<u>Multivariable Differentiation – Multivariable Functions</u>

So far, we have mostly worked with single variable functions. In other words, the output, which may be a multidimensional vector, depends on a single independent variable. On the other hand, for multivariable functions the output depends on more than one independent variable. For clarity here are some examples of different types of functions.

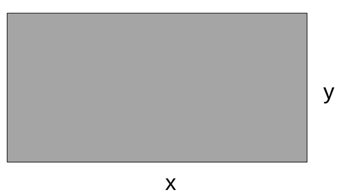
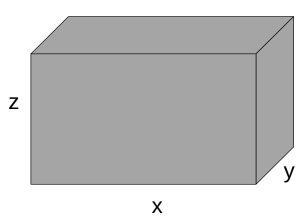
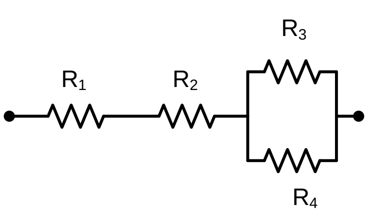
Function Type	***Example***
Single Variable Function	$f(x) = 3x^2 + 7x + 4$
Single Variable Vector-Valued Function	$\boldsymbol{r}(t) = \langle 3t^2, 7t, 4 \rangle$
Multivariable Function	$f(x, y) = 3x^2y + 7xy + 4 + \sqrt{y}$
Multivariable Vector-Valued Function	$\boldsymbol{r}(t, u) = \langle 3t^2u, 7tu, 4 + \sqrt{u} \rangle$

In this section we study real-valued multivariable functions. We have already seen these functions when we studied quadric surfaces. As an example, the elliptical paraboloid can be represented as follows

$$f(x, y) = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$$

Which is an example of a function of two variables.

Of course, there are many other examples of functions that depend on more than one variable.

Area of Rectangle 	Two Variable function $A = f(x, y) = xy$
Volume of a Rectangular Box 	Three Variable function $V = f(x, y, z) = xyz$
Total Resistance of the Circuit 	Four Variable function $R_T = f(R_1, R_2, R_3, R_4) = R_1 + R_2 + \dfrac{R_3R_4}{R_3 + R_4}$

Multivariable Functions
A multivariable function is one that takes n real variables as inputs, $(x_1, x_2, \dots, x_n)$, and assigns a single value, y, to each n-tuple $(x_1, x_2, \dots, x_n)$ in a domain in R^n. The range is the set of all y values for the $(x_1, x_2, \dots, x_n)$ in the domain. • $(x_1, x_2, \dots, x_n)$ are called the independent variables. • y is the dependent variable. The function is represented as $$y = f(x_1, x_2, \dots, x_n)$$

Since we focus on functions of two or three variables, we will mostly use (x, y) or (x, y, z) respectively as the independent variables.

Example 1: Find the domain of the following multivariable functions.

a. $f(x, y) = \sqrt{4 - x^2 - y}$ *b.* $f(x, y, z) = x\sqrt{y} + ln(z - 1)$

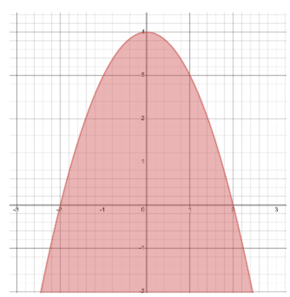
Solution:

a. In this case, the function outputs real values when

$$4 - x^2 - y \geq 0$$

With a little algebra we can write:

$$y \leq 4 - x^2$$



b. In this case we have separate conditions that define the domain related to invalid y and z values.

$$y \geq 0$$

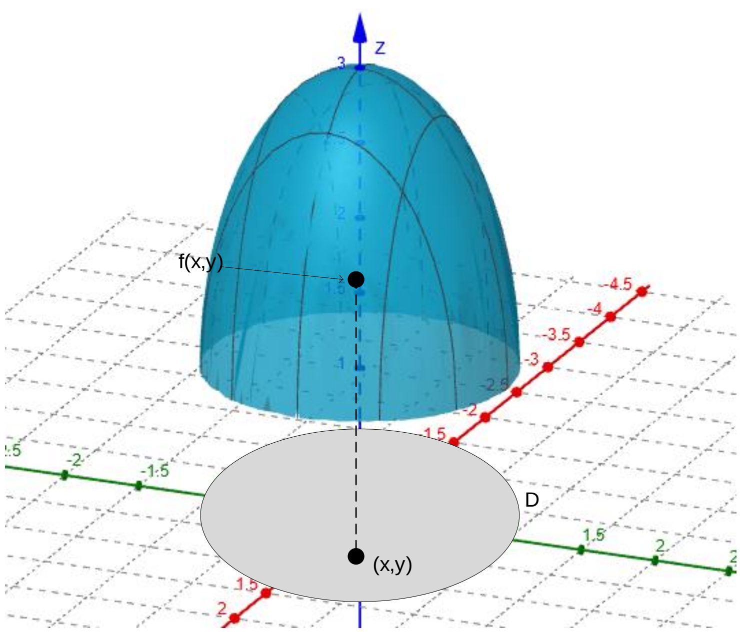
$$z - 1 > 0$$
$$z > 1$$

The domain can then be given by

$$D = \{(x, y, z) : y \geq 0, z > 1\}$$

Graphing Functions

Graphs of single variable functions provided us with insight into function behavior that the equation alone was not able to easily provide. Similarly, graphing multivariable functions allows us to gain insight into function behavior that may not be as obvious from just the expression. However, graphing multivariable functions by hand can be difficult. Single variable functions are represented on a two dimensional coordinate system and are generally easy to plot by hand on a sheet of paper. Two-variable functions can be graphed as surfaces in three dimensions, which are generally more difficult to represent on a sheet of paper. Unfortunately, higher dimensions functions are impossible graph. To visualize three-variable function we can use level surfaces, which are analogous to level curves. For example, to visualize the temperature in a certain volume of space, $T(x, y, z)$, we can draw level surfaces, $T(x, y, z) = c$, which represent locations where the temperature was constant. For functions of four or more variables it is impossible to visualize. In this section we'll focus on two variable functions, which are graphed as surfaces in R^3. Below is a computer generated example of a surface.



In the figure above D represents the set of points, (x, y), from R^2 in the domain of the two variable function, $f(x, y)$. The variables, x and y, are represented as points on a two dimensional horizontal plane, and the output value, $z = f(x, y)$, is represented as the vertical distance from the horizontal plane at $z = 0$.

Fortunately, computer algebra systems can be used to explore multivariable functions graphically and should be used whenever possible. However, alternate techniques exist that can be used without, (or even with), a computer algebra system as a useful way to analyze the graph of $f(x, y)$. We examine these techniques next.

Traces, Level Curves, and Contour Maps

We saw traces when we studied quadric surfaces in an earlier lesson. In this lesson, however, we will make the concept of traces more precise, starting by defining the following terms.

- *Vertical Trace*
  - The intersection of the graph with a vertical plane obtained by setting x or y to a.
    - Vertical trace parallel with the y-z plane: Consists of all points $(a, y, f(a, y))$.
    - Vertical trace parallel with the x-z plane: Consists of all points $(x, a, f(x, a))$.
- *Horizontal Trace*
  - The intersection of the graph with a horizontal plane obtained by setting $f(x, y)$ to c.
    - Horizontal traces are parallel to the x-y plane and consist of all points (x, y, c).
- *Level Curve*
  - The projection of a horizontal trace in the x-y plane.
    - The curve $f(x, y) = c$ in the x-y plane.
- *Contour Map*
  - A plot in the x-y plane showing level curves $f(x, y) = c$ for equally spaced values of c.
- *Contour Interval*
  - The interval, m, between the level curves in a contour map.
  - When moving from one level curve to the next, the value of $f(x, y)$ changes by $\pm m$.

We explore these concepts with examples.

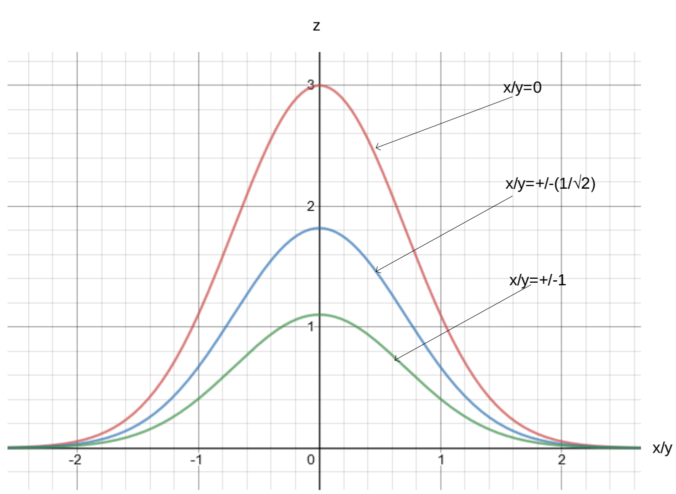
Example 3: Describe the vertical traces of $f(x, y) = 3e^{-x^2-y^2}$.

Solution: To create vertical traces we freeze either the x or y and observe the resulting curve in the appropriate vertical plane.

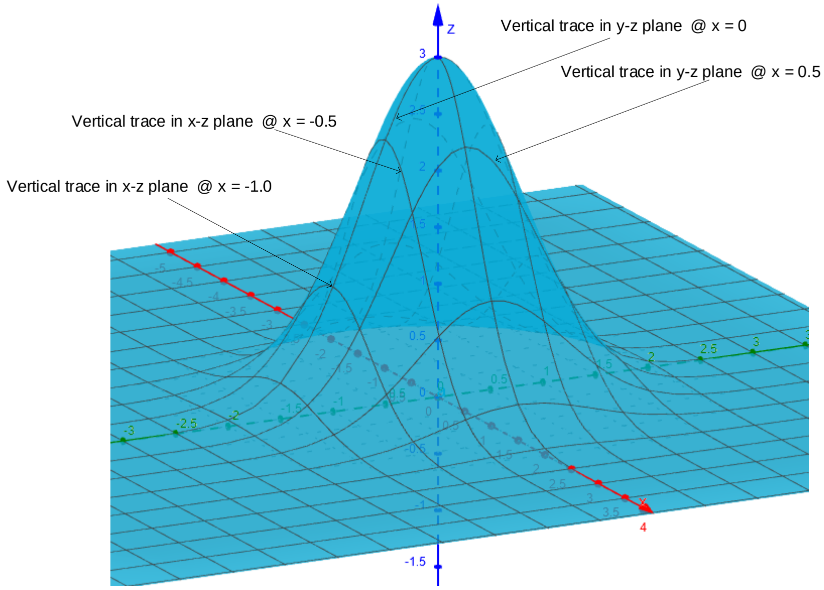
For the y-z plane curve we let $x = a$ resulting in.

$$z = 3e^{-a^2-y^2} = 3e^{-y^2+(-a^2)} = \left(\frac{3}{e^{a^2}}\right)e^{-y^2}$$

The traces are versions of the parent graph $z = 3e^{-y^2}$ that are compressed in the vertical direction. For the x-z plane curve we let $y = a$ resulting in similarly behavior.



For illustration we show the surface graph produced by a computer algebra system highlighting some vertical traces.



Example 4: Sketch the contour map of $f(x, y) = x^2 + 3y^2$, an elliptical paraboloid.

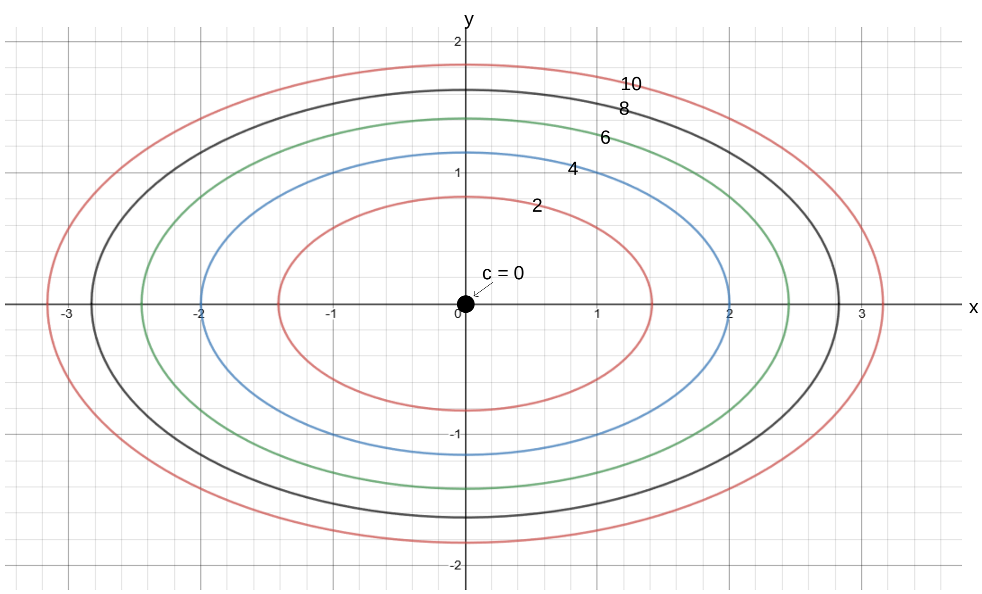
Solution: A contour map is a plot of level curves in the x-y plane, and level curves are the projection of horizontal traces, which are intersections of the graph with a horizontal plane obtained by setting $f(x, y)$ to c. With that in mind we set $f(x, y) = c$.

$$f(x, y) = x^2 + 3y^2 = c$$

With a little algebra we have the following

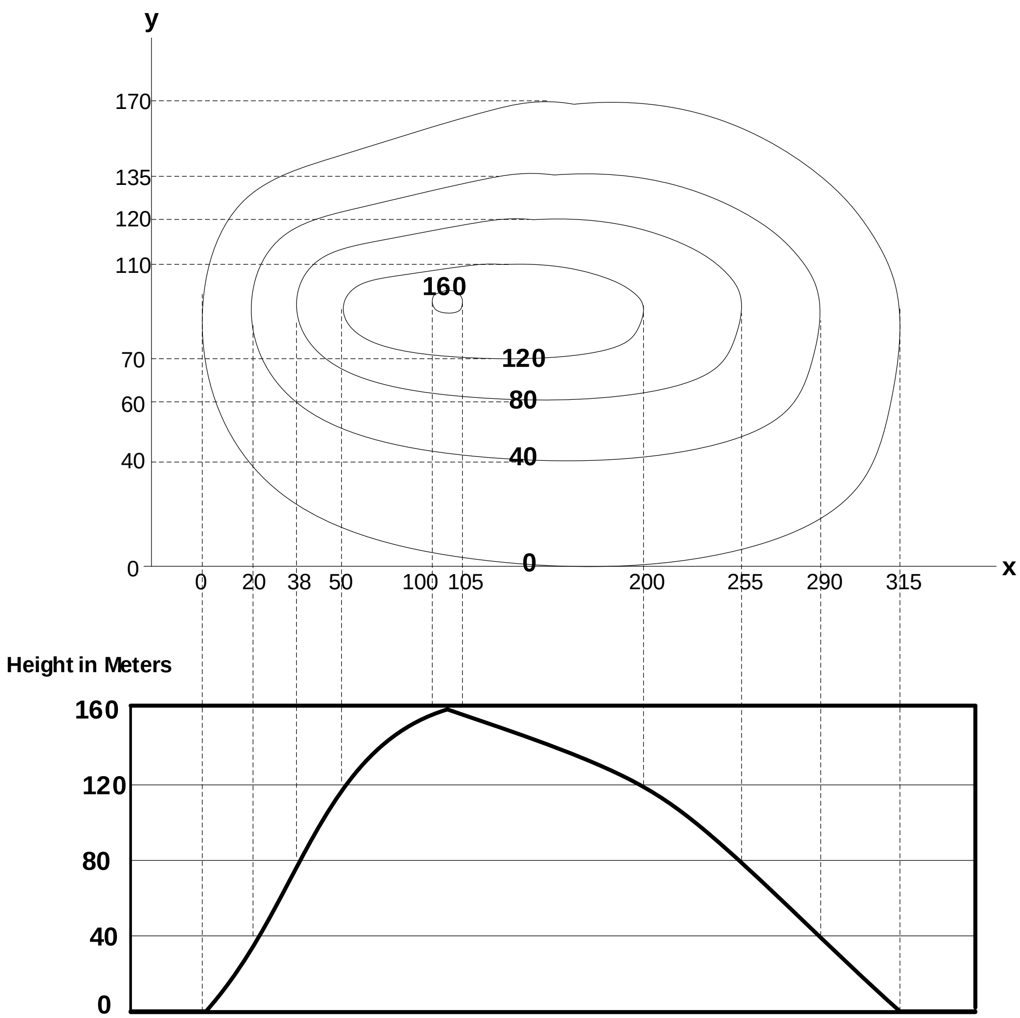
$$\left(\frac{x}{\sqrt{c}}\right)^2 + \left(\frac{y}{\sqrt{c/3}}\right)^2 = 1$$

Which describes an ellipse centered at $(0,0)$ with semi-major axis $\sqrt{c}$, and semi-minor axis $\sqrt{c/3}$. The contour map is shown below with a contour interval of $m = 2$.



Rate of Change using Contour Maps

In the next lesson we formally introduce differentiation for multivariable functions. With this in mind we now introduce the precursor to the derivative, i.e. the average rate of change. As an illustration imagine a person who desires to climb a mountain but is not sure the best way to traverse the mountain. In practice contour maps are frequently provided to the climber so that they can get an idea of the terrain before beginning their journey. The map, instead of being based on an explicit two variable function, will generally be developed by engineers making physical measurements. An example is shown below. A sketch of the mountain is shown for illustration purposes, where various levels are shown "mapped" onto the contour map above. The axes of the contour map represent the x-y plane and the level curves represent the locations on the mountain that were measured to be at the specific height indicated.



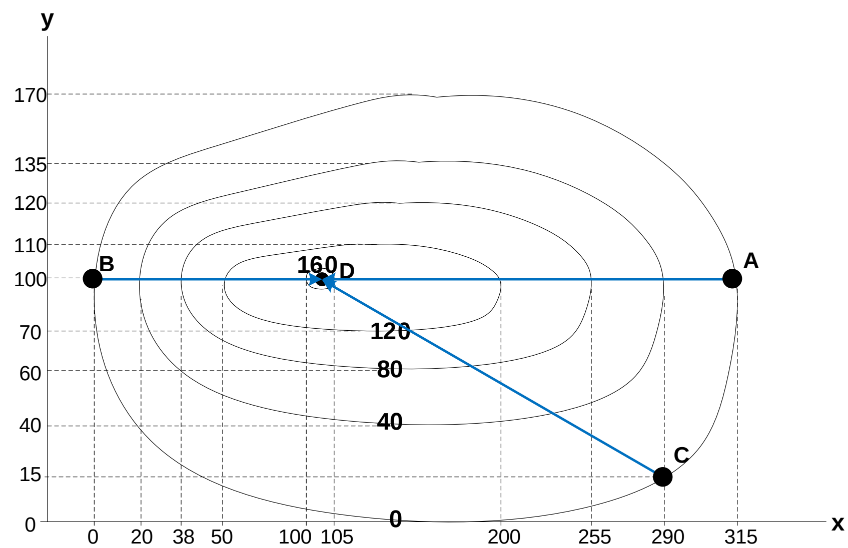
Based on the contour map alone the climber would like to decide which side to climb the mountain from, likely the side that is less steep. The steepness is represented at the rate of change in height with respect to the distance moved in the horizontal plane. The steepness from point P to point Q on the x-y plane is given as:

$$Steepness = Average\ Rate\ of\ Change\ from\ P\ to\ Q = \frac{\Delta\ Altitude}{\Delta\ Horizontal}$$

In the next example we redraw the contour map from above and compute the average rate of change for various trajectories along the mountain.

Example 5: Compute the average rate of change from the following points based on the contour map below.

a. $A \text{ to } D$ b. $B \text{ to } D$ c. $C \text{ to } D$



Solution:

a.

$$\text{Average Rate of Change from } A \text{ to } D = \frac{\Delta \, Altitude}{\Delta \, Horizontal} = \frac{160-0}{212.5} \cong 0.753$$

b.

$$\text{Average Rate of Change from } B \text{ to } D = \frac{\Delta \, Altitude}{\Delta \, Horizontal} = \frac{160-0}{102.5} \cong 1.56$$

c.

$$\text{Average Rate of Change from } B \text{ to } D = \frac{\Delta \, Altitude}{\Delta \, Horizontal} = \frac{160-0}{\sqrt{212.5^2+85^2}} \cong 0.70$$

Note that in *a.* the horizontal distance changed in the x direction while the y distance was held constant. The same happened in case *b.* with the y direction changing and the x distance being held constant. We'll see in the next lesson we refer to this as a *partial derivative*. In *c.* both the x and y coordinates changed. To compute the change in the horizontal distance we needed to take the direction of change into consideration. Taking direction into account should bring to mind the topics of vectors. We'll also see in an upcoming lesson that for this type of derivative we indeed call on vectors and we refer to as the operation as a *directional derivative*.

Final Summary for Multivariable Differentiation – Multivariable Functions

Multivariable Functions
A multivariable function is one that takes n real variables as inputs, $(x_1, x_2, ..., x_n)$, and assigns a single value, y, to each n-tuple $(x_1, x_2, ..., x_n)$ in a domain in R^n. The range is the set of all y values for the $(x_1, x_2, ..., x_n)$ in the domain. • $(x_1, x_2, ..., x_n)$ are called the independent variables. • y is the dependent variable. The function is represented as $y = f(x_1, x_2, ..., x_n)$
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Contour Maps and Rate of Change
• The level curves on a contour map are drawn at equally spaced changes in $f(x, y)$. • The spacing between level curves on a contour map indicates the "steepness" of the change in $f(x, y)$. • The average rate of change from a point P to a point Q on a contour map, $A\Delta_{P \to Q}$,is $A\Delta_{P \to Q} = \frac{\Delta\, Functuion\ value}{\Delta\, Horizonatl\ Distance}$ When the function represents the physical height of an area, we usually say $A\Delta_{P \to Q} = \frac{\Delta\, Altitude}{\Delta\, Horizonatl\ Distance}$

By: ferrantetutoring