Vector Calculus – Differentiation and Integration of Vector-Valued Functions

After being introduced to vector-valued functions in the previous section we are now ready to explore how we can do calculus, i.e. differentiate and integrate, on these functions. Even though most of the techniques from single-variable calculus carryover, it's important we understand the physical and/or geometrical interpretations when doing calculus on vector-valued functions. We start below with differentiation.

Differentiation of Vector-Valued Functions

We start by stating the following theorem without proof.

Derivative of Vector-Valued Functions
The derivative of vector-valued functions are computed component-wise. The derivative of the vector-valued function $\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$, is computed component-wise as $$\frac{d}{dt}\big(\boldsymbol{r}(t)\big) = \boldsymbol{r}'(t) \ = \langle x'(t), y'(t),\ z'(t)\rangle$$ Provided each component is differentiable.

Let's do an example to clarify.

Example 1: Compute the derivative of the following functions.

a. $\boldsymbol{r}(t) = \langle t\,cos(t)\,,\ t^3,\ sin(4t)\rangle$ *b.* $\boldsymbol{r}(t) = \langle cos(t)\,, -3,\ e^{2t}\rangle$

Solution: According the above theorem we can differentiate component-wise as shown.

a.

$$\begin{aligned}\boldsymbol{r}'(t) &= \langle \frac{d}{dt}(t\,cos(t)), \frac{d}{dt}(t^3), \frac{d}{dt}(sin(4t))\rangle \\ &= \langle (cos(t) - t\,sin(t)), 3t^2, 4\,cos(4t)\rangle\end{aligned}$$

b.

$$\begin{aligned}\boldsymbol{r}'(t) &= \langle \frac{d}{dt}(cos(t)), \frac{d}{dt}(-3), \frac{d}{dt}(e^{2t})\rangle \\ &= \langle -\,sin(t)\,, 0, 2e^{2t}\rangle\end{aligned}$$

Note when differentiating the individual components, we used the basic differentiation rules from single variable calculus, e.g. product rule, chain rule.

Furthermore, the following single variable differentiation rules apply to vector-valued functions as summarized below.

Differentiation Rules for Vector-Valued Functions	
Sum Rule:	$\frac{d}{dt}(\boldsymbol{r_1}(t) + \boldsymbol{r_2}(t)) = \frac{d}{dt}(\boldsymbol{r_1}(t)) + \frac{d}{dt}(\boldsymbol{r_2}(t))$
Constant Multiple Rule:	$\frac{d}{dt}(c\boldsymbol{r}(t)) = c\frac{d}{dt}(\boldsymbol{r}(t))$
Scalar Product Rule:	$\frac{d}{dt}(f(t)\boldsymbol{r}(t)) = f'(t)\boldsymbol{r}(t) + f(t)\boldsymbol{r}'(t)$
Chain Rule:	$\frac{d}{dt}\Big(\boldsymbol{r}\big(f(t)\big)\Big) = \boldsymbol{r}'\big(f(t)\big)f'(t)$

Let's look at another example to illustrate the above rules.

Example 2: With $\boldsymbol{r}(t) = \langle t^2, 5t, 1\rangle$ and $f(t) = e^{3t}$, compute the following derivatives.

a. $\frac{d}{dt}(f(t)\boldsymbol{r}(t))$ *b.* $\frac{d}{dt}\Big(\boldsymbol{r}\big(f(t)\big)\Big)$

Solution:

a. Using the product rule as stated above we have:

$$\begin{aligned}\frac{d}{dt}(e^{3t}\langle t^2, 5t, 1\rangle) &= \left(\frac{d}{dt}(e^{3t})\right)\langle t^2, 5t, 1\rangle + e^{3t}\left(\frac{d}{dt}(\langle t^2, 5t, 1\rangle)\right)\\ &= (3e^{3t})\langle t^2, 5t, 1\rangle + e^{3t}\langle 2t, 5, 0\rangle\\ &= \langle e^{3t}(3t^2 + 2t), 5e^{3t}(3t+1), 3e^{3t}\rangle\end{aligned}$$

Note we obtain the same answer by distributing the e^{3t} first and the differentiating.

$$\begin{aligned}\frac{d}{dt}(\langle e^{3t}t^2, 5e^{3t}t, e^{3t}1\rangle) &= \langle\frac{d}{dt}(e^{3t}t^2), \frac{d}{dt}(5e^{3t}t), \frac{d}{dt}(e^{3t})\rangle\\ &= \langle 3e^{3t}t^2 + e^{3t}2t, 5(3e^{3t}t + e^{3t}), 3e^{3t}\rangle\\ &= \langle e^{3t}(3t^2 + 2t), 5e^{3t}(3t+1), 3e^{3t}\rangle\end{aligned}$$

b. With $\boldsymbol{r}(t) = \langle t^2, 5t, 1\rangle$ and $f(t) = e^{3t}$ we can directly use the chain rule from above as

$$\begin{aligned}\frac{d}{dt}\Big(\boldsymbol{r}\big(f(t)\big)\Big) &= \boldsymbol{r}'\big(f(t)\big)f'(t)\\ &= \langle 2e^{3t}, 5, 0\rangle 3e^{3t}\\ &= \langle 6e^{6t}, 15e^{3t}, 0\rangle\end{aligned}$$

We can also create the composite function first and then differentiate.

$$\boldsymbol{r}\big(f(t)\big) = \langle (e^{3t})^2, 5e^{3t}, 1\rangle = \langle e^{6t}, 5e^{3t}, 1\rangle$$

Therefore,

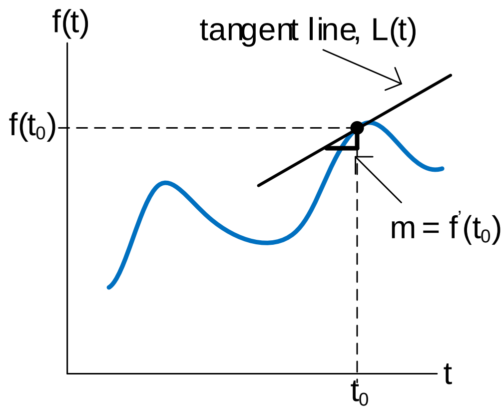
$$\begin{aligned}\frac{d}{dt}\Big(\boldsymbol{r}\big(f(t)\big)\Big) &= \frac{d}{dt}(\langle e^{6t}, 5e^{3t}, 1\rangle)\\ &= \langle 6e^{6t}, 15e^{3t}, 0\rangle\end{aligned}$$

Note that the product rule stated above is applicable to the product of a vector-valued function and a scalar function. However, as we know, there are two key vector products which we learned in an earlier lesson, i.e. Dot Product and Cross Product. The rules for these vector products are given below.

Product Rule for Dot and Cross Product	
Dot Product:	$\frac{d}{dt}\big(\boldsymbol{r_1}(t)\cdot\boldsymbol{r_2}(t)\big) = \boldsymbol{r_1}'(t)\cdot\boldsymbol{r_2}(t) + \boldsymbol{r_1}(t)\cdot\boldsymbol{r_2}'(t)$
Cross Product:	$\frac{d}{dt}\big(\boldsymbol{r_1}(t)\times\boldsymbol{r_2}(t)\big) = \boldsymbol{r_1}'(t)\times\boldsymbol{r_2}(t) + \boldsymbol{r_1}(t)\times\boldsymbol{r_2}'(t)$

The Derivative as the Tangent Vector

We start by recalling the slope of the tangent line interpretation for the derivative of a single variable scalar function.



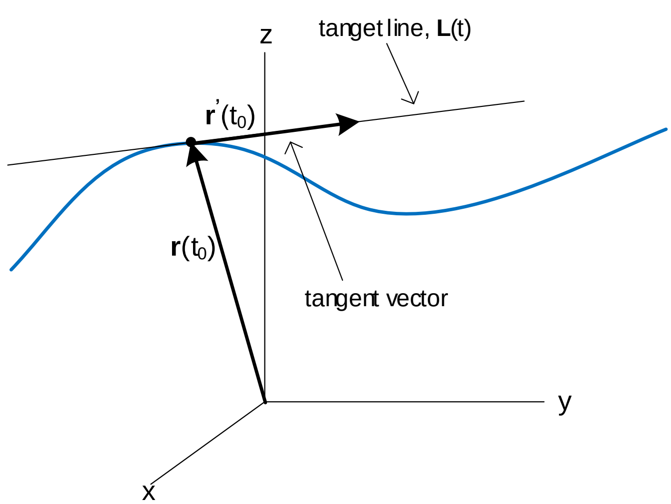
Given a scalar function, $f(t)$, the slope of the tangent line to the curve at the point, t_0, is given by the derivative evaluated at t_0.

$$m = f'(t_0)$$

Next, we can use the point-slope formula to write the equation of the tangent line, $L(t)$, to the curve at t_0 as follows:

$$L(t) - f(t_0) = m(t - t_0)$$
$$L(t) = f'(t_0)(t - t_0) + f(t_0)$$

The derivative of a vector-valued function has an analogous interpretation. In the case above, the slope of the tangent line, i.e. $f'(t_0)$, is a scalar value that tells us how to move from the point t_0 in order to continue along the tangent line. Similarly, derivative of a vector-valued function, $\boldsymbol{r}'(t_0)$, tells us how to move from the point t_0 in order to continue along the tangent line. However, in the previous case the derivative is a scalar value, whereas in this case the derivative is a vector. We refer to this as the tangent vector or the direction vector and it points in the direction that is tangent to the path at the point t_0.



Next, we recall the vector parameterization of a line in R^3 as shown below.

$$\boldsymbol{r}(t) = \boldsymbol{r}(t_0) + t\boldsymbol{v}$$

Where, $\boldsymbol{r}(t_0)$ is a point along the path at t_0 and $\boldsymbol{v}$ is the direction vector which points in a direction parallel to the line.

We can use this directly, with $\boldsymbol{r}'(t_0)$ as the direction vector, to write the equation of the tangent line, $\boldsymbol{L}(t)$, as

$$\boldsymbol{L}(t) = \boldsymbol{r}(t_0) + t\boldsymbol{r}'(t_0)$$

We summarize below:

Derivative of a Vector-Valued Function as the Tangent Vector
The derivative of a vector-valued function at $t_0, \boldsymbol{r}'(t_0)$, is a vector that is tangent to the path, $\boldsymbol{r}(t)$, at t_0. The tangent line to the path, $\boldsymbol{r}(t)$, at t_0 can be written as $$\boldsymbol{L}(t) = \boldsymbol{r}(t_0) + t\boldsymbol{r}'(t_0)$$

Before we look at integration let's do some additional examples with derivatives.

Example 3: Find an equation of the line that is tangent to the curve at the indicated point.

a. $\boldsymbol{r}(t) = \langle ln(t), 2\sqrt{t}, t^2 \rangle$ at $(x_0, y_0, z_0) = (0, 2, 1)$

b. $\boldsymbol{r}(t) = \langle sin(t), cos(t), tan(t) \rangle$ at $t = \pi$

Solution:

a. We start by finding t_0 that corresponds to the (x_0, y_0, z_0).

$ln(t_0) = 0$ $t_0 = 1$	$2\sqrt{t_0} = 2$ $t_0 = 1$	${t_0}^2 = 1$ $t_0 = 1$

The direction vector is then

$$\boldsymbol{r}'(t)|_{t=1} = \langle \frac{1}{t}, \frac{1}{\sqrt{t}}, 2t \rangle_{\,t=1} = \langle 1, 1, 2 \rangle$$

Finally, the tangent line is given as

$$\begin{aligned} \boldsymbol{L}(t) &= \boldsymbol{r}(1) + t\boldsymbol{r}'(1) \\ &= \langle 0, 2, 1 \rangle + t\langle 1, 1, 2 \rangle \\ &= \langle t, 2 + t, 1 + 2t \rangle \end{aligned}$$

b. In this case, since we are given t_0, we can directly write the tangent line as

$$\begin{aligned} \boldsymbol{L}(t) &= \boldsymbol{r}(\pi) + t\boldsymbol{r}'(\pi) \\ &= \langle sin(\pi), cos(\pi), tan(\pi)\rangle + t\langle cos(\pi), -sin(\pi), sec^2(\pi) \rangle \\ &= \langle 0, -1, 0\rangle + t\langle -1, 0, 1 \rangle \\ &= \langle -t, -1, t\rangle \end{aligned}$$

Example 4: Find all the points on the curve, $\boldsymbol{r}(t) = \langle t, t^2, t^3\rangle$, where its tangent line is parallel to the vector, $\boldsymbol{a} = \langle 2, 8, 24\rangle$.

Solution: We know that $\boldsymbol{r}'(t)$ points in a direction that is tangent to the curve. This vector will be parallel to the vector, $\boldsymbol{a}$, when $\boldsymbol{r}'(t) \times \boldsymbol{a} = \boldsymbol{0}$.

$$\boldsymbol{r}'(t) = \frac{d}{dt}(\langle t, t^2, t^3\rangle) = \langle 1, 2t, 3t^2\rangle$$

$$\begin{aligned} \boldsymbol{r}'(t) \times \boldsymbol{a} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 1 & 2t & 3t^2 \\ 2 & 8 & 24 \end{vmatrix} \\ &= \hat{\boldsymbol{\imath}} \begin{vmatrix} 2t & 3t^2 \\ 8 & 24 \end{vmatrix} - \hat{\boldsymbol{\jmath}} \begin{vmatrix} 1 & 3t^2 \\ 2 & 24 \end{vmatrix} + \hat{\boldsymbol{k}} \begin{vmatrix} 1 & 2t \\ 2 & 8 \end{vmatrix} \\ &= \hat{\boldsymbol{\imath}}\,(48t - 24t^2) - \hat{\boldsymbol{\jmath}}\,(24 - 6t^2) + \hat{\boldsymbol{k}}\,(8 - 4t) = \boldsymbol{0} \end{aligned}$$

Finally, we solve the equation for each component simultaneously.

$$48t - 24t^2 = 0 \qquad\qquad 24 - 6t^2 = 0 \qquad\qquad 8 - 4t = 0$$

$$24t(2 - t) = 0 \qquad\qquad 4 - t^2 = 0 \qquad\qquad 2 - t = 0$$

$$(2 - t)(2 + t) = 0$$

$$t = 0, \quad t = 2 \qquad\qquad t = -2, \quad t = 2 \qquad\qquad t = 2$$

Therefore, the point on the curve where the tangent line is parallel to $\boldsymbol{a}$ is at $t = 2$.

$$\boldsymbol{r}(2) = \langle 2, 4, 8\rangle$$

Furthermore, the corresponding parallel vector $\boldsymbol{v}$, is

$$\boldsymbol{v} = \boldsymbol{r}'(2) = \langle 1, 4, 12\rangle$$

Example 5: The path for each of two particles moving in space is described by the following

$$\boldsymbol{r_1}(t) = \langle t^2, 2t+3, t^2 \rangle$$

$$\boldsymbol{r_2}(t) = \langle 5t-6, t^2, 9 \rangle$$

At some moment in time the two particles collide.

a. Find the moment when they collide and the location at which the collision occurs.
b. What is the angle between the paths of the particles at the moment of collision?

Solution:

a. To find when the two particles collide, we equate the two path equations.

$$\boldsymbol{r_1}(t) = \boldsymbol{r_2}(t)$$

To solve we must solve the component equations simultaneously.

$$\begin{aligned} t^2 &= 5t-6 \\ t^2-5t+6 &= 0 \\ (t-3)(t-2) &= 0 \end{aligned} \qquad \begin{aligned} 2t+3 &= t^2 \\ t^2-2t-3 &= 0 \\ (t-3)(t+1) &= 0 \end{aligned} \qquad t^2 = 9$$

$$t = 3, \quad t = 2 \qquad\qquad t = 3, \quad t = -1 \qquad\qquad t = -3, t = 3$$

Therefore, the particles collide at $t = 3$ at the point $\boldsymbol{r_1}(3) = \boldsymbol{r_2}(3) = (9, 9, 9)$

b. To find the angle between the two paths at this point we need to first find the direction vectors.

$$\boldsymbol{r_1}'(t)|_{t=3} = \langle 2t, 2, 2t \rangle|_{t=3} = \langle 6, 2, 6 \rangle \qquad \boldsymbol{r_2}'(t)|_{t=3} = \langle 5, 2t, 0 \rangle|_{t=3} = \langle 5, 6, 0 \rangle$$

Finally, the angle between these two vectors is found as follows

$$\begin{aligned} \theta &= cos^{-1}\left(\frac{\boldsymbol{r_1}'(3) \cdot \boldsymbol{r_2}'(3)}{\|\boldsymbol{r_1}'(3)\|\|\boldsymbol{r_2}'(3)\|}\right) \\ &= cos^{-1}\left(\frac{(30+12+0)}{(\sqrt{36+4+36})(\sqrt{25+36+0})}\right) \\ &= cos^{-1}\left(\frac{42}{(\sqrt{76})(\sqrt{61})}\right) \approx 52^\circ \end{aligned}$$

Example 6: Prove the following Theorem.

Orthogonality of r and r' when r has a Constant Length
If $\boldsymbol{r}(t)$ is a differentiable vector-valued function in R^2 or R^3, and if $\|\boldsymbol{r}(t)\|$ is constant for all t, then $\boldsymbol{r}(t) \cdot \boldsymbol{r}'(t) = 0$. That is, $\boldsymbol{r}(t)$ and $\boldsymbol{r}'(t)$ are orthogonal vectors for all t.

Solution: To prove this we start with the following relationship

$$\boldsymbol{r}(t) \cdot \boldsymbol{r}(t) = \|\boldsymbol{r}(t)\|^2$$

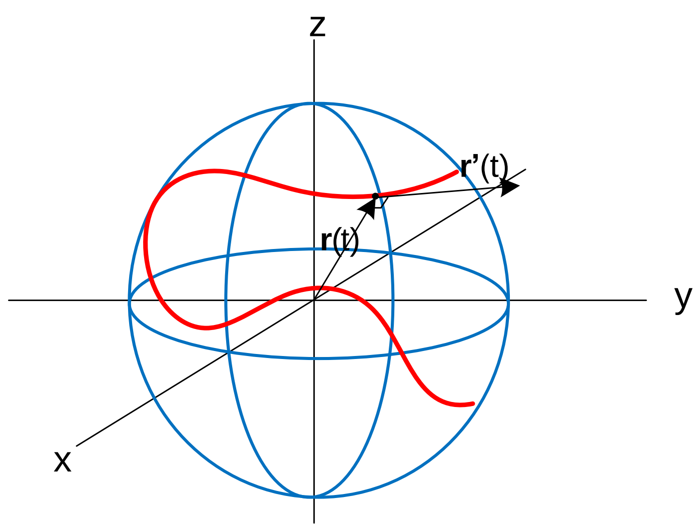
Then we differentiate both sides.

$$\frac{d}{dt}\big(\boldsymbol{r}(t) \cdot \boldsymbol{r}(t)\big) = \frac{d}{dt}(\|\boldsymbol{r}(t)\|^2)$$
$$\boldsymbol{r}'(t) \cdot \boldsymbol{r}(t) + \boldsymbol{r}(t) \cdot \boldsymbol{r}'(t) = \frac{d}{dt}(\|\boldsymbol{r}(t)\|^2)$$
$$2\big(\boldsymbol{r}(t) \cdot \boldsymbol{r}'(t)\big) = \frac{d}{dt}(\|\boldsymbol{r}(t)\|^2)$$

And since $\|\boldsymbol{r}(t)\|$ is constant, $\frac{d}{dt}(\|\boldsymbol{r}(t)\|^2) = \mathbf{0}$. Therefore,

$$2\big(\boldsymbol{r}(t) \cdot \boldsymbol{r}'(t)\big) = 0$$
$$\boldsymbol{r}(t) \cdot \boldsymbol{r}'(t) = 0$$

An interesting example of this is a particle traveling on the surface of a sphere. Because the radius of the sphere is constant, and equal to $\|\boldsymbol{r}(t)\|$, then, regardless of the path the particle takes, $\boldsymbol{r}(t)$ is orthogonal to $\boldsymbol{r}'(t)$ for all t.



Vector-Valued Integration

As we might expect, since differentiation works for vector-valued functions so does integration. Furthermore, it is computed component-wise just like the derivative.

Indefinite and Definite Integral of a Vector-Valued Function
The Indefinite Integral of a vector-valued function is defined as $$\int \boldsymbol{r}(t)dt = \langle \int x(t)dt\,, \int y(t)dt\,, \int z(t)dt\rangle + \boldsymbol{c}$$ Where, $\boldsymbol{c}$ is a constant vector that can be determined from initial conditions. The Definite Integral of a vector-valued function is defined as $$\int_a^b \boldsymbol{r}(t)dt \ = \langle \int_a^b x(t)dt\,, \int_a^b y(t)dt\,, \int_a^b z(t)dt\rangle$$

Example 7: Evaluate the following integrals.

a. $\int \langle (2t+1)^5, -\frac{1}{t}\rangle\, dt$ *b.* $\int_0^{ln(3)} \langle e^t, e^{2t}\ \rangle dt$

Solution:

a.

$$\int \langle (2t+1)^5, -\frac{1}{t}\rangle\, dt = \langle \int (2t+1)^5 dt\,, \int -\frac{1}{t} dt\rangle + \langle c_x, c_y\rangle$$

$$= \langle \frac{(2t+1)^6}{12}, -\,ln|t|\rangle + \langle c_x, c_y\rangle$$

b.

$$\int_0^{ln(3)} \langle e^t, e^{2t}\ \rangle dt = \langle \int_0^{ln(3)} e^t dt\,, \int_0^{ln(3)} e^{2t} dt\rangle$$

$$= \langle e^t\big|_0^{ln(3)}, \frac{1}{2}e^{2t}\Big|_0^{ln(3)}\ \rangle$$

$$= \langle 2, 4\rangle$$

Final Summary for Vector Calculus – Differentiation and Integration of Vector-Valued Functions

Derivative of Vector-Valued Function	
The derivative of the vector-valued function $\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$, is computed component-wise as $$\frac{d}{dt}\big(\boldsymbol{r}(t)\big) = \boldsymbol{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$$ Provided each component is differentiable.	
Differentiation Rules for Vector-Valued Functions	
Sum Rule:	$\frac{d}{dt}\big(\boldsymbol{r_1}(t) + \boldsymbol{r_2}(t)\big) = \frac{d}{dt}\big(\boldsymbol{r_1}(t)\big) + \frac{d}{dt}\big(\boldsymbol{r_2}(t)\big)$
Constant Multiple Rule:	$\frac{d}{dt}\big(c\boldsymbol{r}(t)\big) = c\frac{d}{dt}\big(\boldsymbol{r}(t)\big)$
Scaler Product Rule:	$\frac{d}{dt}\big(f(t)\boldsymbol{r}(t)\big) = f'(t)\boldsymbol{r}(t) + f(t)\boldsymbol{r}'(t)$
Dot Product Rule:	$\frac{d}{dt}\big(\boldsymbol{r_1}(t)\cdot\boldsymbol{r_2}(t)\big) = \boldsymbol{r_1}'(t)\cdot\boldsymbol{r_2}(t) + \boldsymbol{r_1}(t)\cdot\boldsymbol{r_2}'(t)$
Cross Product Rule:	$\frac{d}{dt}\big(\boldsymbol{r_1}(t)\times\boldsymbol{r_2}(t)\big) = \boldsymbol{r_1}'(t)\times\boldsymbol{r_2}(t) + \boldsymbol{r_1}(t)\times\boldsymbol{r_2}'(t)$
Chain Rule:	$\frac{d}{dt}\Big(\boldsymbol{r}\big(f(t)\big)\Big) = \boldsymbol{r}'\big(f(t)\big)f'(t)$
Derivative of Vector-Valued Function as a Tangent Vector	
The derivative at t_0, $\boldsymbol{r}'(t_0)$, is a vector that is tangent to the path, $\boldsymbol{r}(t)$, at t_0. The tangent line to the path, $\boldsymbol{r}(t)$, at t_0 can be written as $$\boldsymbol{L}(t) = \boldsymbol{r}(t_0) + t\boldsymbol{r}'(t_0)$$	
Orthogonality of $\boldsymbol{r}$ and $\boldsymbol{r}'$ when $\boldsymbol{r}$ has a Constant Length	
If $\boldsymbol{r}(t)$ is a differentiable vector-valued function in R^2 or R^3, and if $\lVert\boldsymbol{r}(t)\rVert$ is constant for all t, then $\boldsymbol{r}(t)\cdot\boldsymbol{r}'(t) = 0$. That is, $\boldsymbol{r}(t)$ and $\boldsymbol{r}'(t)$ are orthogonal vectors for all t.	
Indefinite and Definite Integral of a Vector-Valued Function	
The Indefinite Integral of a vector-valued function is defined as $$\int \boldsymbol{r}(t)dt = \langle\int x(t)dt, \int y(t)dt, \int z(t)dt\rangle + \boldsymbol{c}$$ The Definite Integral of a vector-valued function is defined as $$\int_a^b \boldsymbol{r}(t)dt = \langle\int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt\rangle$$	

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