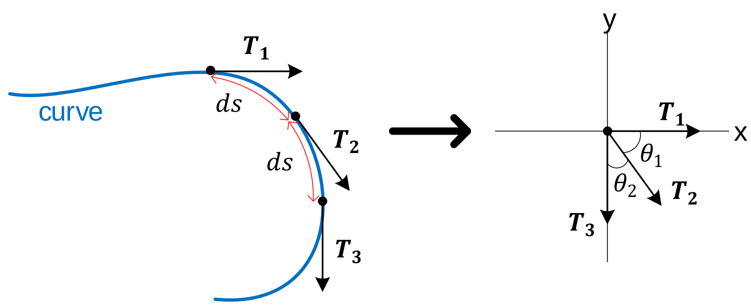
Vector Calculus – Curvature

In the previous section we introduced the so-called arc length parameterization. This is a specific parametrization of a curve that ensures $\|\boldsymbol{v}(t)\| = 1$. We also mentioned that this parametrization allows one to focus on the shape of the curve only and not on the particular way in which it is traversed. In this section we will define a specific measure that describes how much a curve bends. We refer to this as the *curvature*, and we will use the arc length parameterization in its definition.

Curvature

A vector that is tangent to a curve, e.g. $\boldsymbol{r}'(t)$, seems a good parameter to use to measure curvature as it indicates the direction at each point along the curve. However, to avoid being influenced by the length of this tangent vector we instead use the unit tangent vector, $\boldsymbol{T}$, as shown in the figure below. Therefore, the rate of change of $\boldsymbol{T}$ as we traverse along the curve seems a good measure of curvature. Once again, to focus on the underlying curve only and not on the way in which it is traversed, we'll use the rate of change *with respect to a unit length, i.e.* ds. Finally, as we would like to have the curvature be a positive quantity, we use the magnitude. With this the curvature, κ, is defined as

$$\kappa(s) = \left\|\frac{d\boldsymbol{T}}{ds}\right\|$$



Assuming a curve is described by an arc-length parameterizing this derivative can be computed in a fairly straightforward manner.

Curvature
Let $\boldsymbol{r}(s)$ be an arc length parameterization and $\boldsymbol{T} = \boldsymbol{T}(s)$ be the unit tangent vector. The curvature at $\boldsymbol{r}(s)$ is defined as follows: $$\kappa(s) = \left\|\frac{d\boldsymbol{T}}{ds}\right\|$$ Where, $$\boldsymbol{T} = \boldsymbol{T}(s) = \frac{\boldsymbol{r}'(s)}{\|\boldsymbol{r}'(s)\|} = \frac{\boldsymbol{r}'(s)}{1} = \boldsymbol{r}'(s)$$ Note: This assumes $\boldsymbol{r}'(t) \neq 0$ for all t.

The next two examples will serve to illustrate the curvature.

Example 1: One would expect that a line has zero curvature. To show this is true compute the curvature of an arbitrary line defined as.

$$\boldsymbol{r}(t) = \langle x_0, y_0, z_0 \rangle + t\boldsymbol{u}$$

Where $\boldsymbol{u}$ is a unit vector, i.e. $\|\boldsymbol{u}\| = \mathbf{1}$.

Solution: We start by noting that since the direction vector, $\boldsymbol{u}$, is a unit vector, the parametrization, $\boldsymbol{r}(t)$, is an arc length parameterization. We verify below.

$$\begin{aligned} \boldsymbol{v}(t) &= \boldsymbol{r}'(t) \\ &= \frac{d}{dt}(x_0, y_0, z_0) + \frac{d}{dt}(t\boldsymbol{u}) \\ &= \langle 0, 0, 0 \rangle + \boldsymbol{u} \\ \boldsymbol{v}(t) &= \boldsymbol{u} \end{aligned}$$

And since

$$\|\boldsymbol{v}(t)\| = \|\boldsymbol{u}\| = 1$$

The parameterization has unit speed.

The curvature can then be computed as follows:

$$\kappa(t) = \left\|\frac{d\boldsymbol{T}}{dt}\right\| = \left\|\frac{d}{dt}\left(\frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|}\right)\right\| = \left\|\frac{d}{dt}\left(\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}\right)\right\| = 0$$

So, as expected the curvature of a line is zero for all t.

Example 2: For a circular path one can imagine that the curvature increases as the radius decreases. To investigate this, compute the curvature for a circle of radius R.

Solution: We can define a circle in R^2 as follows:

$$\boldsymbol{r}(t) = \langle R\,cos(t)\,, R\,sin(t) \rangle$$

We need to start by creating an arc length parametrization by first finding $s = g(t)$.

$$\begin{aligned} g(t) &= \int_0^t \|\boldsymbol{r}'(u)\|\,du \\ &= \int_0^t \sqrt{\left(\frac{d}{du}(R\,cos(u))\right)^2 + \left(\frac{d}{du}(\,R\,sin(u))\right)^2}\,du \\ &= \int_0^t \sqrt{R^2\,sin^2(u) + R^2\,cos^2(u)}\,du \\ &= \int_0^t R\,du \\ g(t) &= Rt \end{aligned}$$

Therefore, $t = g^{-1}(s) = \frac{s}{R}$, and the new parameterization is.

$$\boldsymbol{r}(s) = \boldsymbol{r}\left(g^{-1}(s)\right) = \langle R\cos\left(\frac{s}{R}\right), R\sin\left(\frac{s}{R}\right)\rangle$$

The unit tangent vector is then

$$\begin{aligned}\boldsymbol{T} &= \boldsymbol{r}'(s) \\ &= \cancel{R}\langle -\frac{1}{\cancel{R}}\sin\left(\frac{s}{R}\right), \frac{1}{\cancel{R}}\cos\left(\frac{s}{R}\right)\rangle \\ &= \langle -\sin\left(\frac{s}{R}\right), \cos\left(\frac{s}{R}\right)\rangle\end{aligned}$$

The curvature is then found as follows:

$$\begin{aligned}\kappa(s) &= \left\|\frac{d\boldsymbol{T}}{ds}\right\| \\ &= \sqrt{\left(-\frac{1}{R}\cos\left(\frac{s}{R}\right)\right)^2 + \left(-\frac{1}{R}\sin\left(\frac{s}{R}\right)\right)^2} \\ &= \frac{1}{R}\sqrt{\cos^2\left(\frac{s}{R}\right) + \sin^2\left(\frac{s}{R}\right)} \\ \kappa &= \frac{1}{R}\end{aligned}$$

As you can see the curvature is inversely related to the radius as expected.

As we mentioned in the previous section, the arc length parameterization is, in most cases, impossible to find explicitly. Fortunately, by modifying the formula for curvature, we can still compute the curvature with *any* parameterization. Although we may not be able to compute the arc length integral explicitly, we know that the arc length parametrization does exist, i.e. t is a function of s. Therefore, the unit tangent vector can be expressed as a composite function, $\boldsymbol{T}\big(t(s)\big)$, for which we can then use the chain rule to differentiate as shown below.

$$\kappa = \left\|\frac{d\boldsymbol{T}}{ds}\right\| = \left\|\frac{d\boldsymbol{T}}{dt}\frac{d\boldsymbol{t}}{ds}\right\| = \left\|\frac{d\boldsymbol{T}}{dt}\Big/\frac{ds}{d\boldsymbol{t}}\right\| = \left\|\frac{d\boldsymbol{T}}{dt}/v(t)\right\| = \frac{1}{v(t)}\left\|\frac{d\boldsymbol{T}}{dt}\right\|$$

Where, we used the fact the speed, $v(t)$, is the rate of change of the distance, i.e. $\frac{ds}{dt} = v(t)$.

As desired, the curvature κ is now expressible as a function of the original parameter, t.

To illustrate, let's use this formula to recompute the curvature from example 2.

Example 3: Compute the curvature for a circle of radius R given in example 2, without creating an arc length parametrization.

$$\boldsymbol{r}(t) = \langle R\,cos(t)\,, R\,sin(t)\,\rangle$$

Solution: We can start by computing the unit tangent vector as a function of t.

$$\begin{aligned}\boldsymbol{T} &= \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} \\ &= \frac{\cancel{R}\langle -\,sin(t)\,, cos(t)\,\rangle}{\cancel{R}\sqrt{sin^2(t) + cos^2(t)}} \\ &= \langle -\,sin(t)\,, cos(t)\,\rangle\end{aligned}$$

Also note the following:

$$v(t) = \|\boldsymbol{v}(t)\| = \|\boldsymbol{r}'(t)\| = R$$

We can now directly apply the 'new' curvature formula from above.

$$\begin{aligned}\kappa(t) &= \frac{1}{v(t)}\left\|\frac{d\boldsymbol{T}}{dt}\right\| \\ &= \frac{1}{R}\sqrt{(-cos(t))^2 + (-\,sin(t))^2} \\ &= \frac{1}{R}\sqrt{cos^2(t) + sin^2(t)} = \frac{1}{R}\end{aligned}$$

Which is the same result from example 2, as expected.

We can also use our 'new' curvature expression to derive yet another option for calculating curvature. We start by manipulating the expression for the unit tangent vector as follows.

$$\frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} = \boldsymbol{T}(t) \rightarrow \boldsymbol{r}'(t) = \boldsymbol{T}(t)\|\boldsymbol{r}'(t)\| = \boldsymbol{T}(t)\|\boldsymbol{v}(t)\| = \boldsymbol{T}(t)v(t)$$

Next, we take the derivative of the right-hand side using the scalar product rule.

$$\boldsymbol{r}''(t) = \frac{d}{dt}\big(\boldsymbol{T}(t)v(t)\big) = \boldsymbol{T}'(t)v(t) + \boldsymbol{T}(t)v'(t)$$

Now, we compete the cross product between $\boldsymbol{r}'(t)$ and $\boldsymbol{r}''(t)$.

$$\begin{aligned}\boldsymbol{r}'(t) \times \boldsymbol{r}''(t) &= \big(\boldsymbol{T}(t)v(t)\big) \times \big(\boldsymbol{T}'(t)v(t) + \boldsymbol{T}(t)v'(t)\big) \\ &= \big(\boldsymbol{T}(t)v(t)\big) \times \big(\boldsymbol{T}'(t)v(t)\big) + \big(\boldsymbol{T}(t)v(t)\big) \times \big(\boldsymbol{T}(t)v'(t)\big) \\ &= v^2(t)\big(\boldsymbol{T}(t) \times \boldsymbol{T}'(t)\big) + \cancel{v(t)v'(t)\big(\boldsymbol{T}(t) \times \boldsymbol{T}(t)\big)} \\ &= v^2(t)\big(\boldsymbol{T}(t) \times \boldsymbol{T}'(t)\big)\end{aligned}$$

Where, we used the fact that $\left(\boldsymbol{T}(t) \times \boldsymbol{T}(t)\right) = \boldsymbol{0}$, since $\boldsymbol{T}$ is, of course, parallel to itself.

Next, we take the magnitude on both sides to finish the derivation below as follows:

$$\begin{aligned}
\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\| &= \left\|v^2(t)\left(\boldsymbol{T}(t) \times \boldsymbol{T}'(t)\right)\right\| \\
&= v^2(t)\left\|\left(\boldsymbol{T}(t) \times \boldsymbol{T}'(t)\right)\right\| \\
&= v^2(t)\|\boldsymbol{T}(t)\|\|\boldsymbol{T}'(t)\| \, sin\left(\frac{\pi}{2}\right) \\
&= v^{\not 2}(t)\left\|\frac{\boldsymbol{r}'(t)}{\cancel{v(t)}}\right\|\left\|\frac{d\boldsymbol{T}}{dt}\right\| \\
&= v(t)\|\boldsymbol{v}(t)\|\left(\kappa(t)v(t)\right) \\
\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\| &= v^3(t)\kappa(t) \\
\kappa(t) &= \frac{\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\|}{v^3(t)}
\end{aligned}$$

Where, we used the fact that $\boldsymbol{T}(t)$ is orthogonal to $\boldsymbol{T}'(t)$**,** which is proved as follows.

Since $\boldsymbol{T}(t)$ is a unit vector we have.

$$\boldsymbol{T}(t) \cdot \boldsymbol{T}(t) = \boldsymbol{1}$$

Next, we differentiate both sides using the product rule.

$$\begin{gathered}
\frac{d}{dt}\left(\boldsymbol{T}(t) \cdot \boldsymbol{T}(t)\right) = \frac{d}{dt}(\boldsymbol{1}) \\
\boldsymbol{2T}'(t) \cdot \boldsymbol{T}(t) = \boldsymbol{0}
\end{gathered}$$

Since the dot product is zero the two vectors must be orthogonal, (assuming neither is $\boldsymbol{0}$).

Curvature Defined for Arbitrary Parameterizations
If $\boldsymbol{r}(t)$ is an arbitrary parameterization, the curvature can be computed with either of the two formulas: $\kappa(t) = \frac{1}{v(t)}\left\|\frac{d\boldsymbol{T}}{dt}\right\|$ $\qquad$ $\kappa(t) = \frac{\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\|}{\|\boldsymbol{r}'(t)\|^3}$

Let's illustrate these formulas with the example below.

Example 4: Compute the curvature using the parametrization given; which is not and arc length parametrization. Use both formulas and verify the answers are identical.

$$\boldsymbol{r}(t) = \langle 1,2t,3t^2\rangle$$

Solution: We start using the first formula, which requires the computation of $\boldsymbol{T}(t)$.

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} = \frac{\langle 0,2,6t\rangle}{\sqrt{4+36t^2}} = \frac{\cancel{2}\langle 0,1,3t\rangle}{\cancel{2}\sqrt{1+9t^2}} = \langle 0, \frac{1}{\sqrt{1+9t^2}}, \frac{3t}{\sqrt{1+9t^2}}\rangle$$

Next, we compute the derivative, which we leave for the reader to verify.

$$\frac{d\boldsymbol{T}}{dt} = \langle 0, \frac{9t}{(1+9t^2)^{3/2}}, \frac{3}{(1+9t^2)^{3/2}}\rangle$$

Now we take the magnitude.

$$\left\|\frac{d\boldsymbol{T}}{dt}\right\| = \sqrt{\left(\frac{1}{(1+9t^2)^{3/2}}\right)^2 (81t^2+9)}$$

$$= \sqrt{\frac{9}{(1+9t^2)^{\cancel{3}2}}\cancel{(1+9t^2)}} = \frac{3}{(1+9t^2)}$$

Finally, the curvature is computed as follows:

$$\kappa(t) = \frac{1}{v(t)}\left\|\frac{d\boldsymbol{T}}{dt}\right\|$$

$$= \left(\frac{1}{2\sqrt{1+9t^2}}\right)\left(\frac{3}{(1+9t^2)}\right)$$

$$= \frac{3}{2(1+9t^2)^{3/2}}$$

For the second formula we start with the cross product.

$$\boldsymbol{r}'(t) \times \boldsymbol{r}''(t) = \left\|\begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ 0 & 2 & 6t \\ 0 & 0 & 6 \end{vmatrix}\right\| = \|\langle 12,0,0\rangle\| = 12$$

Furthermore,

$$\|\boldsymbol{r}'(t)\|^3 = \left(2\sqrt{1+9t^2}\right)^3 = 8(1+9t^2)^{3/2}$$

The curvature is then computed as follows:

$$\begin{aligned}\kappa(t) &= \frac{\|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\|}{\|\boldsymbol{r}'(t)\|^3} \\ &= \frac{12}{8(1+9t^2)^{3/2}} \\ &= \frac{3}{2(1+9t^2)^{3/2}}\end{aligned}$$

Which is identical to the curvature found using the first formula.

One last formula for curvature pertains to the special case where the curve is in a plane and can be expressed as $y = f(x)$, in which case we can express the curve in parametrized from as:

$$\boldsymbol{r}(t) = \langle t, f(t), 0\rangle$$

Using the second formula we have

$$\boldsymbol{r}'(t)\times\boldsymbol{r}''(t) = \left\|\begin{vmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0\end{vmatrix}\right\| = \|\langle 0,0,f''(t)\rangle\| = |f''(t)|$$

The curvature can then be expressed as:

$$\kappa(t) = \frac{\|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)\|}{\|\boldsymbol{r}'(t)\|^3} = \frac{|f''(t)|}{\left(\sqrt{1+\left(f'(t)\right)^2}\right)^{\mathbf{3}}}$$

Curvature of a Graph in the Plane
The curvature of the graph of $y = f(x)$ is equal to $$\kappa(x) = \frac{|f''(x)|}{\left(1+\left(f'(x)\right)^2\right)^{3/2}}$$

Example 5: Compute the curvature of $f(x) = x^3 - 3x^2 + 4$.

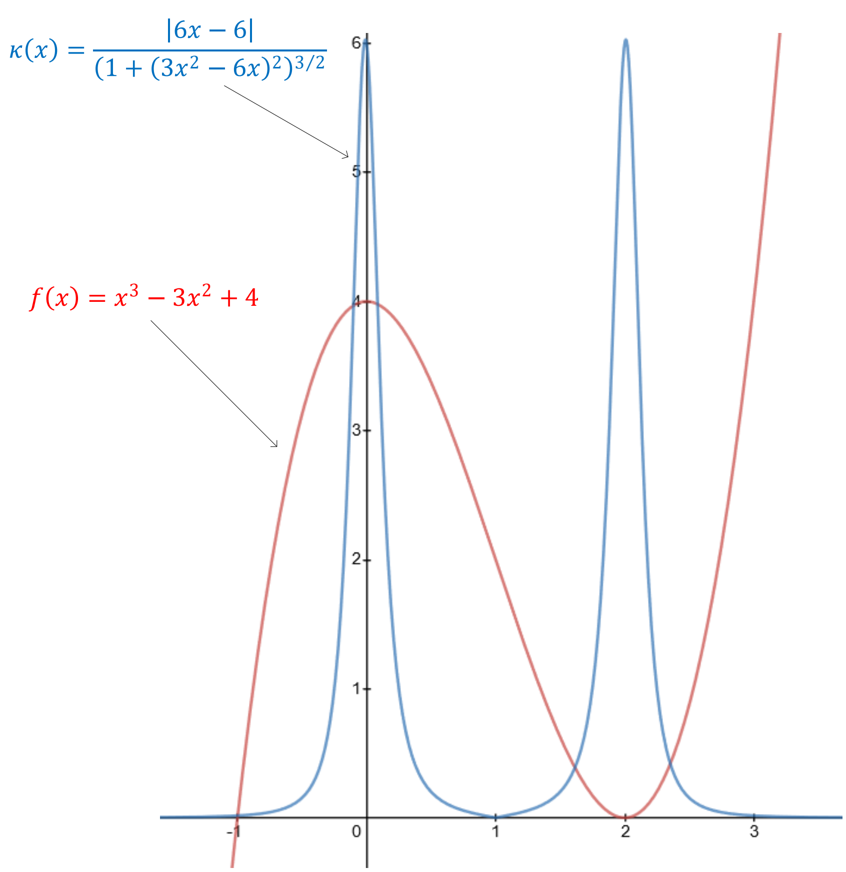
Solution:

$$f'(x) = 3x^2 - 6x \qquad\qquad f''(x) = 6x - 6$$

$$\kappa(x) = \frac{|f''(x)|}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}}$$

$$\kappa(x) = \frac{|6x - 6|}{(1 + (3x^2 - 6x)^2)^{3/2}}$$

The graphs of $f(x)$ and $\kappa(x)$ are shown below for illustrative purposes.



We finish this section with a few more examples before summarizing the results.

Example 6: Compute the curvature for the following curves.

a. $\boldsymbol{r}(t) = \langle 1, e^t, t\rangle$

b. $\boldsymbol{r}(t) = \langle {}^1/_t, {}^1/_{t^2}, t^2\rangle$, at $t = -1$

c. $f(x) = e^x$

d. $f(x) = cos(x)$

Solution: For *a.* and *b.,* we use the following formula.

$$\kappa(t) = \frac{\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\|}{\|\boldsymbol{r}'(t)\|^3}$$

a.

$$\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\| = \left\|\begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ 0 & e^t & 1 \\ 0 & e^t & 0 \end{vmatrix}\right\| = \|\langle -e^t, 0,0\rangle\| = e^t \qquad \|\boldsymbol{r}'(t)\|^3 = \left(\sqrt{e^{2t}+1}\right)^{\mathbf{3}}$$

$$\kappa(t) = \frac{e^t}{(e^{2t}+1)^{3/2}}$$

b.

$$\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\| = \left\|\begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ -{}^1/_{t^2} & -{}^2/_{t^3} & 2t \\ {}^2/_{t^3} & {}^6/_{t^4} & 2 \end{vmatrix}\right\|$$

$$= \left\|\left\langle\left(-{}^4/_{t^3} - {}^{12}/_{t^3}\right), \left({}^4/_{t^2} + {}^2/_{t^2}\right), \left(-{}^6/_{t^6} + {}^4/_{t^6}\right)\right\rangle\right\|$$

$$= \left\|\left\langle\left(-{}^{16}/_{t^3}\right), \left({}^6/_{t^2}\right), \left(-{}^2/_{t^6}\right)\right\rangle\right\|$$

$$= \sqrt{{}^{256}/_{t^6} + {}^{36}/_{t^4} + {}^4/_{t^{12}}}$$

$$\|\boldsymbol{r}'(t)\|^3 = \left(\sqrt{{}^1/_{t^4} + {}^4/_{t^6} + 4t^2}\right)^{\mathbf{3}}$$

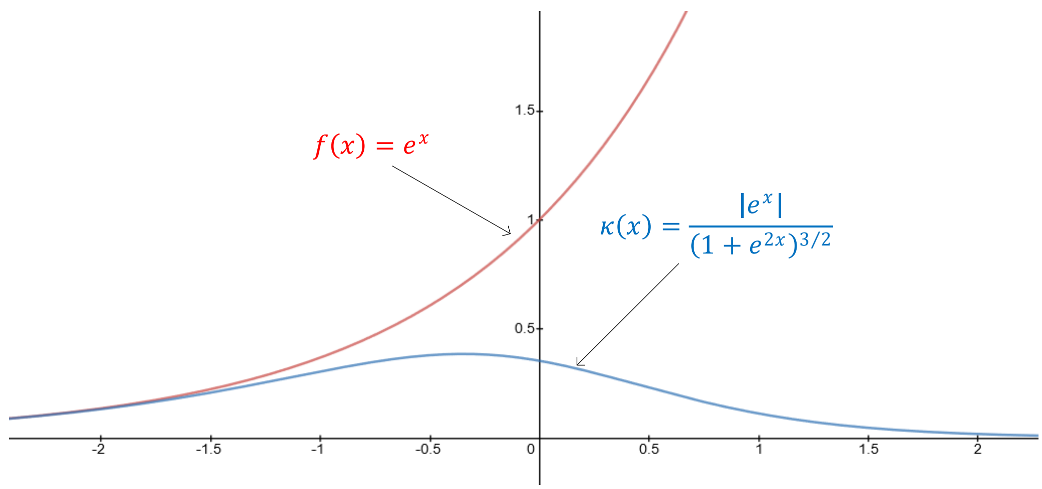
$$\kappa(-1) = \left.\frac{\frac{2}{t^2}\sqrt{{}^{64}/_{t^2} + 9 + {}^1/_{t^8}}}{\left(\sqrt{{}^1/_{t^4} + {}^4/_{t^6} + 4t^2}\right)^{\mathbf{3}}}\right|_{t=-1} = \frac{2\sqrt{74}}{27}$$

For *c.* and d*.,* we use the following formula. We also plot the original function and the curvature for illustration.

$$\kappa(x) = \frac{|f''(x)|}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}}$$

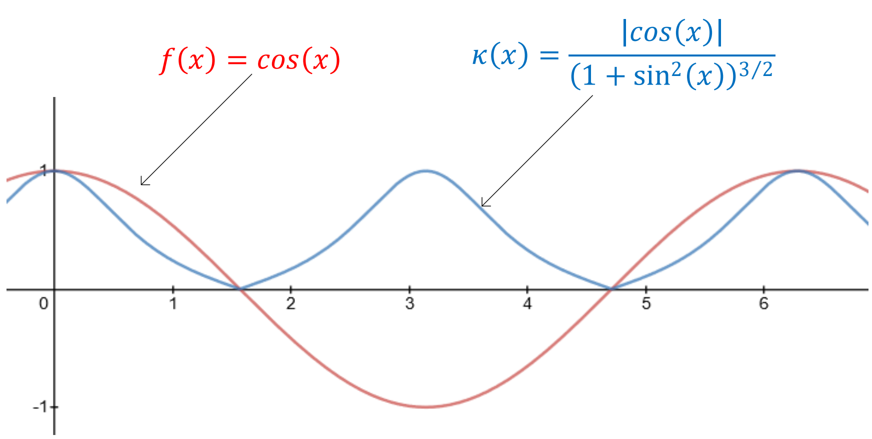
c.

$$\kappa(x) = \frac{|e^x|}{(1 + (e^x)^2)^{3/2}} = \frac{|e^x|}{(1 + e^{2x})^{3/2}}$$



d.

$$\kappa(x) = \frac{|cos(x)|}{(1 + \sin^2(x))^{3/2}}$$



Final Summary for Vector Calculus – Curvature

Curvature
Curvature is a positive numerically positive value that measures how a curve bends. It is defined based using the arc length parametrization of a curve as specified below. Let $\boldsymbol{r}(s)$ be an arc length parameterization and $\boldsymbol{T} = \boldsymbol{T}(s)$ be the unit tangent vector. The curvature at $\boldsymbol{r}(s)$ is defined as follows: $$\kappa(s) = \left\lVert \frac{d\boldsymbol{T}}{ds} \right\rVert$$ Where, $$\boldsymbol{T} = \boldsymbol{T}(s) = \boldsymbol{r}'(s)$$ Note: This assumes $\boldsymbol{r}'(t) \neq 0$ for all t.
Curvature Defined for Arbitrary Parameterizations
Alternate forms for computing the curvature can be derived without using the arc length parametrization as shown below. If $\boldsymbol{r}(t)$ is an arbitrary parameterization, the curvature can be computed with either of the two formulas: $$\kappa(t) = \frac{1}{v(t)} \left\lVert \frac{d\boldsymbol{T}}{dt} \right\rVert \qquad \kappa(t) = \frac{\lVert \boldsymbol{r}'(t) \times \boldsymbol{r}''(t) \rVert}{\lVert \boldsymbol{r}'(t) \rVert^3}$$
Curvature of a Graph in a Plane
The curvature of the graph of $y = f(x)$ is equal to $$\kappa(x) = \frac{\lvert f''(x) \rvert}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}}$$

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