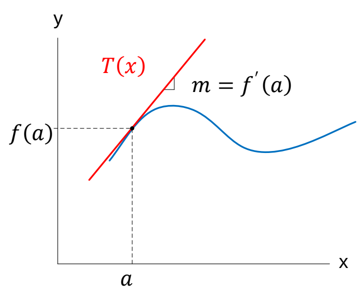
Multivariable Differentiation – Tangent Planes and Linear Approximation

The previous lesson ended with an example where we developed an equation of a tangent plane for a specific two variable function at a given point. In this lesson we will develop a general equation for a tangent plane that can be used for an arbitrary function at an arbitrary point. We will also look at linear approximation for multivariable functions. We studied linear approximation for single variable functions in calculus 1, and of course the idea is very much analogous.

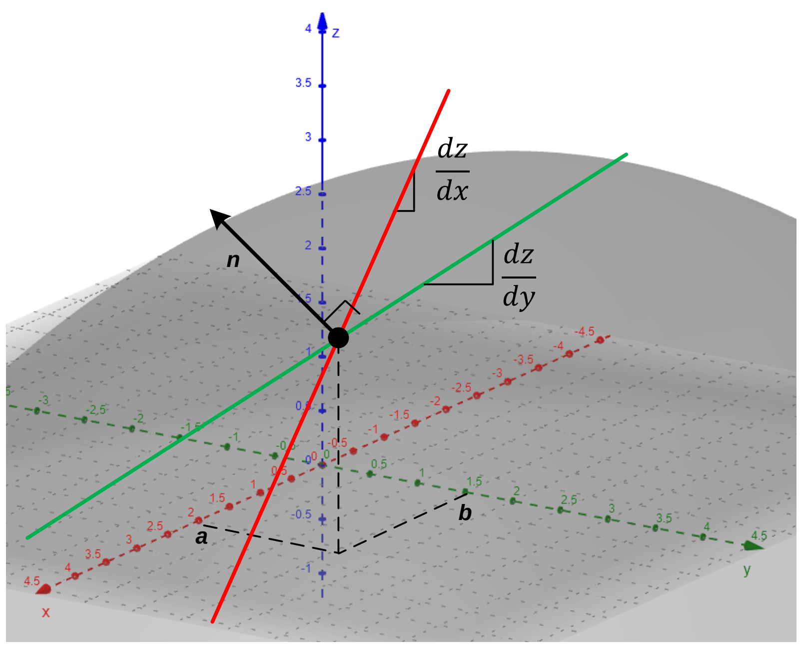
Tangent Planes and Normal Lines to a Surface

The tangent plane for a two variable function is analogous to the tangent line for a single variable function. The general equation of the tangent line for the curve $y = f(x)$ at the point $(a, f(a))$ is a function of the given point and the derivative evaluated at that point. It can be written using the point slope formula as follows:



$$T(x) = f'(a)(x - a) + f(a)$$

We would expect to find a similar dependency for the equation of the tangent plane of the surface, $z = f(x, y)$, i.e. a dependence on the given point and the partial derivatives evaluated at that point. To start we consider the point, $P_0 = (x_0, y_0, z_0)$ on the surface $f(x, y)$. The figure below shows the tangent lines for the surface parallel to the x-z and y-z planes in red and green respectively. The two lines lie in the tangent plane with a normal vector, $\boldsymbol{n}$, which is also shown in the figure below.



The general equation for this tangent plan can be written as follows:

$$\boldsymbol{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle \; = \; 0$$

To complete this expression, we need to derive an expression for the normal vector, which we can find by computing the cross product between the two direction vectors associated with the tangent lines. We can derive the direction vectors, starting with the red line, as follows:

We start at an arbitrary position on the line, then in order to stay on the line we can move dx units in the x direction, dz units in the z direction, and zero units in the y direction. The resulting direction vector is then.

$$\boldsymbol{v}_x \; = \; \langle dx, 0, dz \rangle$$

Next, we can scale this vector by the $\frac{1}{dx}$, which still maintains the direction. Therefore, the final direction vector can be written as follows:

$$\boldsymbol{v}_x \; = \; \langle \frac{dx}{dx}, \frac{0}{dx}, \frac{dz}{dx} \rangle$$

$$\boldsymbol{v}_x \; = \; \langle 1, 0, f_x(x_0, y_0) \rangle$$

A similar argument gives the direction vector for the green line

$$\boldsymbol{v}_y \; = \; \langle 0,1, f_y(x_0, y_0) \rangle$$

Next, we compute the normal vector as

$$\boldsymbol{n} = \boldsymbol{v}_y \times \boldsymbol{v}_x$$

$$= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ 0 & 1 & f_y(x_0, y_0) \\ 1 & 0 & f_x(x_0, y_0) \end{vmatrix} \; = \; \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle$$

With this, the tangent plane is given as

$$0 = \langle f_x(x_0, y_0), -f_y(x_0, y_0), -1 \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$$

$$0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(\, y - y_0) - (\, z - z_0)$$

$$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(\, y - y_0) + z_0$$

Note the similarity between this equation and the equation of the tangent line from above.

We can also derive an expression for the normal line to the surface using the normal vector found, $\boldsymbol{n}$, as the direction vector, and the given point on the surface.

$$\boldsymbol{n}(t) = \boldsymbol{n}(0) + \boldsymbol{n}\boldsymbol{t}$$

$$\boldsymbol{n}(t) = \langle x_0, y_0, z_0 \rangle + \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle t$$

$$\boldsymbol{n}(t) = \langle (x_0 + f_x(x_0, y_0)t), \big(y_0 + f_y(x_0, y_0)t\big), (z_0 - t) \rangle$$

Equation of the Tangent Plane and Normal Line
The tangent plane to the surface $f(x,y)$ at the point (x_0, y_0, z_0) is given by the equation $z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0$ The normal line to the surface is given as $\boldsymbol{n}(t) = \langle (x_0 + f_x(x_0,y_0)t), \big(y_0 + f_y(x_0,y_0)t\big), (z_0 - t)\rangle$

Example 1: Find an equation of the tangent plane to the surface at the given point.

a. $f(x,y) = x^2 + y^{-2}, (4,1)$ *b.* $f(x,y) = x^2y^{-1/2} + y^{-3}, (2,1)$

Solution:

a. Based on the formula derived above we start by finding the partial derivatives.

$$f_x(4,1) = \frac{\partial}{\partial x}(x^2+y^{-2})\bigg|_{(4,1)} = 2x|_{(4,1)} = 8$$

$$f_y(4,1) = \frac{\partial}{\partial y}(x^2+y^{-2})\bigg|_{(4,1)} = -2y^{-3}|_{(4,1)} = -2$$

Then, since $z_0 = f(4,1) = 4^2+1^{-2} = 17$, the tangent plane is given as

$$z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0$$

$$z = 8(x-4) - 2(y-1) + 17$$

$$z = 8x - 2y - 13$$

b.

$$f_x(2,1) = \frac{\partial}{\partial x}\left(x^2y^{-1/2} + y^{-3}\right)\bigg|_{(2,1)} = 2xy^{-1/2}\big|_{(2,1)} = 4$$

$$f_y(2,1) = \frac{\partial}{\partial y}\left(x^2y^{-1/2} + y^{-3}\right)\bigg|_{(2,1)} = \frac{-x^2y^{-3/2}}{2} - 3y^{-4}\bigg|_{(2,1)} = -5$$

Again, with $z_0 = f(2,1) = 2^2(1)^{-1/2} + (1)^{-3} = 5$, the tangent plane is given as

$$z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0$$

$$z = 4(x-2) - 5(y-1) + 5$$

$$z = 4x - 5y + 2$$

Example 2: Find the points on the graph of $f(x, y) = 3x^2 - 4y^2$ where the vector $\boldsymbol{v} = \langle 3, 2, 2\rangle$ is normal to the tangent plane.

Solution: The equation from which we derived the formula for a tangent plane to a surface was

$$\boldsymbol{n} \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$$

Substituting the normal vector given and performing some algebra we get the following

$$0 = \langle 3, 2, 2\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle$$

$$0 = 3(x - x_0) + 2(y - y_0) + 2(z - z_0)$$

$$z = -\frac{3}{2}(x - x_0) - 1(y - y_0) + z_0$$

Next, recall the general formula we derived for the tangent plane

$$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + z_0$$

Comparing the two equations we see that we must find x_0 and y_0 such that $f_x(x_0, y_0) = -\frac{3}{2}$ and $f_y(x_0, y_0) = -1$

$$f_x(x_0, y_0) = -\frac{3}{2}$$

$$6x_0 = -\frac{3}{2}$$

$$x_0 = -\frac{1}{4}$$

$$f_y(x_0, y_0) = -1$$

$$-8y_0 = -1$$

$$y_0 = \frac{1}{8}$$

From there we can find z_0

$$z_0 = 3{x_0}^2 - 4{y_0}^2 = \frac{1}{8}$$

Finally, we can say that the point on the graph is

$$P = \left(-\frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right)$$

Example 3: Find the points on the graph of $f(x, y) = xy^3 + 8y^{-1}$ where the tangent plane is parallel to $2x + 7y + 2z = 0$.

Solution: Two planes are parallel if their normal vectors are parallel. A vector $\boldsymbol{n}_2$ is parallel to $\boldsymbol{n}_1$ if $\boldsymbol{n}_2 = C\boldsymbol{n}_1$. The tangent plane associated with this normal vector is

$$0 = \cancel{C}\boldsymbol{n}_1 \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$$

$$0 = \langle 2, 7, 2 \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$$

$$0 = 2(x - x_0) + 7(y - y_0) + 2(z - z_0)$$

$$z = -1(x - x_0) - \frac{7}{2}(y - y_0) + z_0$$

As in the previous example, we write the general formula for the tangent plane as follows:

$$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + z_0$$

Therefore, we must find x_0 and y_0 such that $f_x(x_0, y_0) = -1$ and $f_y(x_0, y_0) = -\frac{7}{2}$

$$f_x(x_0, y_0) = -\frac{3}{2}$$

$$y_0{}^3 = -1$$

$$y_0 = -1 \quad \rightarrow$$

$$f_y(x_0, y_0) = -1$$

$$3x_0 y_0{}^2 - \frac{8}{y_0{}^2} = -\frac{7}{2}$$

$$3x_0(-1)^2 - \frac{8}{(-1)^2} = -\frac{7}{2}$$

$$3x_0 = \frac{-\frac{7}{2} + 8}{3}$$

$$x_0 = \frac{3}{2}$$

From there we can find z_0

$$z_0 = xy^3 + 8y^{-1}$$

$$z_0 = \frac{3}{2}(-1)^3 + \frac{8}{-1} = -\frac{19}{2}$$

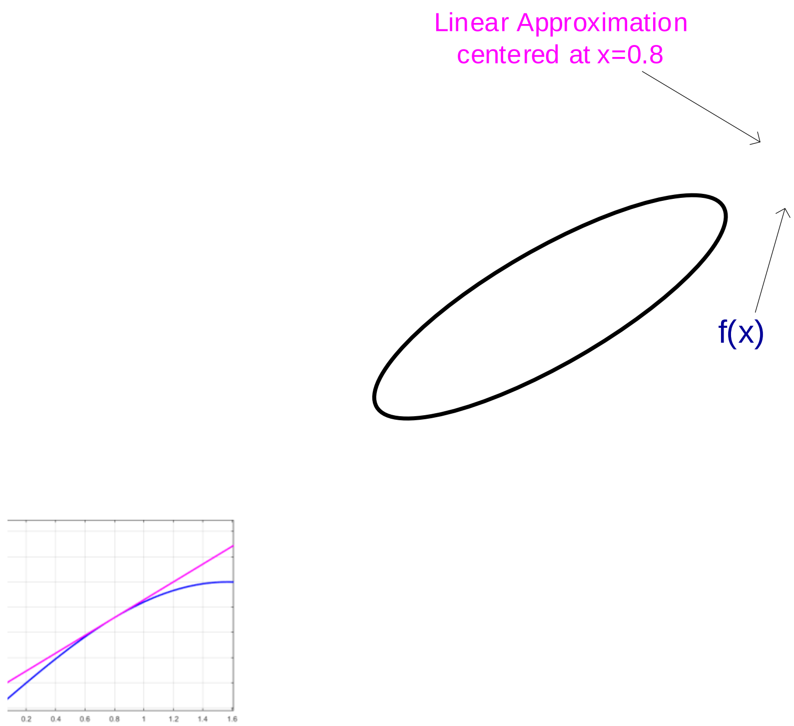
Finally, we can say that the point on the graph is

$$P = \left(\frac{3}{2}, -1, -\frac{19}{2}\right)$$

Linear Approximation

For single variable functions linear approximation allowed us to approximate the value of an otherwise complex function using a linear function instead. As the figure below shows the approximation is valid over a small region around where the linear function is created. The linear function is, of course, the equation of the tangent line evaluated at a point, e.g. $x = a$. The linear approximation of $f(x)$ around the point $(a, f(a))$ is sometimes referred to as the linearization of $f(x)$ at $x = a$.

$$L(x) = f'(a)(x - a) + f(a)$$



The linear approximation for two variable function is completely analogous. In other words, the linear approximation of $f(x, y)$ around the point $(a, b, f(a, b))$ is given by the equation of the tangent plane at that point.

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

Example 4: Given $f(x, y) = x^2/(y^2 + 1)$, approximate the value of $f(4.01, 0.98)$.

Solution: We'll start by creating the linear approximation of $f(x, y)$ at $(4,1)$.

$$L(x, y) = f_x(4,1)(x - 4) + f_y(4,1)(y - 1) + f(4,1)$$

Next, we compute the partials.

$$f_x(4,1) = \frac{\partial}{\partial x}\left(\frac{x^2}{(y^2 + 1)}\right)\Bigg|_{(4,1)} = \left(\frac{2x}{(y^2 + 1)}\right)\Bigg|_{(4,1)} = 4$$

$$f_y(4,1) = \frac{\partial}{\partial y}\left(\frac{x^2}{(y^2 + 1)}\right)\Bigg|_{(4,1)} = \left(-\frac{2yx^2}{(y^2 + 1)^2}\right)\Bigg|_{(4,1)} = -8$$

And since $f(4,1) = 8$, the linear approximation is

$$L(x, y) = 4(x - 4) - 8(y - 1) + 8$$

Therefore, we can approximate $f(4.01, 0.98)$ as follows:

$$f(4.01, 0.98) \cong L(4.01, 0.98) = 4(4.01 - 4) - 8(0.98 - 1) + 8 = 8.2$$

For comparison the actual value computed in a calculator is

$$f(4.01, 0.98) = 8.202458682$$

Linear approximation of Δf, (Differentials)

As we have seen above, to more easily compute values of $f(x, y)$ near a point, (a, b), we can use the linear approximation to the function around that point.

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

Once this function is found we can use it to approximate values near (a, b). In other words, $f(a + \Delta x, b + \Delta y) \cong L(a + \Delta x, b + \Delta y)$, where

$$\begin{aligned} L(a + \Delta x, b + \Delta y) &= f_x(a, b)(\cancel{a} + \Delta x \cancel{- a}) + f_y(a, b)(\cancel{b} + \Delta y \cancel{- b}) + f(a, b) \\ &= f_x(a, b)\Delta x + f_y(a, b)\Delta y + f(a, b) \end{aligned}$$

What if instead of the actual value of the function we just wanted to approximate the change in the output for a change in the inputs? The exact value would be computed as follows:

$$\Delta f \; = f(a + \Delta x, b + \Delta y) - f(a, b)$$

However, using the approximated value of $f(a + \Delta x, b + \Delta y)$ from above we have

$$\Delta f \; \cong \left(f_x(a, b)\Delta x + f_y(a, b)\Delta y + \cancel{f(a, b)}\right) \cancel{- f(a, b)}$$

$$\Delta f \; \cong f_x(a, b)\Delta x + f_y(a, b)\Delta y$$

With this expression, we can define another convenient function called the *differential* of f, df. By allowing both Δx and Δy to become exceedingly small, i.e. $\Delta x \to dx$ and $\Delta y \to dy$, we can write

$$\begin{aligned} df &= f_x(x, y)dx + f_y(x, y)dy \\ &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \end{aligned}$$

Geometrically df refers to a change in the height of the tangent plane for a given change in x and y, i.e. dx and dy.

Linear Approximation and Differentials

The linear approximation of $f(x, y)$ around the point $\left(a, b, f(a, b)\right)$ is given by the equation of the tangent plane at that point.

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(\, y - b) + f(a, b)$$

The value of a function, $f(x, y)$, at $(a + \Delta x, b + \Delta y)$ can be approximated by this linearization, $L(x, y)$, as

$$f(a + \Delta x, b + \Delta y) \cong f_x(a, b)\Delta x + f_y(a, b)\Delta y + f(a, b)$$

Note: This can be extended to any number variables. In three variables we have:

$$f(a + \Delta x, b + \Delta y, c + \Delta z) \cong f_x(a, b, c)\Delta x + f_y(a, b, c)\Delta y + f_z(a, b, c)\Delta z + f(a, b)$$

If Δx and Δy are sufficiently small, then we can approximate Δf as

$$\Delta f \; \cong f_x(a, b)\Delta x + f_y(a, b)\Delta y$$

The differential of $f(x, y)$ is defined as

$$\begin{aligned} df &= f_x(x, y)dx + f_y(x, y)dy \\ &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \end{aligned}$$

Example 5: Use a linear approximation to estimate the value of the following expressions. Then compare the approximation with the value computed with a calculator.

a. $\sqrt{3.01^2 + 3.99^2}$

b. $\sqrt{(1.9)(2.02)(4.05)}$

Solution:

a. We can use the function $f(x,y) = \sqrt{x^2 + y^2}$ with $a = 3$, and $b = 4$. The x and y values we would like to estimate are $(3.0 + 0.01)$ and $\big(4.0 + (-0.01)\big)$. Therefore, we have

$$f(a + \Delta x, b + \Delta y,) \cong f_x(a,b)\Delta x + f_y(a,b)\Delta y + f(a,b)$$
$$f\big(3.0 + 0.01, 4.0 + (-0.01)\big) \cong f_x(3,4)(0.01) + f_y(3,4)(-0.01) + f(3,4)$$

$$f_x(3,4) = \left.\frac{x}{\sqrt{x^2 + y^2}}\right|_{(3.4)} = \frac{3}{\sqrt{25}} = 0.6$$

$$f_y(3,4) = \left.\frac{y}{\sqrt{x^2 + y^2}}\right|_{(3.4)} = \frac{4}{\sqrt{25}} = 0.8$$

$$f(3,4) = \sqrt{25} = 5$$

Therefore,

$$\sqrt{3.01^2 + 3.99^2} \cong 0.6(0.01) + 0.8(-0.01) + 5$$
$$\cong 4.998$$

With a calculator we get

$$\sqrt{3.01^2 + 3.99^2} = 4.998019608$$

b. We can use the function $f(x,y,z) = \sqrt{xyz}$ with $a = 2$, $b = 2$, and $c = 4$. In this case we have a three variable function that we would like to estimate at x, y, and z of $(2.0 - 0.1)$, $(2.0 + 0.02)$, and $(4.0 + 0.05)$ respectively. Therefore, we have

$$f(a + \Delta x, b + \Delta y, c + \Delta z) \cong f_x(a,b,c)\Delta x + f_y(a,b,c)\Delta y + f_z(a,b,c)\Delta z + f(a,b,c)$$
$$\cong f_x(2,2,4)(-0.1) + f_y(2,2,4)(0.02) + f_z(2,2,4)(0.05) + f(2,2,4)$$

$$f_x(2,2,4) = \left.\frac{yz}{2\sqrt{xyz}}\right|_{(2,2,4)} = \frac{8}{2\cdot 4} = 1 \qquad f_y(2,2,4) = \left.\frac{xz}{2\sqrt{xyz}}\right|_{(2,2,4)} = \frac{8}{2\cdot 4} = 1 \qquad f_z(2,2,4) = \left.\frac{xy}{2\sqrt{xyz}}\right|_{(2,2,4)} = \frac{4}{2\cdot 4} = 0.5$$

And, $f_z(2,2,4) = \sqrt{(2)(2)(4)} = 4$.

Therefore,

$$\begin{aligned}\sqrt{(1.9)(2.02)(4.05)} &\cong (1)(-0.1) + (1)(0.02) + (0.5)(0.05) + 4 \\ &\cong 3.945\end{aligned}$$

And with a calculator we get

$$\sqrt{(1.9)(2.02)(4.05)} = 3.9425753$$

Example 6: A person's body mass index, BMI, is defined as $I = W/H^2$, where W is the body weight in kilograms and H is the body height in meters.

a. Estimate the change in BMI if the weight and height change from $(40, 1.45)$ to $(41.5, 1.47)$.

b. Estimate the change in height that will decrease I by 2.0 if $(W, H) = (84, 1.7)$. Then estimate the change in weight that will do the same.

Solution:

a. In this case we can use the concept of differentials to compute the differential of BMI.

$$dI = \frac{\partial I}{\partial W}dW + \frac{\partial I}{\partial H}dH$$

We start by computing the partials at $(40, 1.45)$

$$\left.\frac{\partial I}{\partial W}\right|_{(40,1.45)} = \frac{1}{1.45^2} \cong 0.48 \qquad \left.\frac{\partial I}{\partial H}\right|_{(40,1.45)} = \frac{-2(40)}{1.45^3} \cong -26.24$$

Also, we have

$$dW = 41.5 - 40 = 1.5 \qquad\qquad dH = 1.47 - 1.45 = 0.02$$

Therefore,

$$dI = (0.48)(1.5) + (-26.24)(0.02) \cong 0.2$$

b. We start with the same differential relationship.

$$dI = \frac{\partial I}{\partial W}dW + \frac{\partial I}{\partial H}dH$$

However, in this case we solve for dH and dW respectively.

$$dH = \frac{dI - \frac{\partial I}{\partial W}dW}{\frac{\partial I}{\partial H}} \qquad\qquad dW = \frac{dI - \frac{\partial I}{\partial H}dH}{\frac{\partial I}{\partial W}}$$

The partials are then computed at $(84, 1.7)$.

$$\left.\frac{\partial I}{\partial W}\right|_{(84,1.7)} = \frac{1}{1.7^2} \cong 0.346 \qquad\qquad \left.\frac{\partial I}{\partial H}\right|_{(84,1.7)} = \frac{-2(84)}{1.7^3} \cong -34.195$$

In both cases we would like the BMI to decrease by 2, i.e. $dI = -2$. Furthermore, in the first case $dW = 0$, and in the second case $dH = 0$.

Therefore,

$$dH = \frac{(-1) - \cancel{(0.346)(0)}}{(-34.195)} \cong 0.06\ m \qquad\qquad dW = \frac{(-1) - \cancel{(-34.195)(0)}}{(0.346)} \cong -5.78\ kg$$

Which is equivalent to about $2\frac{1}{2}$ inches

Which is equivalent to about 13 pounds.

Final Summary for Multivariable Differentiation – Tangent Planes and Linear Approximation

Equation of the Tangent Plane and Normal Line
The tangent plane to the surface, $f(x, y)$, at the point (x_0, y_0, z_0) is given by $$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(\,y - y_0) + z_0$$ The normal line to the surface is given as $$\boldsymbol{n}(t) = \langle (x_0 + f_x(x_0, y_0)t), \big(y_0 + f_y(x_0, y_0)t\big), (z_0 - t) \rangle$$
Linear Approximation and Differentials
The linear approximation of $f(x, y)$ around the point $\big(a, b, f(a, b)\big)$ is given by the equation of the tangent plane at that point. $$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(\,y - b) + f(a, b)$$ The value of a function, $f(x, y)$, at $(a + \Delta x, b + \Delta y)$ can be approximated by this linearization, $L(x, y)$, as $$f(a + \Delta x, b + \Delta y) \cong f_x(a, b)\Delta x + f_y(a, b)\Delta y + f(a, b)$$ Note: This can be extended to any number variables. In three variables we have: $$f(a + \Delta x, b + \Delta y, c + \Delta z) \cong f_x(a, b, c)\Delta x + f_y(a, b, c)\Delta y + f_z(a, b, c)\Delta z + f(a, b)$$ If Δx and Δy are sufficiently small, then we can approximate Δf as $$\Delta f \;\cong f_x(a, b)\Delta x + f_y(a, b)\Delta y$$ The differential of $f(x, y)$ is defined as $$df = f_x(x, y)dx + f_y(x, y)dy$$ $$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

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