Vector Geometry – Quadric Surfaces

Quadric surfaces are analogs of conic sections, which we covered in the calculus 2 section. Conic sections are described by quadradic equation in two dimensions. The general form of which is

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$$

When $C = D = E = 0$ the conic axes are parallel to the coordinate axes and the shape is centered at $(0,0)$. Recall the four major categories of conics.

1. **Circle**: Centered at $(0,0)$ with a radius r.

$$x^2 + y^2 = r^2$$

2. **Ellipse**: Centered at $(0,0)$ with semi-major axis, a, and semi-minor axis, b.

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

3. **Hyperbola**: Centered at $(0,0)$ with inner rectangle length $2a$ and height $2b$.

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$

4. **Parabola**: With vertex at $(0,0)$ and focus at $(0,c)$.

$$y = \frac{1}{4c}x^2$$

A quadric surface is defined by a quadratic equation in three variables. The general form of which is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + ax + by + cz + d = 0$$

Similarly, when $D = E = F = a = b = c = 0$, the quadric axes are parallel to the coordinate axes and the surface is centered at $(0,0)$. The four surfaces which are analogs of the conic section above are the sphere, ellipsoid, hyperboloid, and the paraboloid.

The goal of this section is not to memorize the equations for the different surfaces, but rather to learn how to recognize and roughly sketch the surface using cross sections, or traces. A trace is the intersection of a surface with a given plane. A trace can be obtained by ‘freezing’ one of the three variables and sketching the resulting 2D equation. For example, a horizontal trace is created by setting $z = z_0$. Let’s introduce each of the four basic surfaces mentioned above and explore traces along the way.

Sphere – Analog of the Conic Section, Circle

The standard equation for a sphere is given as

$$x^2 + y^2 + z^2 = r^2$$

Where r is the radius.

Example 1: Sketch a sphere with a radius of 4.

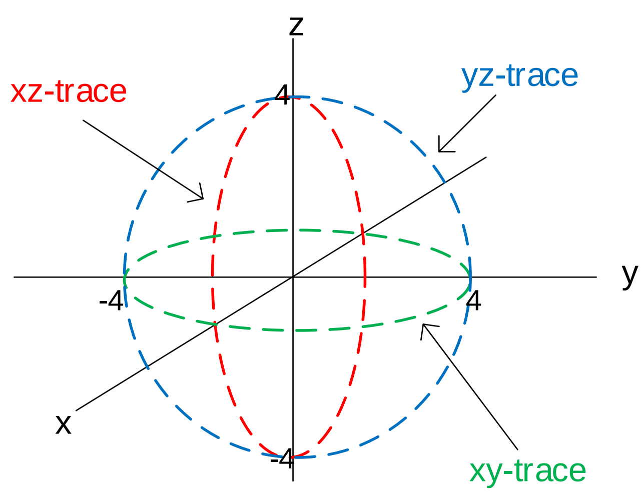
Solution: The equation is given by

$$x^2 + y^2 + z^2 = 4^2$$

Although we are well aware of what a sphere looks like let's explore creating the surface using traces.

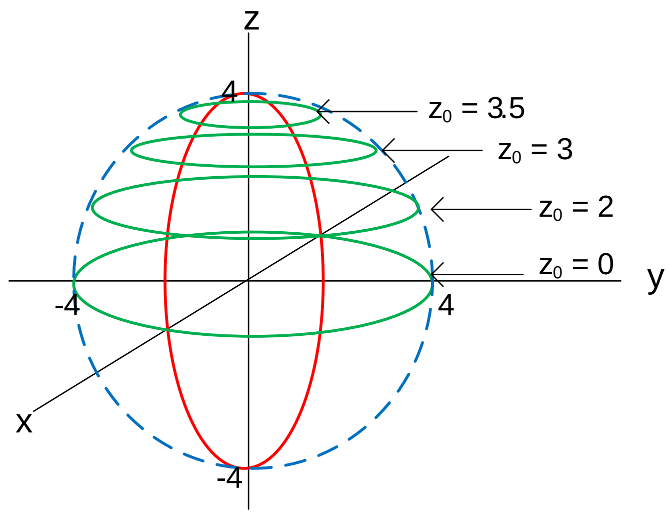
We start by observing that the traces on the coordinate planes are all circles of radius 4.

xy- trace: Set $z = 0$	$x^2 + y^2 = 4^2$
yz- trace: Set $x = 0$	$y^2 + z^2 = 4^2$
xz- trace: Set $y = 0$	$x^2 + z^2 = 4^2$



Furthermore, we can see that by increasing the height of the horizontal traces the radius gets smaller until we reach $z_0 = 4$, where we reach the top of the sphere.

$z = 0$	$x^2 + y^2 = 4^2$
$z = 2$	$x^2 + y^2 = \left(\sqrt{12}\right)^2$
$z = 3$	$x^2 + y^2 = \left(\sqrt{7}\right)^2$
$z = 3.5$	$x^2 + y^2 = \left(\sqrt{3.75}\right)^2$



Similar behavior occurs for all 3 orientations of traces and results in the surface of a sphere.

Ellipsoid – Analog of the Conic Section, Ellipse

The standard equation for an ellipsoid is given as

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

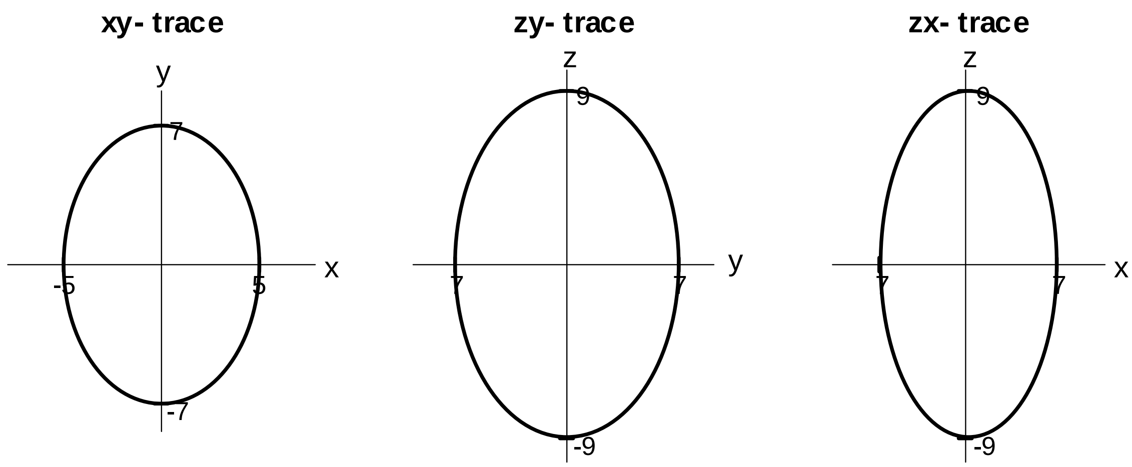
Where, the 'radius' in in the x, y, and z direction equal to a, b, and c respectively.

Example 2: Graph the xy, yz, and the xz traces for the following quadric surface.

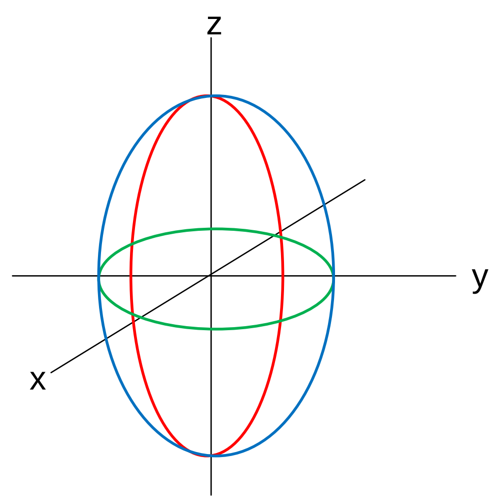
$$\frac{x^2}{25} + \frac{y^2}{49} + \frac{z^2}{81} = 1$$

Solution: The 2D traces for each coordinate axes are found by setting one of the variables to zero as follows:

xy- trace: Set $z = 0$	$\frac{x^2}{25} + \frac{y^2}{49} = 1$	Ellipse with x axis 'radius' equal to 5 and y axis 'radius' equal to 7
yz- trace: Set $x = 0$	$\frac{y^2}{49} + \frac{z^2}{81} = 1$	Ellipse with y axis 'radius' equal to 7 and z axis 'radius' equal to 9
xz- trace: Set $y = 0$	$\frac{x^2}{25} + \frac{z^2}{81} = 1$	Ellipse with x axis 'radius' equal to 5 and z axis 'radius' equal to 9



From the traces we can deduce the shape of the ellipsoid shown below.



Hyperboloid – Analog of the Conic Section, Hyperbola

The hyperboloid comes in three different forms as summarized below.

Hyperboloids	
One Sheet	$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 + 1$
Two Sheets	$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 - 1$
Elliptical Cone (limiting case of one sheet)	$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 + 0$

Example 3: Graph all three cases for a hyperboloid using $a = 2$, $b = 3$, and $c = 4$.

One Sheet:

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \left(\frac{z}{4}\right)^2 + 1$$

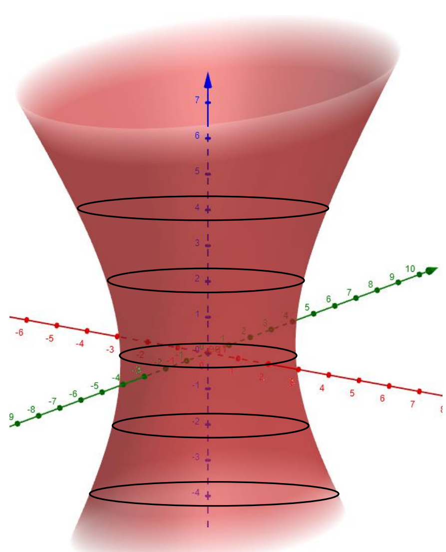
In order to visualize traces in a horizontal plane we first rearrange the equation as follows:

$$16\left(\frac{x^2}{4} + \frac{y^2}{9}\right) = z^2 + 16$$
$$\frac{x^2}{4/16} + \frac{y^2}{9/16} = z^2 + 16$$

Now let's look at a few values of z.

$z = 0$	$\frac{x^2}{4/16} + \frac{y^2}{9/16} = 16$ $\frac{x^2}{4} + \frac{y^2}{9} = 1$	Ellipse with x axis 'radius' equal to $\sqrt{4}$ and y axis 'radius' equal to $\sqrt{9}$
$z = \pm 2$	$\frac{x^2}{4/16} + \frac{y^2}{9/16} = 20$ $\frac{x^2}{5} + \frac{y^2}{11.25} = 1$	Ellipse with x axis 'radius' equal to $\sqrt{5}$ and y axis 'radius' equal to $\sqrt{11.25}$
$z = \pm 4$	$\frac{x^2}{4/16} + \frac{y^2}{9/16} = 32$ $\frac{x^2}{8} + \frac{y^2}{18} = 1$	Ellipse with x axis 'radius' equal to $\sqrt{8}$ and y axis 'radius' equal to $\sqrt{18}$

As the table shows, the horizontal traces are ellipses that increase in size as we move away from $z = 0$.



Two Sheets:

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \left(\frac{z}{4}\right)^2 - 1$$

We can start with the same rearrangement of the equation from above.

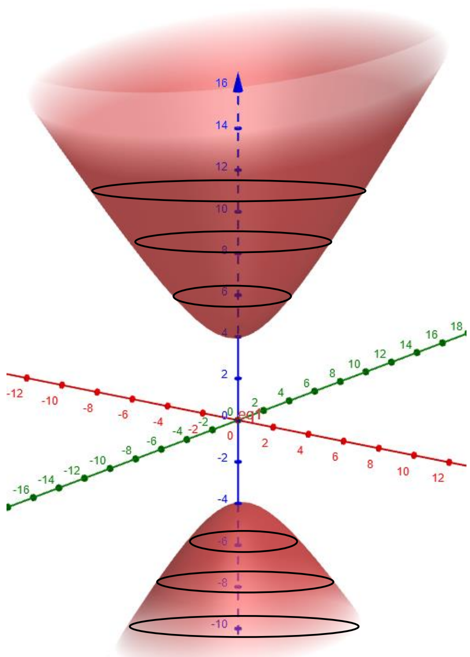
$$\frac{x^2}{4/16} + \frac{y^2}{9/16} = z^2 - 16$$

In this case we see the surface does not contain points when the right hand side is negative.

$$z^2 - 16 < 0$$

$$-4 < z < 4$$

For z values greater than 4, however, result in the same behavior as the one sheet case; i.e. ellipses of increasing size.



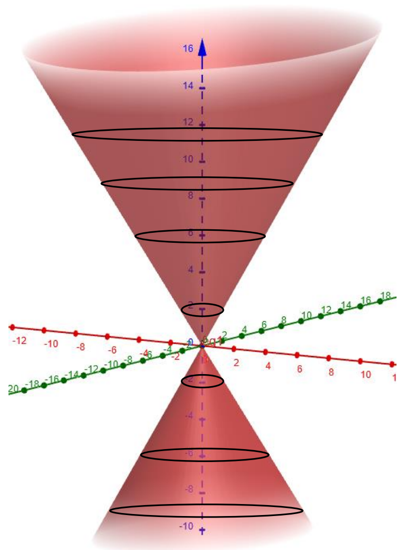
Elliptical Cone:

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \left(\frac{z}{4}\right)^2$$

Once more we can start with the same rearrangement.

$$\frac{x^2}{4/16} + \frac{y^2}{9/16} = z^2$$

In this case the horizontal traces exists for all values of z. When $z = 0$ x and y must also be zero, which results in a 'pinch point' at the origin. The



Paraboloid – Analog of the Conic Section, Parabola

The final surface we will look at is that of a paraboloid. The paraboloid comes in two different forms as summarized below.

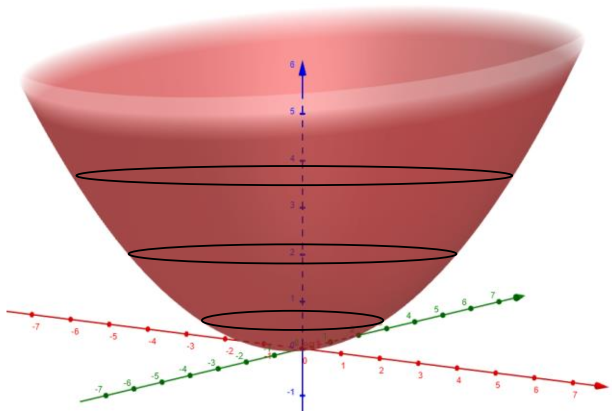
Paraboloid	
Elliptical (bowl)	$z = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$
Hyperbolic (Saddle)	$z = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$

Example 4: Graph the two cases for a paraboloid using $a = 2$ and $b = 3$

Elliptical (bowl):

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = z$$

In this case the surface does not contain values for $z < 0$ since the left hand side can never be negative. Additionally, for $z > 0$, we have increasing size ellipses as we did with hyperboloids above. The result is a bowl shaped surface as shown below.

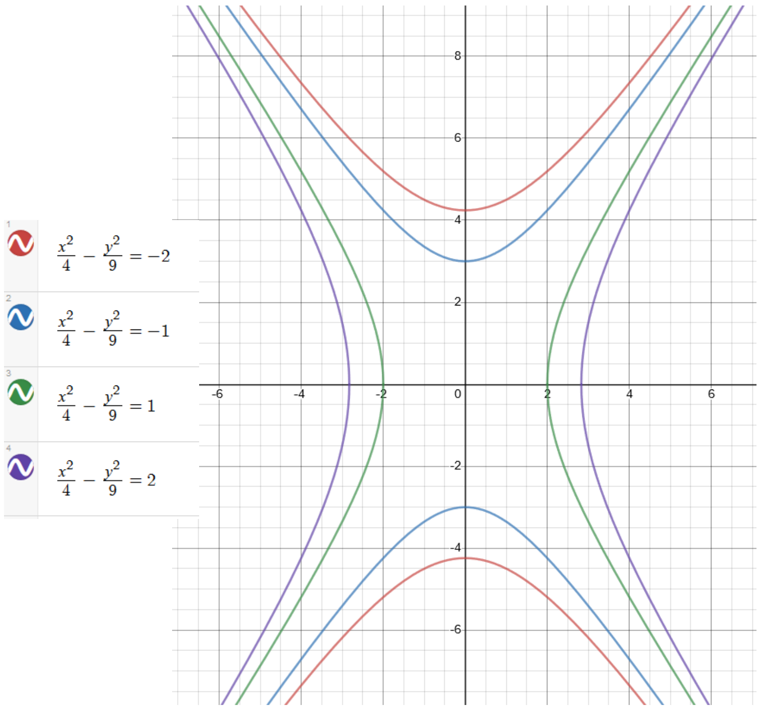


Hyperbolic (Saddle):

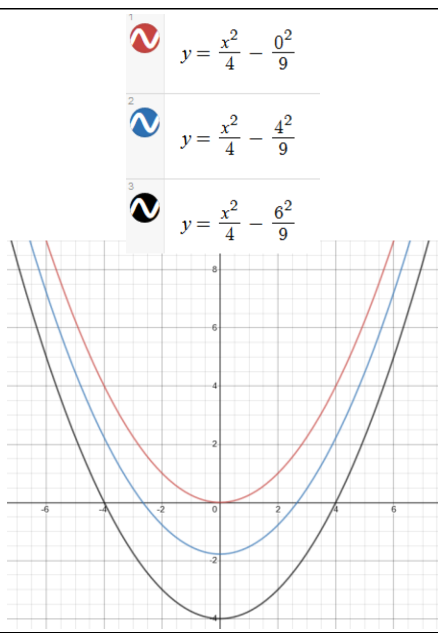
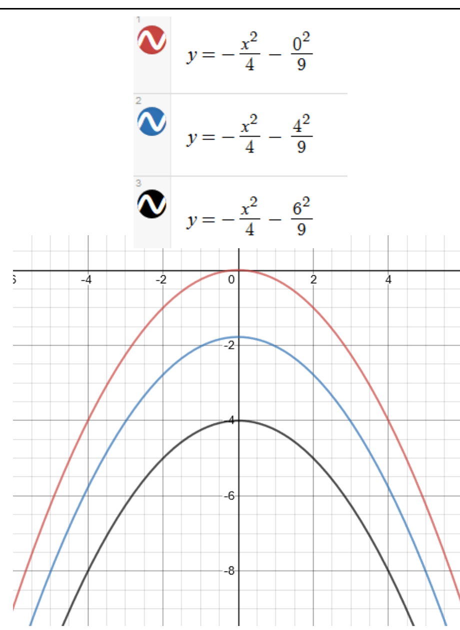
$$\left(\frac{x}{2}\right)^2 - \left(\frac{y}{3}\right)^2 = z$$

In this case the horizontal traces are hyperbolas. When $z > 0$ the hyperboles are oriented with the x-axis and when $z < 0$ they are oriented with the y-axis Let's draw the traces using the table below.

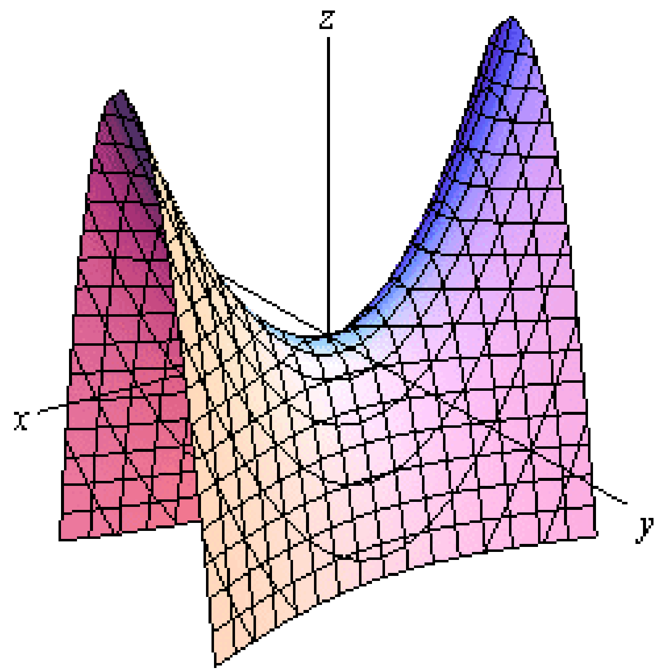
$z = -2$	$\frac{x^2}{4} - \frac{y^2}{9} = -2$ $\frac{y^2}{18} - \frac{x^2}{8} = 1$
$z = -1$	$\frac{x^2}{4} - \frac{y^2}{9} = -1$ $\frac{y^2}{9} - \frac{x^2}{4} = 1$
$z = 1$	$\frac{x^2}{4} - \frac{y^2}{9} = 1$
$z = 2$	$\frac{x^2}{4} - \frac{y^2}{9} = 2$ $\frac{x^2}{8} - \frac{y^2}{18} = 1$



The vertical traces are upward and downward parabolas as shown below.

$zx - traces$	$zy - traces$
$z = \frac{1}{4}x^2 - \left(\frac{y^2}{9}\right)$	$z = -\frac{1}{9}y^2 - \left(\frac{x^2}{4}\right)$
	

The resulting surface is in the shape of a saddle as shown below.



Final Summary for Vector Geometry – Quadric Surfaces

Quadric Surface
A quadric surface is defined by a quadratic equation in three variables. The general form of which is $$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + ax + by + cz + d = 0$$ When $D = E = F = a = b = c = 0$, the quadric axes are parallel to the coordinate axes and the surface is centered at $(0, 0)$. When this is the case the equations are said to be in *standard form*.

Quadric Surfaces in Standard Form	
1. **Sphere**: Centered at $(0, 0, 0)$ with a radius r.	
	$x^2 + y^2 + z^2 = r^2$
2. **Ellipsoid**: Centered at $(0, 0, 0)$ with x, y, and z 'radius' equal to a, b, and c respectively.	
	$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$
3. **Hyperboloid**:	
One Sheet	$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 + 1$
Two Sheets	$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 - 1$
Elliptical Cone (limiting case of one sheet)	$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 + 0$
4. **Paraboloid:**	
Elliptical (bowl)	$z = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$
Hyperbolic (Saddle)	$z = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$

Trace
A trace is the intersection of a surface with a given plane. A trace can be obtained by 'freezing' one of the three variables and sketching the resulting 2D equation. Traces can be used to help us to draw the graph of a surface. *Horizontal Trace:* Setting $z = z_0$ results in a curve in an xy oriented plane. *Vertical Trace:* Setting $x = x_0$ results in a curve in a yz oriented plane. Setting $y = y_0$ results in a curve in an xz oriented plane.

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