As you can see below the Frenet frame is a function of the underlying curve and changes from point to point along the curve. As such, it is very useful in analyzing motion of objects in space.

<table>
<tr><th>Cartesian Frame</th><th>Frenet Frame</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>Coordinate System is fixed in space</td><td>Coordinate System moves along with the curve</td></tr>
</table>

Frenet Frame

The first vector in the Frenet frame, the *unit tangent vector*, defined in the previous lesson, is.

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|}$$

The second vector is called the *unit normal vector*, $\boldsymbol{N}$. In the previous section, when deriving an alternate form for curvature we showed that a unit vector and its derivative are orthogonal. Therefore, we define the unit normal vector as the derivative of the unit tangent vector.

$$\boldsymbol{N}(t) = \frac{\boldsymbol{T}'(t)}{\|\boldsymbol{T}'(t)\|}$$

The unit normal vector points in the direction in which the curve is turning, as seen in the figure above.

Finally, we create a third vector, referred to as the *unit binormal vector*, using the cross product as shown.

$$\boldsymbol{B}(t) = \boldsymbol{T}(t) \times \boldsymbol{N}(t)$$

These three vectors, $(\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B})$**,** are mutually orthogonal and of unit length. Together they form an orthonormal set of vectors, which we refer to as the *Frenet frame*, after the French geometer Jean Frenet. As we move along a curve this frame twists and turns along with the curve. The example below illustrates this concept for a simple curve in the y-z plane.

Example 1: Find the Frenet frame unit vectors, $\boldsymbol{T}, \boldsymbol{N}$, and $\boldsymbol{B}$, for the plane curve, $\boldsymbol{r}(t)$, at $t = 0$.

$$\boldsymbol{r}(t) = \langle 0, t, t^2 \rangle$$

Unit Tangent Vector

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} = \frac{\langle 0, 1, 2t \rangle}{\sqrt{1+4t^2}} \quad \rightarrow \quad \boldsymbol{T}(0) = \frac{\langle 0, 1, 0 \rangle}{\sqrt{1+0}} = \langle 0, 1, 0 \rangle$$

Unit Normal Vector: Since $\boldsymbol{N}(t) = \frac{\boldsymbol{T}'(t)}{\|\boldsymbol{T}'(t)\|}$, we first find $\boldsymbol{T}'(t)$ using the product rule.

$$\begin{aligned} \boldsymbol{T}'(t) &= \frac{d}{dt}\left(\left(\frac{1}{\sqrt{1+4t^2}}\right)\langle 0, 1, 2t \rangle\right) \\ &= \frac{d}{dt}\left(\frac{1}{\sqrt{1+4t^2}}\right)\langle 0, 1, 2t \rangle + \left(\frac{1}{\sqrt{1+4t^2}}\right)\frac{d}{dt}\langle 0, 1, 2t \rangle \\ &= -\frac{8t}{2(1+4t^2)^{3/2}}\langle 0, 1, 2t \rangle + \left(\frac{1}{\sqrt{1+4t^2}}\right)\langle 0, 0\ 2 \rangle \end{aligned}$$

At this point we can substitute $t = 0$

$$\boldsymbol{T}'(0) = -\frac{0}{2(1+0)^{3/2}}\langle 0, 1, 0 \rangle + \left(\frac{1}{\sqrt{1+0}}\right)\langle 0, 0, 2 \rangle = \langle 0, 0, 2 \rangle$$

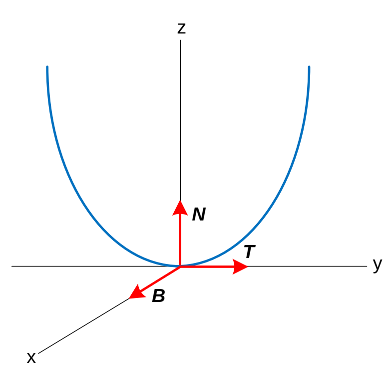
Therefore,

$$\boldsymbol{N}(0) = \frac{\boldsymbol{T}'(0)}{\|\boldsymbol{T}'(0)\|} = \frac{\langle 0, 0, 2 \rangle}{\sqrt{4}} = \langle 0, 0, 1 \rangle$$

Finally, we use the cross product to find the *binormal vector*, $\boldsymbol{B}(0)$.

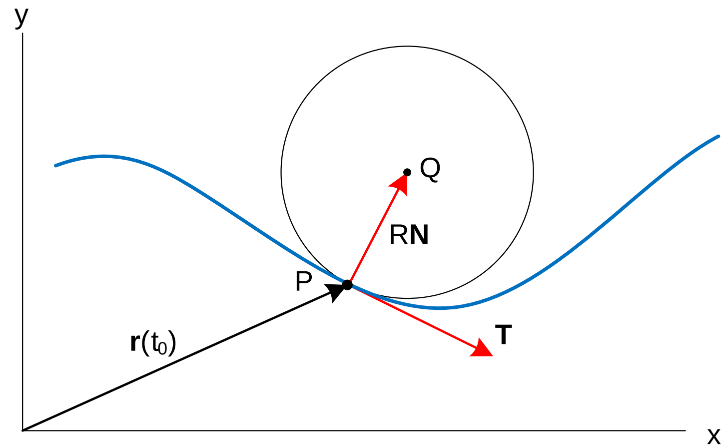
$$\begin{aligned} \boldsymbol{B}(0) &= \boldsymbol{T}(0) \times \boldsymbol{N}(0) \\ &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 1, 0, 0 \rangle \end{aligned}$$

As you can see below, for $t = 0$, the Frenet Frame is equivalent to the Cartesian Frame!



Osculating Circle

Another use of curvature is in a finding circle that "best fits" a curve at a point, P. We refer to this circle as the *osculating circle*. To illustrate we consider the point P on a plane curve as shown below, where the curvature at that point, κ_P, is non-zero, and $R = 1/\kappa_P$ is called the *radius of curvature*.



Recall the equation of a circle of radius R centered at (x_c, y_c)can be written as

$$\boldsymbol{r}(t) = R\langle cos(t)\,, sin(t)\,\rangle + \langle x_c, y_c\rangle$$

For the osculating circle above the radius is $R\|\boldsymbol{N}\|$, and the center of the circle is

$$\langle x_c, y_c\rangle = \boldsymbol{r}(t_0) + R\boldsymbol{N}$$

Therefore, the equation of an osculating circle can be written as

$$\begin{aligned}\boldsymbol{r_c}(t) &= R\langle cos(t)\,, sin(t)\,\rangle + \boldsymbol{r}(t_0) + R\boldsymbol{N} \\ &= \boldsymbol{r}(t_0) + 1/\kappa_P\,(\langle cos(t)\,, sin(t)\,\rangle + \boldsymbol{N})\end{aligned}$$

Although this does extend to general space curves in R^3, we will limit our study to plane curves in the x-y plane.

Osculating Circle
The *osculating circle* to a plane curve, $\boldsymbol{r}(t)$, at the point P is the circle that "best fits" the curve at P. The center of the circle lies in the direction of the normal vector, $\boldsymbol{N}$, to the curve, and the radius of the circle is called the radius of curvature, $R = 1/\kappa_P$. The equation of the osculating circle to the plane curve, $\boldsymbol{r}(t)$, at $P = \boldsymbol{r}(t_0)$ is given as follows: $\boldsymbol{r_c}(t) = \boldsymbol{r}(t_0) + 1/\kappa_P\,(\langle cos(t)\,, sin(t)\,\rangle + \boldsymbol{N}_P)$

Example 2: Find $\boldsymbol{T}, \boldsymbol{N}$, and $\boldsymbol{B}$ for the given curves at the indicated points.

a. $\boldsymbol{r}(t) = \langle cos(t)\,, sin(t)\,, t\rangle \text{ at } t = {}^{\pi}/_{2}$ *b.* $\boldsymbol{r}(t) = \langle\, t\,, t^2, \frac{2}{3}t^3\rangle \text{ at } t = 1$

Solution:

a. Start by finding $\boldsymbol{T}(t)$

$$\begin{aligned}\boldsymbol{T}(t) &= \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} \\ &= \frac{\langle -sin(t)\,, cos(t)\,, 1\,\rangle}{\sqrt{sin^2(t) + cos^2(t) + 1}} \\ &= \frac{1}{\sqrt{2}}\langle -sin(t)\,, cos(t)\,, 1\,\rangle\end{aligned}$$

Next, we compute $\boldsymbol{N}(t)$

$$\begin{aligned}\boldsymbol{N}(t) &= \frac{\boldsymbol{T}'(t)}{\|\boldsymbol{T}'(t)\|} \\ &= \frac{\frac{1}{\sqrt{2}}\langle -cos(t)\,, -sin(t)\,, 0\,\rangle}{\frac{1}{\sqrt{2}}\sqrt{cos^2(t) + sin^2(t)}} \\ &= \langle -cos(t)\,, -sin(t)\,, 0\,\rangle\end{aligned}$$

Finally, we use the cross product to compute $\boldsymbol{B}(t)$

$$\begin{aligned}\boldsymbol{B}(t) &= \boldsymbol{T}(t) \times \boldsymbol{N}(t) \\ &= \frac{1}{\sqrt{2}}\begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ -sin(t) & cos(t) & 1 \\ -cos(t) & -sin(t) & 0 \end{vmatrix} \\ &= \frac{1}{\sqrt{2}}\langle sin(t)\,, -cos(t)\,, 1\rangle\end{aligned}$$

The vectors at the point $t = \frac{\pi}{2}$ are

$$\boldsymbol{T}\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}\langle -1{,}0{,}\,1\,\rangle \qquad \boldsymbol{N}\left(\frac{\pi}{2}\right) = \langle 0, -1{,}0\,\rangle \qquad \boldsymbol{B}\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}\langle 1{,}0{,}\,1\rangle$$

b. Start by finding $\boldsymbol{T}(t)$ and $\boldsymbol{T}(1)$.

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} = \frac{\langle 1,2t,2t^2 \rangle}{\sqrt{1+4t^2+4t^4}} \quad \rightarrow \quad \boldsymbol{T}(1) = \frac{\langle 1,2,2\rangle}{\sqrt{1+4+4}} = \frac{1}{3}\langle 1,2,2\rangle$$

Next, since $\boldsymbol{N}(t) = \frac{\boldsymbol{T}'(t)}{\|\boldsymbol{T}'(t)\|}$, we first find $\boldsymbol{T}'(t)$ using the product rule.

$$\begin{aligned}\boldsymbol{T}'(t) &= \frac{d}{dt}\left(\frac{1}{\sqrt{1+4t^2+4t^4}}\right)\langle 1,2t,2t^2 \rangle + \left(\frac{1}{\sqrt{1+4t^2+4t^4}}\right)\frac{d}{dt}\langle 1,2t,2t^2 \rangle \\ &= -\frac{8t+16t^3}{2(1+4t^2+4t^4)^{3/2}}\langle 1,2t,2t^2 \rangle + \left(\frac{1}{\sqrt{1+4t^2+4t^4}}\right)\langle 0,2,4t \rangle\end{aligned}$$

At this point we can substitute $t = 1$

$$\begin{aligned}\boldsymbol{T}'(1) &= -\frac{8+16}{2(1+4+4)^{3/2}}\langle 1,2,2 \rangle + \left(\frac{1}{\sqrt{1+4+4}}\right)\langle 0,2,4 \rangle \\ &= -\frac{4}{9}\langle 1,2,2 \rangle + \left(\frac{1}{3}\right)\langle 0,2,4 \rangle = \frac{2}{3}\langle -2,-1,2\rangle\end{aligned}$$

Therefore,

$$\boldsymbol{N}(1) = \frac{\boldsymbol{T}'(1)}{\|\boldsymbol{T}'(1)\|} = \frac{\frac{2}{3}\langle -2,-1,2\rangle}{\frac{2}{3}\sqrt{9}} = \frac{1}{3}\langle -2,-1,2\rangle$$

Finally, we use the cross product to find the *binormal vector*, $\boldsymbol{B}(1)$.

$$\begin{aligned}\boldsymbol{B}(1) &= \boldsymbol{T}(1) \times \boldsymbol{N}(1) \\ &= \frac{1}{3}\cdot\frac{1}{3}\begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 1 & 2 & 2 \\ -2 & -1 & 2 \end{vmatrix} = \frac{1}{9}\langle 6,-6,3\rangle = \frac{1}{3}\langle 2,-2,1\rangle\end{aligned}$$

The vectors are listed below.

$$\boldsymbol{T}(1) = \frac{1}{3}\langle 1,2,2\rangle \qquad \boldsymbol{N}(1) = \frac{1}{3}\langle -2,-1,2\rangle \qquad \boldsymbol{B}(1) = \frac{1}{3}\langle 2,-2,1\rangle$$

Example 3: Find the equation of the osculating circle for the plane curve at the indicated point.

$$y = x^2 \ at\ x = 1$$

Solution: We can start by finding the curvature using the formula for plane curves.

$$\kappa_P = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}} = \left.\frac{|2|}{(1+4x^2)^{3/2}}\right|_{x=1} = \frac{2}{5\sqrt{5}}$$

To find $\boldsymbol{r}(t_0)$ and $\boldsymbol{N_P}$**,** we first write the curve as a vector function as follows:

$$\boldsymbol{r}(t) = \langle t, t^2 \rangle$$

In this case, $x = 1$ corresponds to $t = 1$, therefore, $\boldsymbol{r}(t_0) = \boldsymbol{r}(1) = \langle 1, 1 \rangle$.

To find $\boldsymbol{N}(1)$ we start with $\boldsymbol{T}(t)$

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} = \frac{\langle 1,2t \rangle}{\sqrt{1+4t^2}}$$

Next, we find $\boldsymbol{T}'(t)$ using the product rule.

$$\begin{aligned}\boldsymbol{T}'(t) &= \frac{d}{dt}\left(\left(\frac{1}{\sqrt{1+4t^2}}\right)\langle 1,2t \rangle\right) \\ &= \frac{d}{dt}\left(\frac{1}{\sqrt{1+4t^2}}\right)\langle 1,2t \rangle + \left(\frac{1}{\sqrt{1+4t^2}}\right)\frac{d}{dt}\langle 1,2t \rangle \\ &= -\frac{8t}{2(1+4t^2)^{3/2}}\langle 1,2t \rangle + \left(\frac{1}{\sqrt{1+4t^2}}\right)\langle 0,2 \rangle\end{aligned}$$

At this point we can substitute $t = 1$

$$\boldsymbol{T}'(1) = -\frac{8}{2(1+4)^{3/2}}\langle 1,2 \rangle + \left(\frac{1}{\sqrt{1+4}}\right)\langle 0,2,0 \rangle = \frac{2}{5\sqrt{5}}\langle -2,1 \rangle$$

Therefore,

$$\boldsymbol{N}(1) = \frac{\boldsymbol{T}'(1)}{\|\boldsymbol{T}'(1)\|} = \frac{\frac{2}{5\sqrt{5}}\langle -2,1,0 \rangle}{\frac{2}{5\sqrt{5}}\sqrt{5}} = \frac{1}{\sqrt{5}}\langle -2,1,0 \rangle$$

Finally, the equation for the osculating circle is given below.

$$\begin{aligned}\boldsymbol{r_c}(t) &= \boldsymbol{r}(t_0) + 1/\kappa_P\,(\langle cos(t), sin(t) \rangle + \boldsymbol{N_P}) \\ &= \langle 1, 1 \rangle + \frac{5\sqrt{5}}{2}\left(\langle cos(t), sin(t) \rangle + \frac{1}{\sqrt{5}}\langle -2,1,0 \rangle\right) \\ &= \langle 1, 1 \rangle + \langle -5, \frac{5}{2} \rangle + \frac{5\sqrt{5}}{2}(\langle cos(t), sin(t) \rangle) \\ &= \langle -4, \frac{7}{2} \rangle + \frac{5\sqrt{5}}{2}(\langle cos(t), sin(t) \rangle)\end{aligned}$$

Final Summary for Vector Calculus – Frenet Frames

Frenet Frame

A unit vector that is tangent to a space curve, $\boldsymbol{r}(t)$, for all t is called the ***unit tangent vector*** and is given as

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|}$$

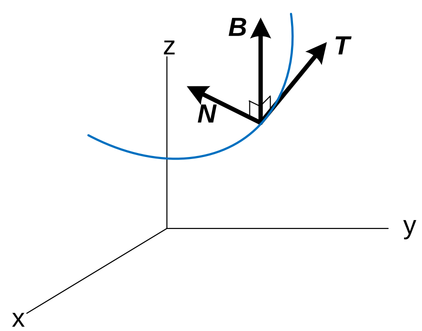
A unit normal vector that is orthogonal to $\boldsymbol{T}(t)$ for all t and points in the direction that the curve is turning is called the ***unit normal vector*** and is given as

$$\boldsymbol{N}(t) = \frac{\boldsymbol{T}'(t)}{\|\boldsymbol{T}'(t)\|}$$

A unit vector that is orthogonal to both $\boldsymbol{T}(t)$ and $\boldsymbol{N}(t)$ is called a ***unit binormal vector*** and is give as

$$\boldsymbol{B}(t) = \boldsymbol{T}(t) \times \boldsymbol{N}(t)$$

The three vectors, $(\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B})$**,** are mutually orthogonal and of unit length. Together they form an orthonormal set of vectors, which we refer to as the ***Frenet Frame****.* The Frenet frame is a function of the underlying curve and changes from point to point along the curve. As such, it is very useful in analyzing motion of objects in space



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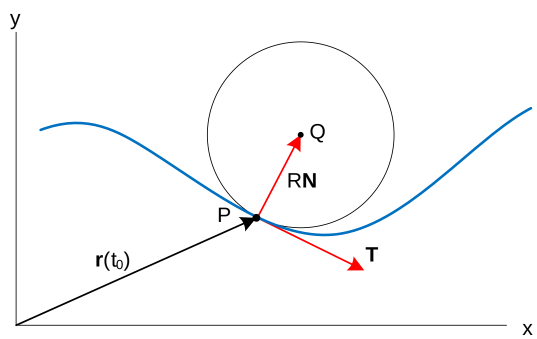
The equation of the osculating circle to the plane curve, $\boldsymbol{r}(t)$, at $P = \boldsymbol{r}(t_0)$ is given as follows:

$$\boldsymbol{r_c}(t) = \boldsymbol{r}(t_0) + 1/\kappa_P\,(\langle cos(t)\,, sin(t)\,\rangle + \boldsymbol{N}_P)$$



By: ferrantetutoring