Vector Calculus – Vector Valued Functions

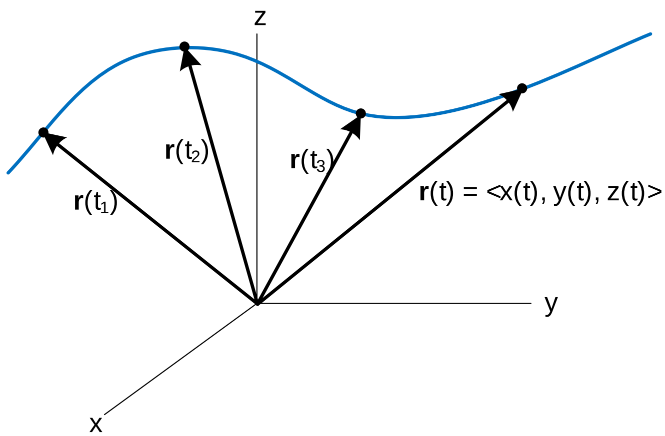
The next few sections will involve using calculus on vector-valued functions. Vector-valued functions are especially useful for studying curves and for analyzing the motion of objects in space. As we know, the analysis of motion relies heavily on the ideas of calculus. However, before we learn how to perform calculus on vector-valued functions we need to learn precisely what vector-valued functions are and how to work with them. Therefore, this section will focus on introducing vector-valued functions, focusing on how they can be used to describe curves and more precisely how they can be used to analyze the motion of objects in space.

Vector-valued Functions

As a practical example let's consider a particle whose position over time in R^3 can be specified for each coordinate axis separately, i.e. $x(t)$, $y(t)$, and $z(t)$. It would be most convenient to combine these coordinate positions into a single position vector as follows:

$$\boldsymbol{r}(t) = \langle x(t), y(t), z(t) \rangle$$

Where, $\boldsymbol{r}(t)$ is a vector-valued function. We can think of $\boldsymbol{r}(t)$ as a vector that points to the position of the particle for any time, t. Tracing out these positions over time we can draw the path of the particle as a curve as shown in the figure below.



In a more general sense a vector-valued function is any function whose domain is a set of real number and whose range is a set of vectors. The variable t is called a parameter, which doesn't necessarily have to represent time, and the functions $x(t), y(t)$ and $z(t)$ are called the components or coordinate functions. As stated, if we trace the terminal point of a vector-valued function as the parameter varies a path is traced in space. With this we refer to $\boldsymbol{r}(t)$ as a path or as a vector parameterization of a path.

We can also represent the vector parameterization of a path as a curve with a set of parametric equations as

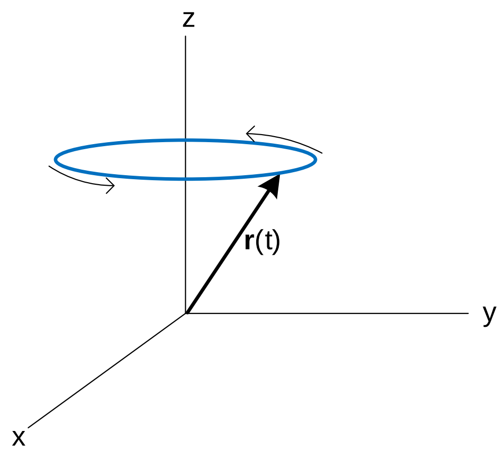
$$c(t) = \big(x(t), y(t), z(t)\big)$$

Although, the two representations seem equivalent it is important to distinguish between the path traced out by $\boldsymbol{r}(t)$, and the underlying curve represented by $c(t)$. The curve is the set of all points, $x(t), y(t), z(t)$, as t varies over its domain. However, the path referred to by $\boldsymbol{r}(t)$ represents the particular way the curve is traversed, e.g. it may traverse the curve several times, reverse direction, move back and forth, etc. Let's look at an example to illustrate this point.

Example 1: Describe the curve as well as the path taken by the following equation.

$$\boldsymbol{r}(t) = \langle cos(t), sin(t), 1\rangle, \quad -\infty < t < \infty$$

Solution: The curve is a unit circle parallel to the x-y plane at a height of one.



However, since $-\infty < t < \infty$, the path is traced in a counter clockwise direction and the vector traverses that path an infinite number of times.

Generally speaking it is difficult to sketch a curve in R^3. The task can sometimes be helped by first drawing projections onto the three planes. We project onto each plane by setting the third coordinate to zero.

Projection onto x-y plane: Let $z(t) = 0$, $\boldsymbol{r}(t) = \langle x(t), y(t), 0\rangle$

Projection onto x-z plane: Let $y(t) = 0$, $\boldsymbol{r}(t) = \langle x(t), 0, z(t)\rangle$

Projection onto y-z plane: Let $x(t) = 0$, $\boldsymbol{r}(t) = \langle 0, y(t), z(t)\rangle$

Let's look at a few examples before we summarize and move onto the next section where we learn how to perform calculus on vector-valued functions.

Example 2: Sketch the following vector-valued functions. Indicate the direction a particle would move along the path. For *c*. draw the $\boldsymbol{r}(0)$ and $\boldsymbol{r}(4)$.

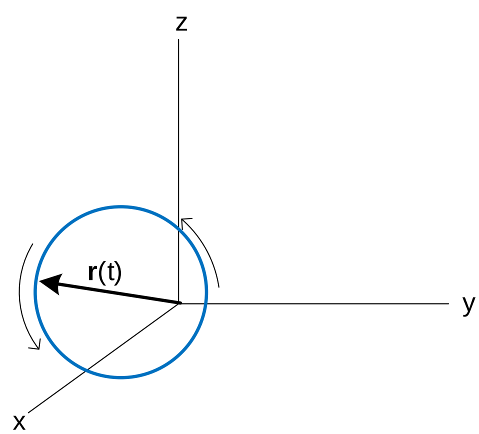
a. $\boldsymbol{r}(t) = \langle 1, 4\,cos(t), 4\,sin(t)\rangle$

b. $\boldsymbol{r}(t) = \langle 4\,cos(t), 4\,sin(t)\,, t\rangle$

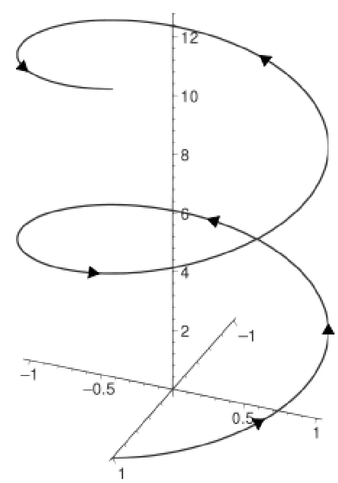
c. $\boldsymbol{r}(t) = \langle t, t^2\rangle,\ 0 \le t \le 4$

Solution:

a. This curve is a circle as in example 1, however it is orientated y-z plane.



b. This curve also traces out a circle oriented in the x-y plane as in example 1. However, the z coordinate linearly increase with t. The curve is known as a helix.



c. In this case we can create a function, $y(x)$, from the given description.

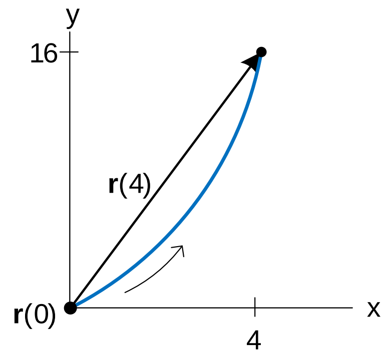
$$\boldsymbol{r}(t) = \langle t, t^2\rangle,\ 0 \le t \le 4$$

By letting $x = t$, we can write the curve as

$$y(t) = t^2$$

$$y(x) = x^2$$

Which is a simple parabola. However, since the path start at $t = 0$ and ends at $t = 4$ we sketch the path as shown.



Example 3: Sketch the following vector-valued function. Indicate the direction a particle would move along the path and draw the vectors $\boldsymbol{r}(0)$, $\boldsymbol{r}(1)$, $\boldsymbol{r}(2)$, and $\boldsymbol{r}(3)$.

$$\boldsymbol{r}(t) = \langle 1 + t, 6 - t, 0 \rangle$$

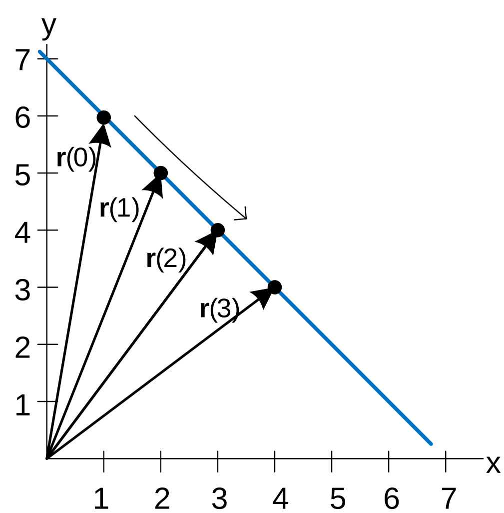
Solution: In a previous lesson we introduced the vector parameterization of a line in R^3 as:

$$\boldsymbol{r}(t) = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle = \langle x_0 + at,\ y_0 + bt,\ z_0 + ct \rangle$$

Which parameterizes a line that passes through $\langle x_0, y_0, z_0 \rangle$ in the direction of $\langle a, b, c \rangle$. In the case above we have a line that passes through $\langle 1, 6, 0 \rangle$ in the direction of $\langle 1, -1, 0 \rangle$

We can draw the line in the x-y plane since $z = 0$ for all t. The vectors evaluated below are also shown in the figure.

$$\boldsymbol{r}(0) = \langle 1 + 0, 6 - 0, 0 \rangle = \langle 1, 6, 0 \rangle$$
$$\boldsymbol{r}(1) = \langle 1 + 1, 6 - 1, 0 \rangle = \langle 2, 5, 0 \rangle$$
$$\boldsymbol{r}(2) = \langle 1 + 2, 6 - 2, 0 \rangle = \langle 3, 4, 0 \rangle$$
$$\boldsymbol{r}(3) = \langle 1 + 3, 6 - 3, 0 \rangle = \langle 4, 3, 0 \rangle$$



Final Summary for Vector Calculus – Vector-Valued Functions

Vector-Valued Function

A vector-valued function is any function whose domain is a set of real number and whose range is a set of vectors. The variable t is called a parameter, which doesn't necessarily have to represent time, and the functions $x(t), y(t)$ and $z(t)$ are called the components or coordinate functions.

$$\boldsymbol{r}(t) = \langle x(t), y(t), z(t) \rangle$$

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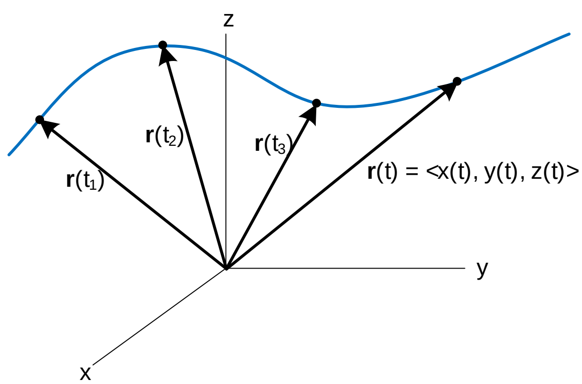

Projections

Projections of $\boldsymbol{r}(t)$ onto a plane can help us sketch the underlying curve. We project onto each plane by setting the third coordinate to zero.

Projection onto x-y plane: Let $z(t) = 0,\ \ \boldsymbol{r}(t) = \langle x(t), y(t), 0 \rangle$

Projection onto x-z plane: Let $y(t) = 0,\ \ \boldsymbol{r}(t) = \langle x(t), 0, z(t) \rangle$

Projection onto y-z plane: Let $x(t) = 0,\ \ \boldsymbol{r}(t) = \langle 0, y(t), z(t) \rangle$

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