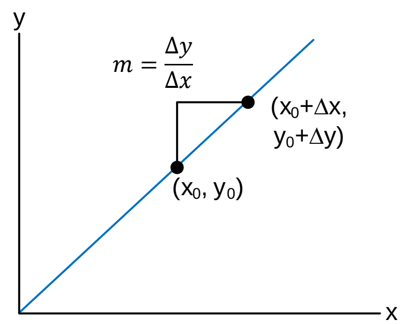
Vector Geometry – Lines and Planes in R^3

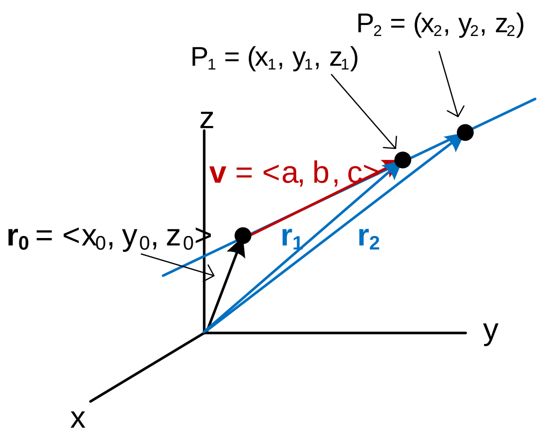
The calculus we have learned to this point is limited to functions in two dimensions (R^2), i.e. single variable calculus. Multivariable calculus, on the other hand, allows us to work with functions of several variables. Although multivariable calculus allows for functions of an arbitrary number of variables, when studying objects in space we are mostly concerned with objects in three dimensions (R^3). In this section, therefore, we will focus on working in R^3. More specifically we'll learn how to describe lines and planes in R^3 using vectors.

Lines in R^3

To describe a line in R^2 we can use the *point-slope form*, $y = m(x - x_0) + y_0$, where (x_0, y_0) is a point on the line and the slope, $m = \frac{\Delta y}{\Delta x}$, tells us how we get from one point to the next.



We can derive a similar "*point-direction form*" for a line in R^3 using vectors as illustrated below.



Similar to the line in R^2, we start with a point described by a vector, $\boldsymbol{r_0} = \langle x_0, y_0, z_0 \rangle$. We then use a direction vector, $\boldsymbol{v} = \langle a, b, c \rangle$, to describe the remaining points on the line. For example, $\boldsymbol{r_1} = \langle x_0, y_0, z_0 \rangle + 1\langle a, b, c \rangle$, $\boldsymbol{r_2} = \langle x_0, y_0, z_0 \rangle + 2\langle a, b, c \rangle$, etc. By introducing a scalar parameter, e.g. t, all points on the line can be described with the following vector equation.

$$\boldsymbol{r}(t) = \boldsymbol{r_0} + t\boldsymbol{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

The above representation is called a vector parameterization of a line. We can also expand this equation and write three parametric equations as shown below.

$x(t) = x_0 + at$	$y(t) = y_0 + bt$	$z(t) = z + ct$

Equation of a Line in R^3 (Point-Direction Form)
The line $\mathcal{L}$ through the point (x_0, y_0, z_0) in the directions of $\boldsymbol{v} = \langle a, b, c\rangle$ can be described in the following ways: ***Vector Parameterization:*** $$\boldsymbol{r}(t) = \boldsymbol{r_0} + t\boldsymbol{v} = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$$ ***Parametric Equations:*** $x(t) = x_0 + at$ $\quad$ $y(t) = y_0 + bt$ $\quad$ $z(t) = z + ct$ Where $(-\infty < t < \infty)$

Example 1:

Find the equation of a line that passes through the points $P_0 = (1, 3, 4)$ and $P_1 = (-2, 1, 7)$.

Solution: We first need to find the direction vector, $\boldsymbol{v}$, which is the vector that points from P_0 to P_1.

$$\boldsymbol{v} = \overrightarrow{P_0P_1} = \langle -2 - 1, 1 - 3, 7 - 4\rangle = \langle -3, -2, 3\rangle$$

Next, we can use the point P_0 to create the vector $\boldsymbol{r_0}$. The vector equation of the line can then be written as

$$\boldsymbol{r}(t) = \boldsymbol{r_0} + t\boldsymbol{v} = \langle 1, 3, 4\rangle + t\langle -3, -2, 3\rangle$$

Or equivalently using the following parametric equations

$$x(t) = 1 - 3t \qquad y(t) = 3 - 2t \qquad z(t) = 4 + 3t$$

Parallel, Intersecting, and Skew Lines

We know that two lines in $\boldsymbol{R}^2$ are parallel when they have the same slopes. Similarly, two lines in $\boldsymbol{R}^3$ are parallel when they have the same direction vector. For example, the line from above: $\boldsymbol{r_1}(t) = \langle 1, 3, 4\rangle + t\langle -3, -2, 3\rangle$ is a parallel to the line $\boldsymbol{r_2}(t) = \langle -7, 3, 6\rangle + t\langle -3, -2, 3\rangle$. More generally we say that two lines are parallel is the cross product of their direction vectors is zero, i.e. $\boldsymbol{v_1} \times \boldsymbol{v_2} = \boldsymbol{0}$.

In $\boldsymbol{R}^2$ if two lines are not parallel, they intersect. However, in $\boldsymbol{R}^3$ there is an additional scenario where the lines pass by each other without ever touching. We refer to these lines as being *skewed*. Two line intersect if there exist parameter values, t_1 and t_2, such that $\boldsymbol{r_1}(t_1) = \boldsymbol{r_2}(t_2)$. Let's illustrate with an example.

Example 2: Determine whether the following two lines intersect.

$$\boldsymbol{r_1}(t) = \langle 1, 14, 5\rangle + t\langle -2, 1, -1\rangle \qquad\qquad \boldsymbol{r_2}(t) = \langle 0, 4, 3\rangle + t\langle 1, 3, 1\rangle$$

Solution:

Setting the two lines equal to each other using parameter values, t_1 and t_2, results in an overdetermined system with three equations and two unknowns. Recall that overdetermined systems in most cases do not have a solution, which in our case means the lines are skewed. However, if a solution can be found the lines do indeed intersect. To make this determination we proceed as shown below.

$$\langle 1, 14, 5\rangle + t_1\langle -2, 1, -1\rangle = \langle 0, 4, 3\rangle + t_2\langle 1, 3, 1\rangle$$

$$1 - 2t_1 = 0 + t_2 \qquad\qquad 14 + t_1 = 4 + 3t_2 \qquad\qquad 5 - t_1 = 3 + t_2$$

Solving the first equation for t_2 and substituting into the second we have

$$\begin{aligned} 14 + t_1 &= 4 + 3(1 - 2t_1) \\ 7 + t_1 &= -6t_1 \\ t_1 &= -1 \end{aligned}$$

And substituting back into the first equation we find

$$\begin{aligned} t_2 &= 1 - 2(-1) \\ t_2 &= 3 \end{aligned}$$

Next, the solution set, $[t_1, t_2] = [-1, 3]$, is verified using the third equation

$$\begin{aligned} 5 - t_1 &= 3 + t_2 \\ 5 - (-1) &= 3 + 3 \\ 6 &= 6 \end{aligned}$$

Finally, we find the intersection point by finding either $\boldsymbol{r_1}(\,t_1)$ or $\boldsymbol{r_2}(t_2)$. For illustrative purposes we'll show both equations evaluate to the same point.

$$\begin{aligned} \boldsymbol{r_1}(-1) &= \langle 1, 14, 5\rangle + (-1)\langle -2, 1, -1\rangle \\ \boldsymbol{r_1}(-1) &= \langle 1 + 2, 14 - 1, 5 + 1\rangle \\ \boldsymbol{r_1}(-1) &= \langle 3, 13, 6\rangle \end{aligned} \qquad\qquad \begin{aligned} \boldsymbol{r_2}(3) &= \langle 0, 4, 3\rangle + (3)\langle 1, 3, 1\rangle \\ \boldsymbol{r_2}(3) &= \langle 0 + 3, 4 + 9, 3 + 3\rangle \\ \boldsymbol{r_2}(3) &= \langle 3, 13, 6\rangle \end{aligned}$$

Let's do another example to illustrate the case when two lines are skewed.

Example 3: Determine whether the following two lines intersect.

$$\boldsymbol{r_1}(t) = \langle 1, 0, 1\rangle + t\langle 3, 3, 5\rangle \qquad \boldsymbol{r_2}(t) = \langle 3, 6, 1\rangle + t\langle 4, -2, 7\rangle$$

Solution: We begin the same way we did in example 2.

$$\langle 1, 0, 1\rangle + t_1\langle 3, 3, 5\rangle = \langle 3, 6, 1\rangle + t_2\langle 4, -2, 7\rangle$$

$$1 + 3t_1 = 3 + 4t_2 \qquad 3t_1 = 6 - 2t_2 \qquad 1 + 5t_1 = 1 + 7t_2$$

Using the first equation again we find

$$t_1 = \frac{3 + 4t_2 - 1}{3} = \frac{2}{3} + \frac{4}{3}t_2$$

Next, we solve for t_1 by substituting into the second equation.

$$\begin{aligned} 3\left(\frac{2}{3} + \frac{4}{3}t_2\right) &= 6 - 2t_2 \\ 2 + 4t_2 &= 6 - 2t_2 \\ 6t_2 &= 4 \\ t_2 &= \frac{2}{3} \end{aligned}$$

Therefore

$$t_1 = \frac{2}{3} + \frac{4}{3}\left(\frac{2}{3}\right) = \frac{14}{9}$$

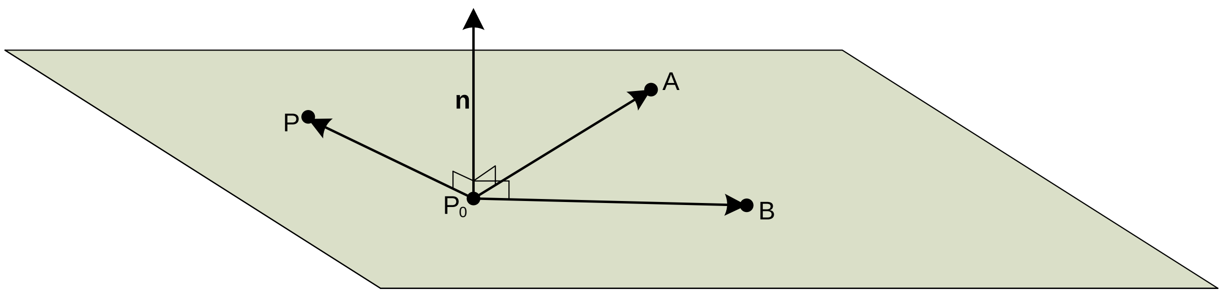
Finally, we check this solution with the third equation.

$$\begin{aligned} 1 + 5\left(\frac{14}{9}\right) &= 1 + 7\left(\frac{2}{3}\right) \\ \frac{79}{9} &\neq \frac{17}{3} \end{aligned}$$

Since the solution set does not satisfy the third equation the lines do not intersect and are skewed.

Planes in $\boldsymbol{R}^3$

A line can be specified with only two points. However, to specify a plane we need three non-colinear points, e.g. P_0, A, and B. To define a plane that contains these points we start by creating two vectors, $\overrightarrow{P_0A}$ and $\overrightarrow{P_0B}$, as shown below.



Next, we find a vector orthogonal to these two vectors, and hence the plane, by computing the cross product, $\overrightarrow{P_0A} \times \overrightarrow{P_0B}$. The resulting vector is referred to as a normal vector, $\boldsymbol{n} = \langle a, b, c \rangle$. We then consider an arbitrary point on the plane, $P = (x, y, z)$, and construct the vector $\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$. Finally, since the vectors $\overrightarrow{P_0P}$ and $\boldsymbol{n}$ must be orthogonal, i.e. $\boldsymbol{n} \cdot \overrightarrow{P_0P} = 0$, we can derive an equation for the plane as shown below.

$$\boldsymbol{n} \cdot \overrightarrow{P_0P} = 0$$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = ax_0 + by_0 + cz_0$$

$$ax + by + cz = d$$

Where, $d = ax_0 + by_0 + cz_0$

This equation defines a plane in $\boldsymbol{R}^3$ that contains the point $P = (x_0, y_0, z_0)$ and has a normal vector $\boldsymbol{n} = \langle a, b, c \rangle$.

Note also, that we can express this equation in vector for as follows:

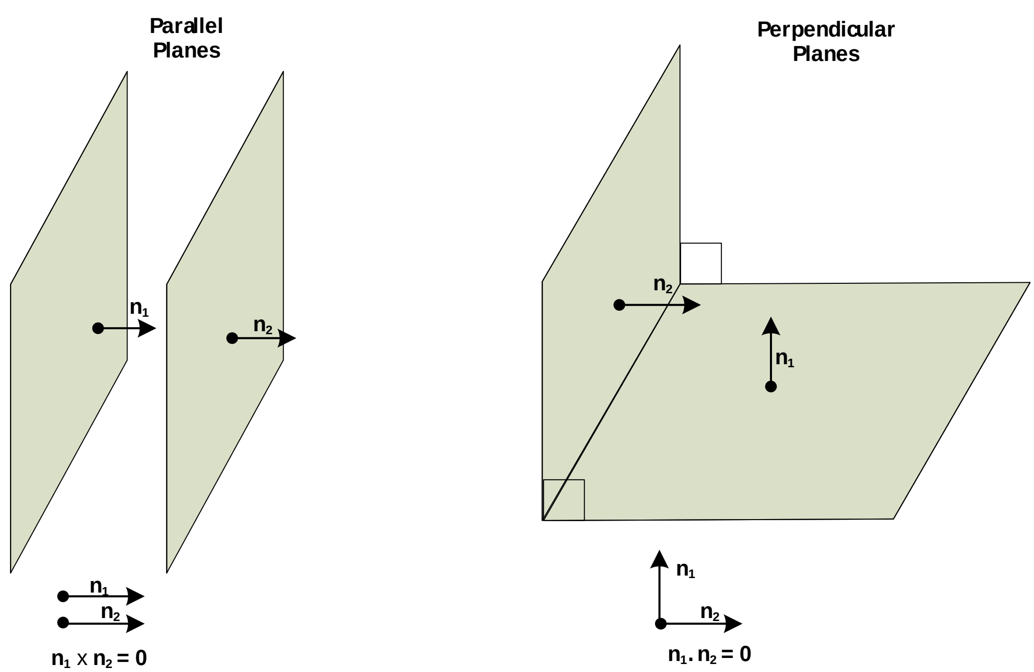
$$\boldsymbol{n} \cdot \langle x, y, z \rangle = d$$

We summarize these results below.

Equation of a Plane in R^3 (Point-Normal Form)
The plane $\mathcal{P}$ through the point (x_0, y_0, z_0) with a normal vector $\boldsymbol{n} = \langle a, b, c\rangle$ can be described in the following ways: ***Vector Form:*** $$\boldsymbol{n} \cdot \langle x, y, z\rangle = d$$ ***Scalar Form:*** $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$ $$ax + by + cz = d$$ Where, $d = ax_0 + by_0 + cz_0$

Parallel and Intersecting Planes

The orientation of a line in $\boldsymbol{R}^3$ can be identified by the direction vector, $\boldsymbol{v}$. For that reason, we used the direction vector to determine whether two lines were parallel, i.e. if $\boldsymbol{v_1} \times \boldsymbol{v_2} = \boldsymbol{0}$ then $\mathcal{L}_1$ is parallel to $\mathcal{L}_2$. In a similar fashion the normal vector, $\boldsymbol{n}$, can be used to determine of two planes are parallel, i.e. if $\boldsymbol{n_1} \times \boldsymbol{n_2} = \boldsymbol{0}$ then $\mathcal{P}_1$ is parallel to $\mathcal{P}_2$. We can also use the dot product to determine if two planes are orthogonal, i.e. if $\boldsymbol{n_1} \cdot \boldsymbol{n_2} = 0$ then $\mathcal{P}_1$ is orthogonal to $\mathcal{P}_2$. These concepts are illustrated in the figure below.



If two planes are not parallel, they will intersect along a line. This line is called the *Line of Intersection* and it can be found using the normal vectors of the two planes. Note the two orthogonal planes in the above figure. A vector that points along the line of intersection is orthogonal to both normal vectors, which can be found using the cross product between the normal vectors, i.e. $\boldsymbol{v} = \boldsymbol{n_1} \times \boldsymbol{n_2}$. This fact is true in general for any two interesting planes, i.e. where $\boldsymbol{n_1} \times \boldsymbol{n_2} \neq \boldsymbol{0}$. The resulting vector is the direction vector for the line of intersection of the two planes. To find the equation of the line we also need a point on the line which can be found by solving the two planes simultaneously. Let's illustrate the procedure with an example.

Example 4: Compute the equation of the line of intersection of the following two planes. intersect.

$$\mathcal{P}_1:\ x + y + z = 7 \qquad\qquad \mathcal{P}_2:\ 2x + 4z = 6$$

Solution: The direction vector of the line of intersection is given by the cross product of the normal vectors. For the first plane the normal vector is directly given by the coefficients as $\boldsymbol{n_1} = \langle 1, 1, 1\rangle$, whereas the second plane we could first divide through by 2. The normal vector is then given by $\boldsymbol{n_2} = \langle 1, 0, 2\rangle$.

$$\begin{aligned}\boldsymbol{v} = \boldsymbol{n_1} \times \boldsymbol{n_2} &= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} \\ &= \hat{\boldsymbol{\imath}}\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} - \hat{\boldsymbol{\jmath}}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + \widehat{\boldsymbol{k}}\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \\ &= \hat{\boldsymbol{\imath}}(2-0) - \hat{\boldsymbol{\jmath}}(2-1) + \widehat{\boldsymbol{k}}(0-1) \\ &= \langle 2, -1, -1\rangle\end{aligned}$$

Next, we need to find a common point between the two planes. We'll start by solving the second plane for x, i.e. $x = 3 - 2z$, and substituting into the first.

$$\begin{aligned}(3 - 2z) + y + z &= 7 \\ y &= z + 4\end{aligned}$$

We can treat z as a free variable and set it to zero so that $y = 0$. Nest we can use the first equation to find $x = 3$. A point on the line of intersection is then

$$P_{\mathcal{L}}: (3, 4, 0)$$

Finally, using the point-direction form of a line, we can write the line of intersection as

$$\boldsymbol{r}(t) = \langle 3, 4, 0\rangle + t\langle 2, -1, -1\rangle$$

Before summarizing this section, let's continue with examples so that we become more familiar with working with lines and planes in $\boldsymbol{R}^3$.

Example 5: Two bugs are walking along lines in $\boldsymbol{R}^3$. The positions of the bugs at time t is given by the following vector equations.

$$\boldsymbol{r_1}(t) = \langle 1, 3, 5\rangle + t\langle 2, 5, 2\rangle \qquad\qquad \boldsymbol{r_2}(t) = \langle 0, 11, 4\rangle + t\langle 1, -1, 1\rangle$$

i. Compute the distance between the bugs initial positions.
ii. Find the point where the paths intersect. Do the bugs collide at this point?

Solution:

i. The initial position are given by $\boldsymbol{r_1}(0)$ and $\boldsymbol{r_2}(0)$ respectively.

$$\boldsymbol{r_1}(0) = \langle 1, 3, 5\rangle \qquad\qquad \boldsymbol{r_2}(0) = \langle 0, 11, 4\rangle$$

The distance between these two points is given as

$$\begin{aligned} d &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \\ &= \sqrt{(-1)^2 + (8)^2 + (-1)^2} \\ &= \sqrt{66} \end{aligned}$$

ii. The lines traced out by the bugs intersect when the positions are equal. We create an equation for each coordinate as follows

$$x\colon 1 + 2t_1 = t_2 \qquad\qquad y\colon 3 + 5t_1 = 11 - t_2 \qquad\qquad z\colon 5 + 2t_1 = 4 + t_2$$

We start by substituting t_2 from the first equation into the second.

$$\begin{aligned} 3 + 5t_1 &= 11 - (1 + 2t_1) \\ 7t_1 &= 7 \\ t_1 &= 1 \end{aligned}$$

Next, we substitute back into equation one to find t_2 and verify equation three.

$$t_2 = 1 + 2(1) = 3 \qquad\qquad \begin{aligned} t_2 &= 1 + 2(1) = 3 \\ 5 + 2(1) &= 4 + (3) \\ 7 &= 7 \end{aligned}$$

Finally, we can use $\boldsymbol{r_1}(t_1)$ to find the point if intersection.

$$\boldsymbol{r_1}(1) = \langle 1, 3, 5\rangle + 1\langle 2, 5, 2\rangle = \langle 3, 8, 7\rangle$$

Since the point of intersection occurs when $t_1 = 1$ and $t_2 = 3$ the bugs do not collide.

Example 6: Find the point of intersection of the of the plane and the line given below.

$$\mathcal{P}:\ 3x - 9y + 2z = 7 \qquad\qquad \mathcal{L}:\ \boldsymbol{r}(t) = \langle 1,2,1\rangle + t\langle -2,0,1\rangle$$

Solution: To find the point of intersection we use the parametric equations of the lines and substitute them into the equation for the plane.

$$\mathcal{L}: \begin{cases} x(t) = 1 - 2t \\ y(t) = 2 \\ z(t) = 1 + t \end{cases}$$

Substituting we have

$$\begin{aligned} 3(1 - 2t\,) - 9(2) + 2(1 + t) &= 7 \\ 3 - 6t - 18 + 2 + 2t &= 7 \\ 4t &= -20 \\ t &= -5 \end{aligned}$$

Therefore, the plane and the line intersect at $\boldsymbol{r}(-5)$.

$$\boldsymbol{r}(-5) = \langle 1,2,1\rangle + -5\langle -2,0,1\rangle = \langle 11,2,-4\rangle$$

Example 7: Determine whether the following are parallel, perpendicular, or neither.

i. $\mathcal{P}_1:\ 5x - 3y + 4z = -1 \qquad \mathcal{P}_2:\ 2x + -2y - 4z = 9$

ii. $\mathcal{P}:\ 5x - 3y + 4z = -1 \qquad \boldsymbol{r}(t) = \langle 2,3,5\rangle + t\langle 2,-2,-4\rangle$

iii. $\mathcal{P}:\ 5x - 3y + 4z = -1 \qquad \boldsymbol{r}(t) = \langle 2,3,5\rangle + t\,\langle \frac{5}{2},\frac{-3}{2},2\rangle$

Solution:

i. If two planes are parallel the cross product of their normal vectors is zero.

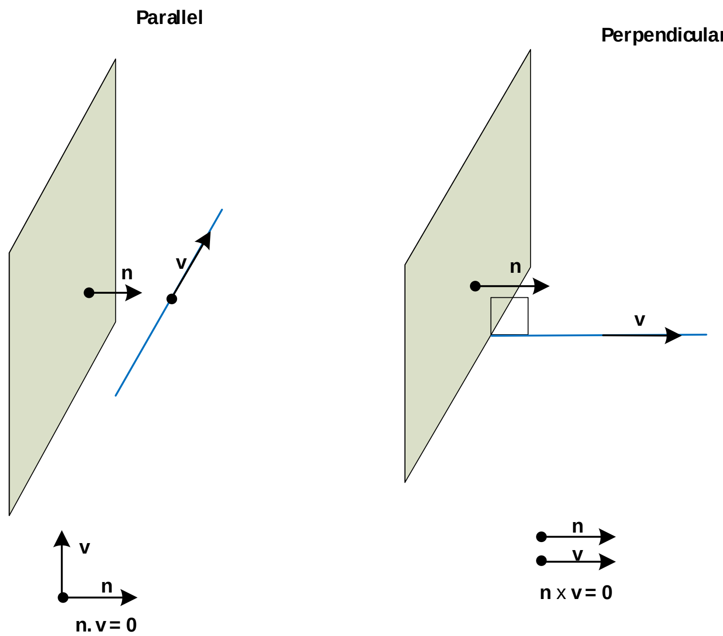
$$\langle 5,-3,4\rangle \times \langle 2,-2,-4\rangle = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 5 & -3 & 4 \\ 2 & -2 & -4 \end{vmatrix}$$
$$= \hat{\boldsymbol{\imath}}(12 + 8)\ldots$$

Since the $\hat{\boldsymbol{\imath}}$ component is not zero there is no need to continue the computation. The planes are not parallel. Next, we look at the dot product to check for orthogonality.

$$\langle 5,-3,4\rangle \cdot \langle 2,-2,-4\rangle = 10 + 6 - 16 = 0$$

Therefore, the planes are perpendicular.

ii. The orientation of a plane is determined by its normal vector; however, the orientation of a line is determined by its direction vector. The figure below illustrates how we can determine if these two different objects are parallel, perpendicular, or neither.



Let's begin by checking if they are parallel.

$$\langle 5, -3, 4\rangle \cdot \langle 2, -2, -4\rangle = 10 + 6 - 16 \ = \ 0$$

Therefore, the plane and the line are parallel.

iii. Let's again check if the plane and line are parallel in this case.

$$\langle 5, -3, 4\rangle \cdot \langle \frac{5}{2}, \frac{-3}{2}, 2\rangle = 12.5 + 4.5 + 8 \ \neq \ 0$$

Now let's check the cross product.

$$\langle 5, -3, 4\rangle \times \langle \frac{5}{2}, \frac{-3}{2}, 2\rangle = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 5 & -3 & 4 \\ \frac{5}{2} & \frac{-3}{2} & 2 \end{vmatrix}$$

$$= \ \hat{\boldsymbol{\imath}}(-6 + 6) \ - \hat{\boldsymbol{\jmath}}(10 - 10) + \hat{\boldsymbol{k}}(-7.5 + 7.5) = \ \mathbf{0}$$

Therefore, the plane and the line are perpendicular.

Example 8: Find an equation of the plane that satisfies the given conditions.

i. The plane which passes through $P(1,2,3)$ and is parallel to the plane $3x - 5y + z = 2$.

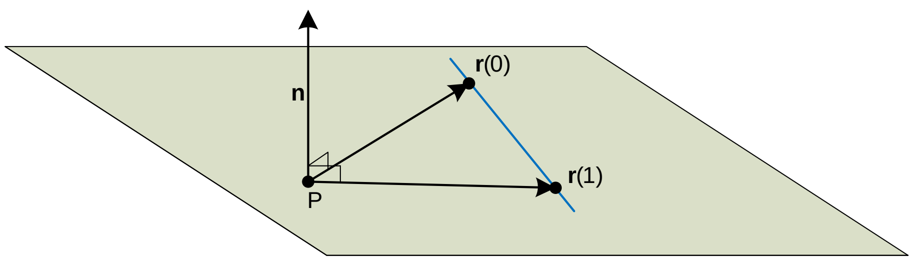
ii. The plane which contains the point $P(-2,-1,3)$ and which contains the line $\boldsymbol{r}(t) = \langle 1,3,0\rangle + t\langle 1,-2,4\rangle$

Solution:

i. Since the second plane is parallel to the first plane, they have the same normal vector, given by $\boldsymbol{n} = \langle 3,-5,1\rangle$. Using the scalar form for the equation of a plane and the point, $P(1,2,3)$, we have.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$
$$3(x-1) + -5(y-2) + 1(z-3) = 0$$
$$3x - 5y + z = -4$$

ii. In this case we assume the given point is not on the line so that we can form two vectors as shown below and find the normal vector to the plane.



The two vectors are given as:

$$\overrightarrow{Pr(0)} = \langle 1-(-2), 3-(-1), 0-3\rangle = \langle 3,4,-3\rangle$$

$$\overrightarrow{Pr(1)} = \langle 2-(-2), 1-(-1), 4-3\rangle = \langle 4,2,1\rangle$$

Therefore, the normal vector is

$$\boldsymbol{n} = \overrightarrow{Pr(1)} \times \overrightarrow{Pr(0)} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 4 & 2 & 1 \\ 3 & 4 & -3 \end{vmatrix}$$
$$= \hat{\boldsymbol{\imath}}(-6-4) - \hat{\boldsymbol{\jmath}}(-12-3) + \hat{\boldsymbol{k}}(16-6)$$
$$\boldsymbol{n} = \langle -10, 15, 10\rangle$$

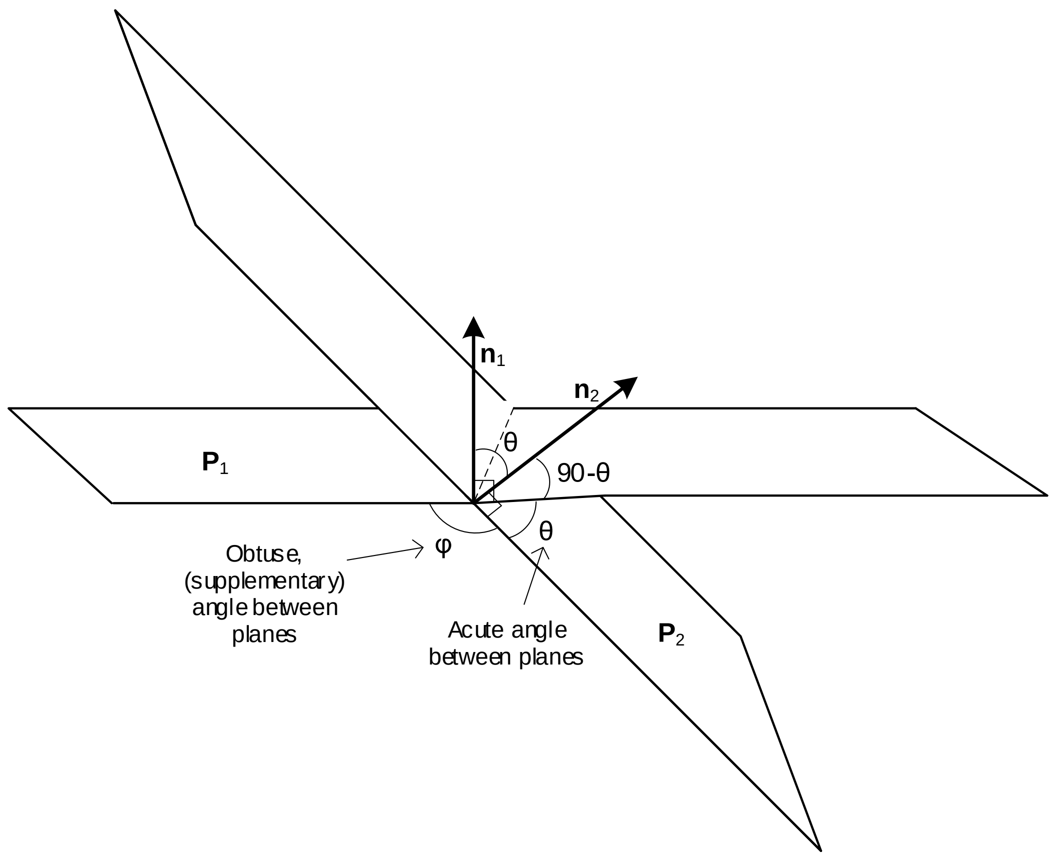
The equation of the plane is then

$$-10(x+2) + 15(y+1) + 10(z-3) = 0$$
$$-10x + 15y + 10z = 35$$
$$-2x + 3y + 2z = 7$$

Example 9: Find the acute angle of intersection of the following two planes.

$\mathcal{P}_1\text{: } 3x - 2y - 5z = 0$ $\qquad\qquad$ $\mathcal{P}_2\text{: } -x + -y + 2x = 3$

Solution: As illustrated in the figure below, the angle between two planes is equal to the angle between their normal vectors.



We can use the dot product to find this angle.

$$\theta = cos^{-1}\left(\frac{\boldsymbol{n_1}\cdot\boldsymbol{n_2}}{\|\boldsymbol{n_1}\|\|\boldsymbol{n_2}\|}\right)$$

$$= cos^{-1}\left(\frac{(-3+2-10)}{\left(\sqrt{3^2+2^2+5^2}\right)\left(\sqrt{1^2+1^2+2^2}\right)}\right)$$

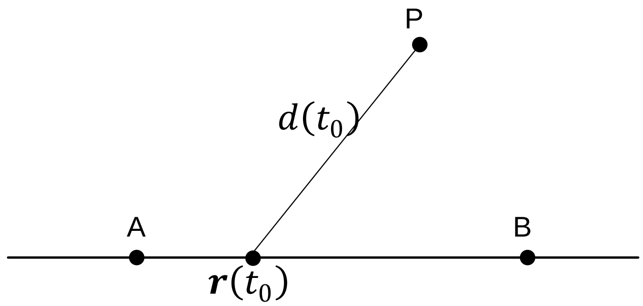
$$= cos^{-1}\left(\frac{-11}{\left(\sqrt{38}\right)\left(\sqrt{6}\right)}\right) \approx 137^{\circ}$$

Finally, we note that this is the larger of the two supplementary angles and since we are asked for the acute angle we subtract this value from 180°.

$$\theta_{acute} = 180^{\circ} - 137^{\circ} = 43^{\circ}$$

Example 9: Consider the points $A = (1,0,1)$, $B = (2,3,1)$, and $P = (5,3,0)$ as shown in the figure below.

i. Compute an equation for the line, $\boldsymbol{r}(t)$.
ii. Compute a function $d(t)$ which gives the distance from the point P to an arbitrary point on the line.
iii. Calculate the value of t which minimizes the distance from point P to the line $\boldsymbol{r}(t)$.
iv. Compute the minimum distance from iii.



Solution:

i. We start by finding the direction vector of the line between A and B.

$$\boldsymbol{v} = \overrightarrow{AB} = \langle 2-1, 3-0, 1-1 \rangle = \langle 1, 3, 0 \rangle$$

Then we use A as the starting point and write.

$$\boldsymbol{r}(t) = \langle 1, 0, 1 \rangle + t\langle 1, 3, 0 \rangle$$

We can write the equation in parametric form as follows.

$$\mathcal{L} = \begin{cases} x(t) = 1 + t \\ y(t) = 3t \\ z(t) = 1 \end{cases}$$

ii. The distance between $\boldsymbol{r}(t)$ and P will be a function of t and is given with the distance formula as

$$\begin{aligned} d(t) &= \sqrt{\left(1 + (t-5)\right)^2 + (3t-3)^2 + (0-1)^2} \\ &= \sqrt{(t-4)^2 + 9(t-1)^2 + 1} \\ &= \sqrt{10t^2 - 26t + 26} \end{aligned}$$

iii. As $d(t)$ is a function of a single variable, t, we can use single variable calculus to minimize the function. Furthermore, to make the computation easier we can minimize $d^2(t)$ instead of $d(t)$ since the minimum value will occur at the same location. Recall, to find the extreme values, (in this case there will be a single minimum value), we set the derivative to zero.

$$\frac{d}{dt}\left(d^2(t)\right) = 20t - 26 = 0$$
$$t = \frac{26}{20} = \frac{13}{10}$$

iv. Finally, the minimum distance occurs at $d\left(\frac{13}{10}\right)$.

$$d\left(\frac{13}{10}\right) = \sqrt{10\left(\frac{13}{10}\right)^2 - 26\left(\frac{13}{10}\right) + 26}$$
$$= \sqrt{\frac{13^2}{10} - \frac{338}{10} + \frac{260}{10}}$$
$$= \sqrt{\frac{91}{10}}$$

Final Summary for Vector Geometry – Lines and Planes in R^3

Equation of a Line in R^3 (Point-Direction Form)
The line $\mathcal{L}$ through the point (x_0, y_0, z_0) in the directions of $\boldsymbol{v} = \langle a, b, c \rangle$ can be described in the following ways: ***Vector Parameterization:*** $$\boldsymbol{r}(t) = \boldsymbol{r_0} + t\boldsymbol{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$ ***Parametric Equations:*** $x(t) = x_0 + at$ $\quad$ $y(t) = y_0 + bt$ $\quad$ $z(t) = z + ct$ Where $(-\infty < t < \infty)$
Parallel, Perpendicular, and Intersecting Lines
Two lines are parallel when the cross product of their direction vectors is zero. $$\boldsymbol{v_1} \times \boldsymbol{v_2} = \boldsymbol{0}$$ Two lines are perpendicular when the dot product of their direction vectors is zero. $$\boldsymbol{v_1} \cdot \boldsymbol{v_2} = 0$$ The point of intersection between two lines can be found by setting the equations of the lines equal to one another for different values of the parameter t. $$\boldsymbol{r_1}(t_1) = \boldsymbol{r_2}(t_2)$$ If an intersection point does not exist *and* the lines are not parallel, we refer to them as skewed.
Equation of a Plane in R^3 (Point-Normal Form)
The plane $\mathcal{P}$ through the point (x_0, y_0, z_0) with a normal vector $\boldsymbol{n} = \langle a, b, c \rangle$ can be described in the following ways: ***Vector Form:*** $$\boldsymbol{n} \cdot \langle x, y, z \rangle = d$$ ***Scalar Form:*** $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$ $$ax + by + cz = d$$ Where, $d = ax_0 + by_0 + cz_0$

Parallel and Intersecting Planes
Two planes are parallel when the cross product of their normal vectors is zero. $$\boldsymbol{n_1} \times \boldsymbol{n_2} = \boldsymbol{0}$$ Two planes are perpendicular when the dot product of their normal vectors is zero. $$\boldsymbol{n_1} \cdot \boldsymbol{n_2} = 0$$ When two planes are not parallel, they intersect along a line, *Line of Intersection (LOS).* The direction vector of the LOS is given as $$\boldsymbol{v} = \boldsymbol{n_1} \times \boldsymbol{n_2}$$
Parallel and Perpendicular Lines and Planes
A plane and a line are parallel when the dot product of the normal and direction vector is zero. $$\boldsymbol{n} \cdot \boldsymbol{v} = 0$$ A plane and a line are perpendicular when the cross product of the normal and direction vector is zero. $$\boldsymbol{n} \times \boldsymbol{v} = \boldsymbol{0}$$

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