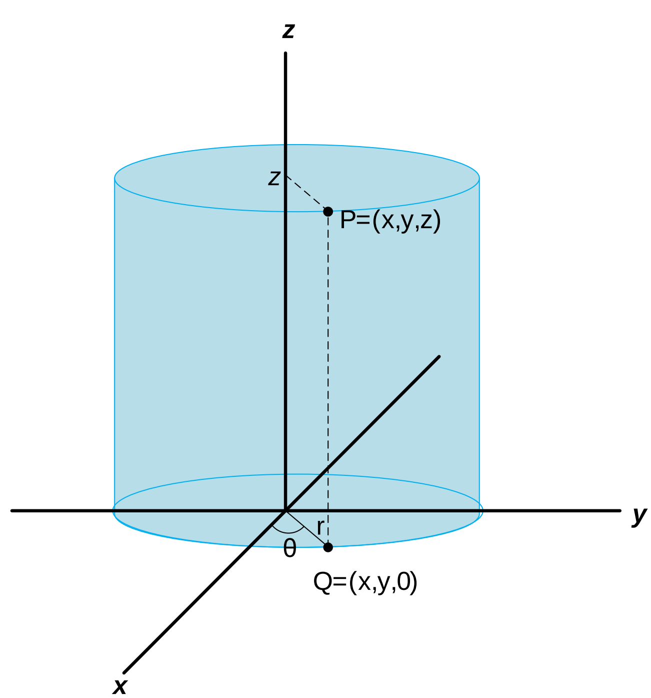
Vector Geometry – Cylindrical and Spherical Coordinates

We first introduced vectors using a rectangular coordinate system. As we'll see this coordinate system is not always the most convenient to work with. Two alternative coordinate systems that become useful for certain problems, (namely ones that display symmetry about a certain axis or rotational symmetry), are the cylindrical and spherical coordinate system. In this section we introduce these two coordinate systems.

To identify a point in three dimensional space one needs to specify three values. In the rectangular coordinate system these three values correspond directly to the x, y, and z location. The cylindrical coordinate system uses polar coordinates to specify the x and y locations but maintains the z value for the third value. This system is best used when the problem displays a symmetry around the z axis. On the other hand, the spherical coordinate system is best used when the problem displays a more general rotational symmetry around the origin. The spherical coordinate system specifies a location in three dimensional space using one value that corresponds to the distance from the origin and two angular values that are specified with respect to the horizontal and vertical planes.

Cylindrical Coordinate System

As mentioned, the cylindrical coordinate system uses polar coordinates to specify a location in the x-y plane and the z coordinate directly for the 'height'. The figure below shows a point, P, with rectangular coordinates of (x, y, z). The same point can be specified using cylindrical coordains as $P = (r\,cos(\theta)\,, r\,sin(\theta)\,, z)$. Note the x and y coordinates correspond to the point Q in the x-y plane.



The table below, which is identical to the one for polar coordinates for x and y, shows how to convert from rectangular to cylindrical and vice-versa.

Rectangular and Cylindrical Coordinate Conversion Formulas	
Cylindrical to Rectangular	***Rectangular to Cylindrical***
$x = r\,cos(\theta)$ $y = r\,sin(\theta)$ $z = z$	$r = \sqrt{x^2 + y^2}$ $\theta = tan^{-1}\left(\frac{y}{x}\right)$ $z = z$

Graphing Equations in Cylindrical Coordinates

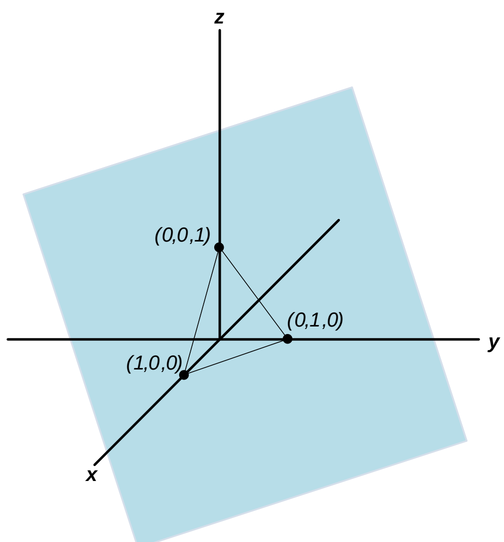
Generally, an equation given in rectangular coordinates can be expressed in cylindrical coordinates, and vice-versa. Sketching the graph may be easier or harder in one or the other coordinate system. This concept can be illustrated with the following examples.

Consider a plane in R^3 given by

$$x + y + z = 1$$

The plane can be sketched by finding where it intersects with the coordinates axes as follows.

x-intercept	y-intercept	z-intercept
$y = z = 0$	$x = z = 0$	$y = x = 0$
$x + 0 + 0 = 1$ $x = 1$	$0 + y + 0 = 1$ $y = 1$	$0 + 0 + z = 1$ $z = 1$



We can also express the equation in cylindrical coordinates using $x = r\,cos(\theta)$ and $y = r\,sin(\theta)$.

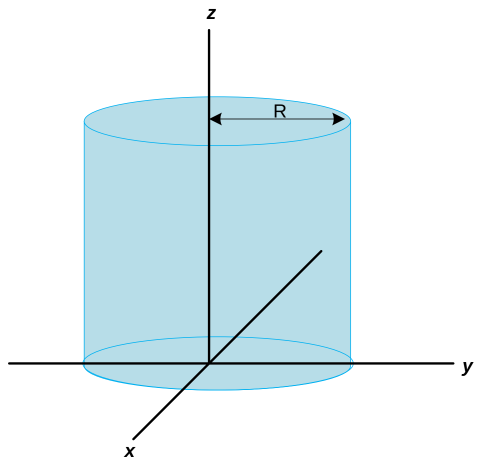
$$\begin{aligned} x + y + z &= 1 \\ r\,cos(\theta) + r\,sin(\theta) + z &= 1 \\ z &= 1 - r(cos(\theta) + sin(\theta)) \end{aligned}$$

However, in this form the equation is not as straightforward to sketch.

At the other extreme we can have the rather simple equation in cylindrical coordinates

$$r = R$$

Since the value of r is fixed for all z and θ, its rather straightforward to see that the equation describes a cylinder with radius R.



Converting this equation to rectangular coordinates we have

$$\begin{aligned} r &= R \\ \sqrt{x^2 + y^2} &= R \\ x^2 + y^2 &= R^2 \end{aligned}$$

Although this equation is also somewhat straightforward to sketch, assuming we are familiar with the equation of a circle, the cylindrical equation is much simpler.

Before introducing spherical coordinates let's look at some examples to become more familiar with using cylindrical coordinates.

Example 1: Convert from cylindrical coordinates to rectangular coordinates.

a. $(4, \pi, 4)$

b. $\left(1, \frac{\pi}{2}, -2\right)$

Solution: Note that cylindrical coordinates are given as (r, θ, z).

a.

$x = r\,cos(\theta) = 4\,cos(\pi) = -4$

$y = r\,sin(\theta) = 4\,sin(\pi) = 0$

$z = 4$

Therefore, $(x, y, z) = (-4, 0, 4)$

b.

$x = r\,cos(\theta) = 1\,cos(\pi/2) = 0$

$y = r\,sin(\theta) = 1\,sin(\pi/2) = 1$

$z = -2$

Therefore, $(x, y, z) = (0, 1, -2)$

Example 2: Convert from rectangular coordinates to cylindrical coordinates.

a. $(1, \sqrt{3}, 7)$ *b.* $\left(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}, 2\right)$

Solution:

a.

$$r = \sqrt{x^2 + y^2} = \sqrt{1+3} = 2$$
$$\theta = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$
$$z = 7$$

Therefore, $(r, \theta, z) = \left(2, \frac{\pi}{3}, 7\right)$

b.

$$r = \sqrt{x^2 + y^2} = \sqrt{\frac{25}{2} + \frac{25}{2}} = 5$$
$$\theta = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(\frac{5/\sqrt{2}}{5/\sqrt{2}}\right) = \frac{\pi}{4}$$
$$z = 2$$

Therefore, $(r, \theta, z) = \left(2, \frac{\pi}{4}, 7\right)$

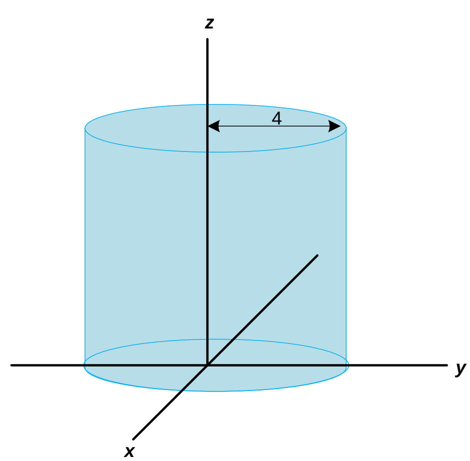
Example 3: Sketch the following surfaces (described in cylindrical coordinates).

a. $r = 4$ *b.* $z = -2$

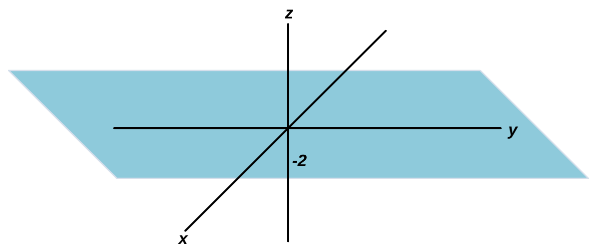
c. $\theta = \frac{\pi}{3}$ *d.* $z^2 + r^2 = 100$

Solution:

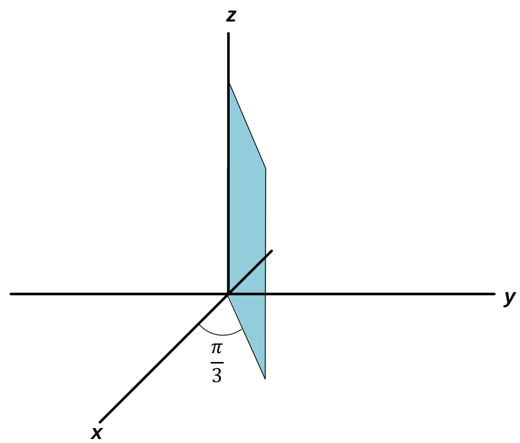
a. Fixing r and allowing the other two variables, θ and z, to vary over all values results in a cylindrical surface with a radius of 4.



b. Fixing z and allowing the other two variables, θ and r, to vary over all values results in a plane oriented with the z-y plane at a height of -2.



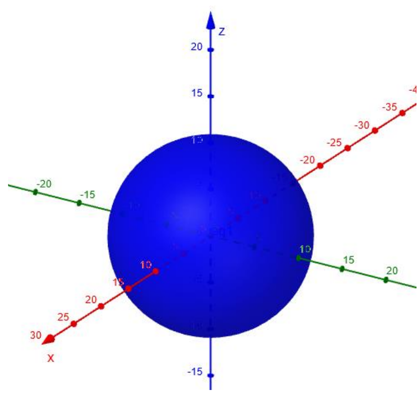
c. Fixing θ and allowing the other two variables, z and r, to vary over all values results in a vertical half plane oriented at an angle of $\frac{\pi}{3}$.



d. In this case we convert the equation to rectangular as follows:

$$z^2 + r^2 = 100$$
$$z^2 + x^2 + y^2 = 100$$
$$x^2 + y^2 + z^2 = 10^2$$

Which is a sphere with radius 10.

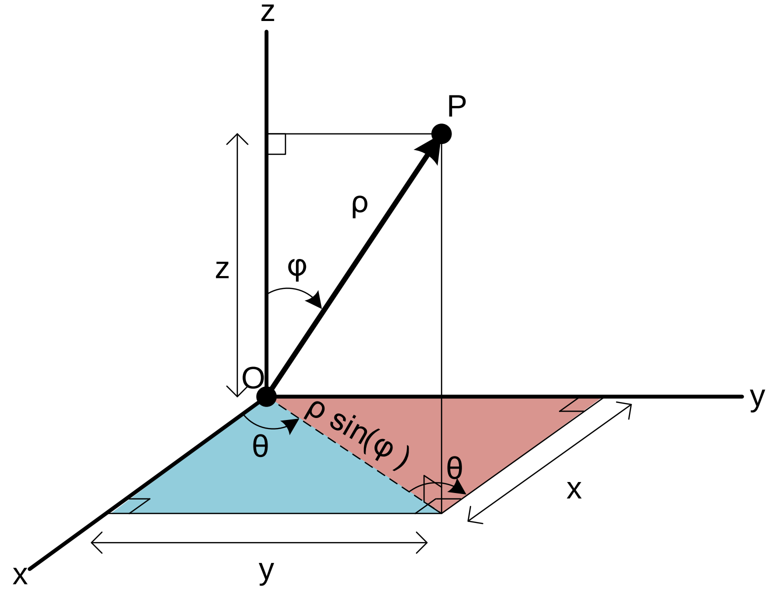


The first three problems are referred to as *level surfaces*. Level surfaces of a coordinate system are surfaces obtained by setting one of the coordinates equal to a constant. In the rectangular system the levels surfaces are planes. Example 6 will show the levels surfaces for spherical coordinates.

Spherical Coordinate System

The spherical coordinate system describes a point, P, in R^3 using the following three values.

- ρ: The distance from the origin to the point, P, where $\rho \geq 0$
- θ: The angle of the projection for $\overrightarrow{0P}$ onto the x-y plane, where $-180^\circ \leq \theta \leq 180^\circ$
- ϕ: The angle of declination, which measures how much the vector, $\overrightarrow{0P}$, declines from the vertical, where $0^\circ \leq \phi \leq 180^\circ$



Let's use the figure above to show how we can convert between rectangular and spherical coordinates.

Rectangular to Spherical: $(x, y, z) \rightarrow (\rho, \theta, \phi)$

The value of ρ is simply given by the extended Pythagorean Theorem as

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

The value of θ is found using the blue right triangle as

$$\theta = tan^{-1}\left(\frac{y}{x}\right)$$

Finally, ϕ is found using the vertical right triangle as

$$\phi = cos^{-1}\left(\frac{z}{\rho}\right)$$

Spherical to Rectangular: $(\rho, \theta, \phi) \rightarrow (x, y, z)$

The x coordinate is found using the red right triangle as

$$cos(\theta) = \frac{x}{\rho\, sin(\phi)} \rightarrow x = \rho\, sin(\phi)\, cos(\theta)$$

The y coordinate is found using the blue right triangle as

$$sin(\theta) = \frac{y}{\rho\, sin(\phi)} \rightarrow y = \rho\, sin(\phi)\, sin(\theta)$$

Finally, the z coordinate is found using the vertical right triangle as

$$cos(\phi) = \frac{z}{\rho} \rightarrow z = \rho\, cos(\phi)$$

The results are summarized in the table below.

Rectangular and Spherical Coordinate Conversion Formulas	
Spherical to Rectangular	***Rectangular to Spherical***
$x = \rho\, sin(\phi)\, cos(\theta)$ $y = \rho\, sin(\phi)\, sin(\theta)$ $z = \rho\, cos(\phi)$	$\rho = \sqrt{x^2 + y^2 + z^2}$ $\theta = tan^{-1}\left(\frac{y}{x}\right)$ $\phi = cos^{-1}\left(\frac{z}{\rho}\right)$

Let's finish with some examples, similar to the ones we did using cylindrical coordinates, using spherical coordinates.

Example 4: Convert from spherical coordinates to rectangular coordinates.

a. $(3, \pi, 0)$

b. $\left(6, \frac{\pi}{6}, \frac{5\pi}{6}\right)$

Solution: Note that spherical coordinates are given as (ρ, θ, ϕ).

a.

$x = \rho\, sin(\phi)\, cos(\theta) = 4\, cos(\pi) = -4$

$y = r\, sin(\theta) = 4\, sin(\pi) = 0$

$z = 4$

Therefore, $(x, y, z) = (-4, 0, 4)$

b.

$x = r\, cos(\theta) = 1\, cos(\pi/2) = 0$

$y = r\, sin(\theta) = 1\, sin(\pi/2) = 1$

$z = -2$

Therefore, $(x, y, z) = (0, 1, -2)$

Example 5: Convert from rectangular coordinates to spherical coordinates.

a. $(1, 1, 1)$

b. $\left(\frac{1}{2}, \frac{\sqrt{2}}{2}, \sqrt{3}\right)$

Solution:

a.

$r = \sqrt{x^2 + y^2} = \sqrt{1 + 3} = 2$

$\theta = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

$z = 7$

Therefore, $(r, \theta, z) = \left(2, \frac{\pi}{3}, 7\right)$

b.

$r = \sqrt{x^2 + y^2} = \sqrt{\frac{25}{2} + \frac{25}{2}} = 5$

$\theta = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(\frac{5/\sqrt{2}}{5/\sqrt{2}}\right) = \frac{\pi}{4}$

$z = 2$

Therefore, $(r, \theta, z) = \left(2, \frac{\pi}{4}, 7\right)$

Example 6: Sketch the following surfaces (described in cylindrical coordinates).

a. $\rho = 4$

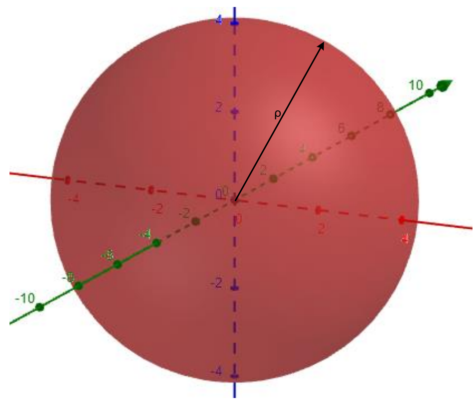
b. $\theta = \frac{\pi}{4}$

c. $\phi = \frac{\pi}{6}$

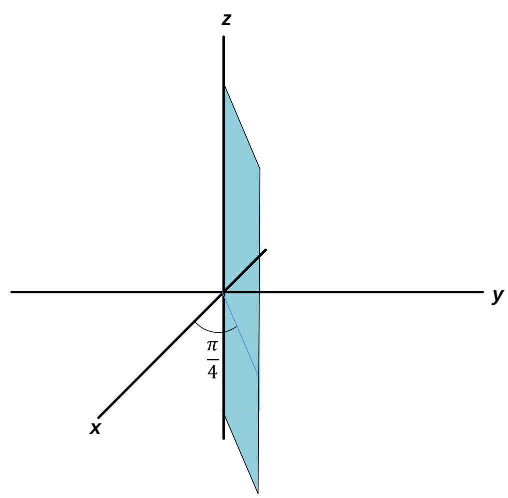
d. $\rho = 2,\ 0 \leq \phi \leq \frac{\pi}{2}$

Solution:

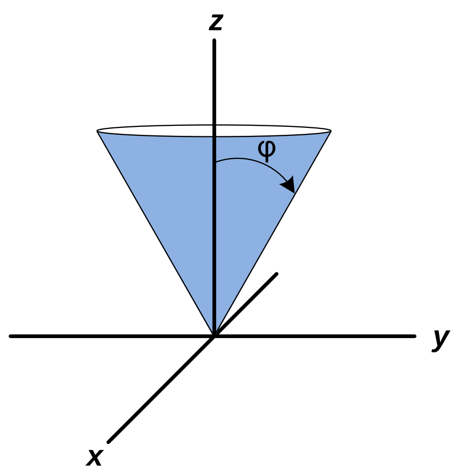
a. Fixing ρ and allowing the other two variables, θ and ϕ, to vary over all values results in a spherical surface with a radius of 4.



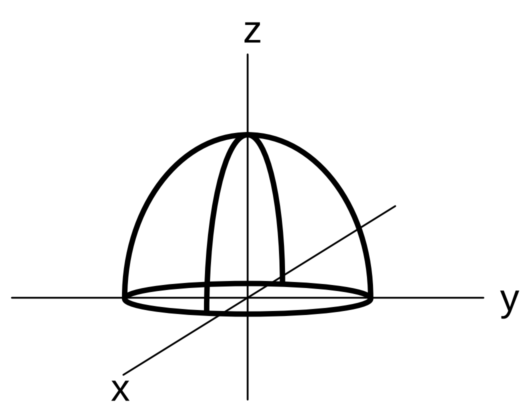
b. Fixing θ and allowing the other two variables, ρ and ϕ, to vary over all values results in a vertical half plane, similar to the cylindrical case, oriented at an angle of $\frac{\pi}{4}$.



c. Fixing ϕ and allowing the other two variables, ρ and θ, to vary over all values results in a right circular cone with an opening of $\frac{\pi}{6}$.

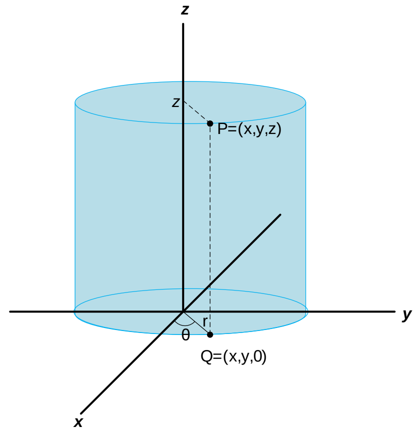


d. This problem is similar to the first problem, except for the fact that the angle of declination is limited to $0 \le \phi \le \frac{\pi}{2}$ resulting in a half sphere as shown.



Final Summary for Vector Geometry – Cylindrical and Spherical Coordinates

Cylindrical Coordinate System



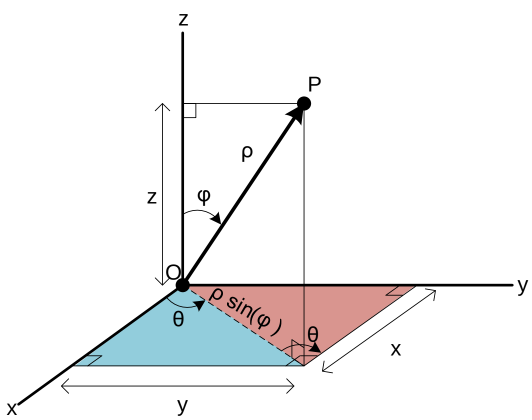
A point in cylindrical coordinates is described by: $P = \ (r, \theta, z)$

- r: The horizontal distance from the origin.
- θ: The polar angle measured from the positive x-axis.
- z: The vertical distance from the origin.

Rectangular and Cylindrical Coordinate Conversion Formulas

Cylindrical to Rectangular	***Rectangular to Cylindrical***
$x = r\,cos(\theta)$ $y = r\,sin(\theta)$ $z = z$	$r = \sqrt{x^2 + y^2}$ $\theta = tan^{-1}\left(\frac{y}{x}\right)$ $z = z$

Spherical Coordinate System



A point in spherical coordinates is described by: $P = \ (\rho, \theta, \phi)$

- ρ: The distance from the origin to the point, P, where $\rho \geq 0$
- θ: The angle of the projection for $\overrightarrow{0P}$ onto the x-y plane, where $-180^{\circ} \leq \theta \leq 180^{\circ}$
- ϕ: The angle of declination, which measures how much the vector, $\overrightarrow{0P}$, declines form the vertical, where $0^{\circ} \leq \phi \leq 180^{\circ}$

Rectangular and Spherical Coordinate Conversion Formulas

Spherical to Rectangular	***Rectangular to Spherical***
$x = \rho\,sin(\phi)\,cos(\theta)$ $y = \rho\,sin(\phi)\,sin(\theta)$ $z = \rho\,cos(\phi)$	$\rho = \sqrt{x^2 + y^2 + z^2}$ $\theta = tan^{-1}\left(\frac{y}{x}\right)$ $\phi = cos^{-1}\left(\frac{z}{\rho}\right)$

Level Surfaces

Level Surfaces are surfaces obtained by setting one of the coordinates to a constant.

Rectangular Coordinate System:

- $x = C:$ Vertically aligned plane parallel to the y-z plane.
- $y = C:$ Vertically aligned plane parallel to the x-z plane.
- $z = C:$ Horizontally aligned plane parallel to the x-y plane

Cylindrical Coordinate System:

- $r = C:$ Cylinder with radius, C.
- $\theta = C:$ Vertical half plane oriented at an angle, C.
- $z = C:$ Horizontally aligned plane parallel to the x-y plane

Spherical Coordinate System:

- $\rho = C:$ Sphere with radius, C
- $\theta = C:$ Vertical half plane oriented at an angle, C.
- $\phi = C:$ *R*ight circular cone with an opening at an angle, C.

By: ferrantetutoring