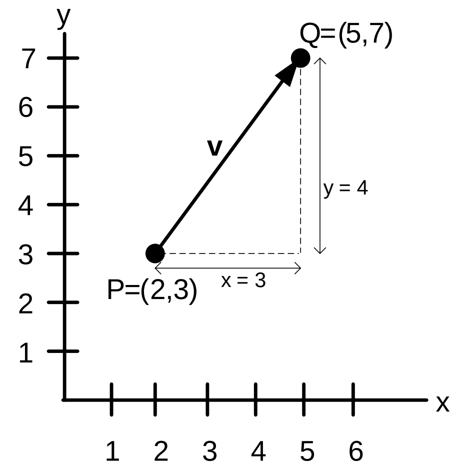
Vector Geometry – Vectors Introduction

Vectors are used to represent quantities that have both a magnitude and direction. Various phenomenon in science and engineering require the use of vectors. For example, two vehicles starting at the same location and traveling at the same speed may end up at vastly different locations based on the direction in which each is traveling. Accordingly, it should be apparent that speed alone is not adequate to describe the motion of objects. Instead we use the vector quantity, v*elocity,* to describe both the speed and the direction of a moving object. In this section we develop the basic properties of vectors, which we will use in the remainder of our studies of calculus 3.

Vectors

Although most of the vector concepts can be easily extended to n-dimensions, starting with the description in 2-dimensions will allow us to better visualize the concepts. A 2-dimensional vector is defined by two points in a plane, e.g. an initial point, P, and a terminal point Q. We draw the vector with an arrow starting at P and ending at Q as shown below.



We write the vector as

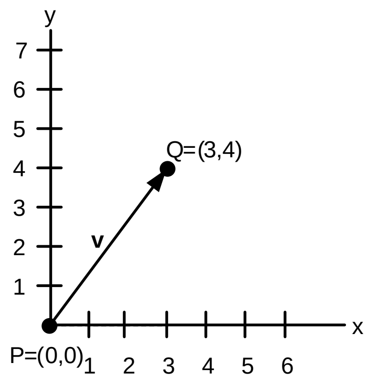
$$\boldsymbol{v} = \overrightarrow{PQ}$$

Where, we use the bolt type, $\boldsymbol{v}$**,** to indicate the quantity is a vector.

We identify a vector based on the length of its so-called components. An n-dimesional vector has n components. In the figure above the x-component of $\boldsymbol{v}$ is 3 and the y-component of $\boldsymbol{v}$ is 4, denoted as $\boldsymbol{v} = \langle 3,4 \rangle$. The components are computed using the points P and Q as follows:

$$\boldsymbol{v} = \overrightarrow{PQ} = \langle Q_x - P_x, Q_y - P_y \rangle = \langle 5-2, 7-3 \rangle = \langle 3,4 \rangle$$

The components of a vector can be used to indicate the length, (magnitude), and the direction of the desired quantity. It's important to note however, that a vector **does not** have a fixed position. In other words, we can move any vector to begin at a new point without changing its length or direction, (called translation). Imagine moving the vector in the figure above so that its staring point is at the origin, i.e. $P = (0,0)$. Since the starting point is, in most cases, arbitrary, we will assume all vectors start at the origin. In this case the components of a vector are simply the coordinates of the endpoint and we refer to this as a vector in *standard position*.



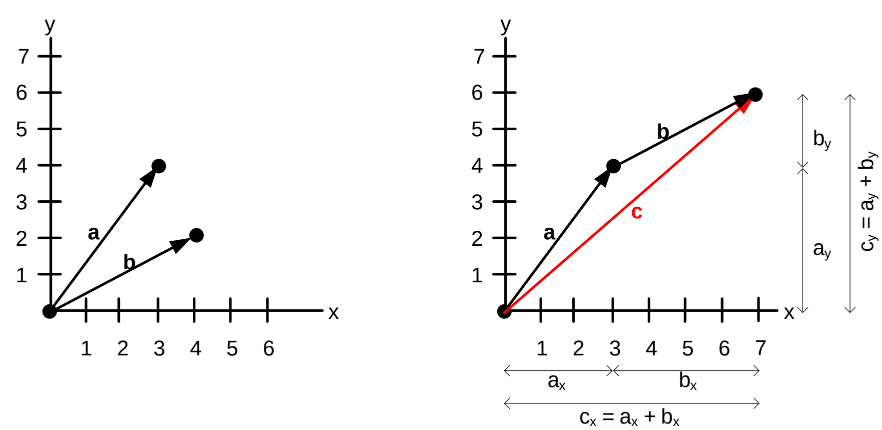
$$\boldsymbol{v} = \langle 3,4 \rangle$$

Operation with Vectors

Just as with ordinary numbers, (scalars), we would like to perform basic operations with vectors, e.g. addition, subtraction, etc. In this section we'll cover basic adding and subtracting of vectors as well as scalar multiplication, which are simple extensions of the scalar operations. Multiplying two vectors, however, is not a simple extension and we will cover that later in our lessons.

Vector Addition/Subtraction

Geometrically, two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$ are added by placing the initial point of $\boldsymbol{b}$ on the terminal point of $\boldsymbol{a}$ and drawing the resultant vector, $\boldsymbol{c}$, as shown below.

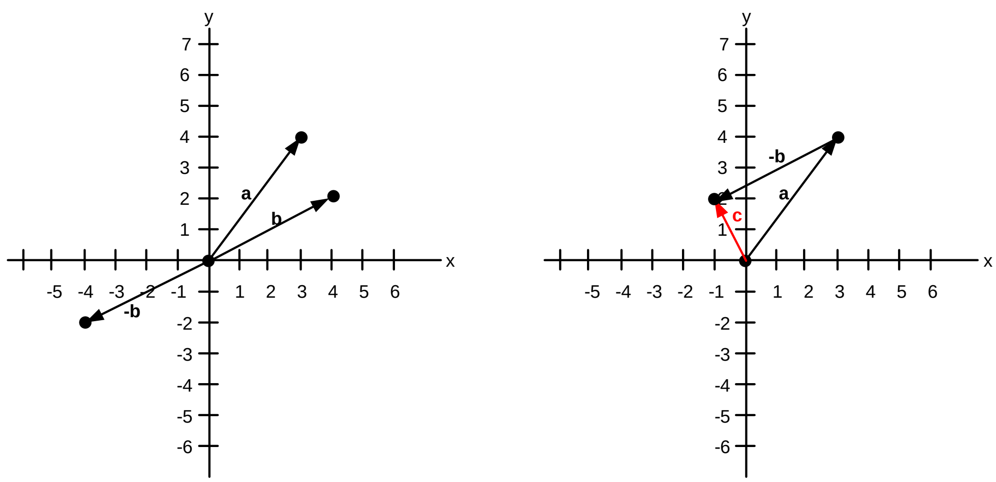


As shown in the figure, the vectors can also be added algebraically by simply adding the components, which of course is much more convenient.

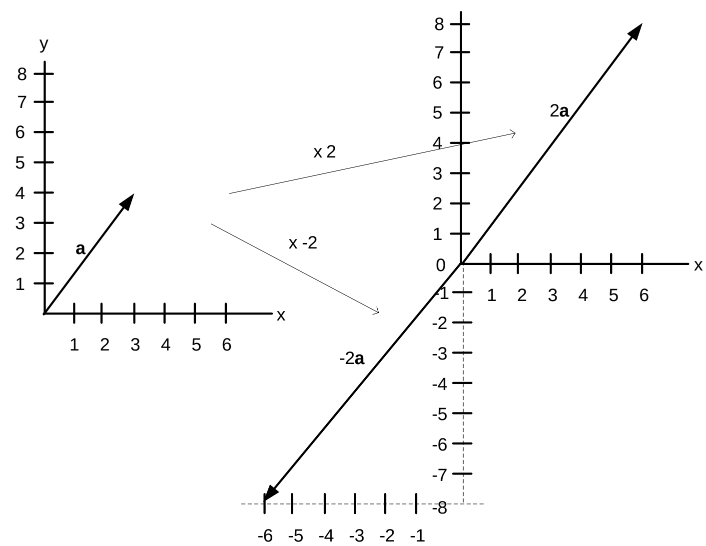
$$\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$$
$$\boldsymbol{c} = \langle 3,4 \rangle + \langle 4,2 \rangle$$
$$\boldsymbol{c} = \langle 3 + 4, 4 + 2 \rangle$$
$$\boldsymbol{c} = \langle 7,6 \rangle$$

Similarly, subtraction can be performed geometrically by adding the negative of the second vector, i.e. $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$**,** where the negative of a vector is computed by negating each component. Subtracting the two vectors above we have.

$$\begin{aligned} \boldsymbol{a} - \boldsymbol{b} &= \boldsymbol{a} + (-\boldsymbol{b}) \\ &= \langle 3,4 \rangle + \langle -4,-2 \rangle \\ &= \langle 3-4, 4-2 \rangle \\ &= \langle -1,2 \rangle \end{aligned}$$



Finally, scalar multiplication simply changes the length of a vector. If the scalar is negative it also reverses the sign of each component, i.e. reverses its direction.



$$2\boldsymbol{a} = 2\langle 3,4 \rangle = \langle 6,8 \rangle \qquad -2\boldsymbol{a} = -2\langle 3,4 \rangle = \langle -6,-8 \rangle$$

Below is a summary of some basic vector operations using components.

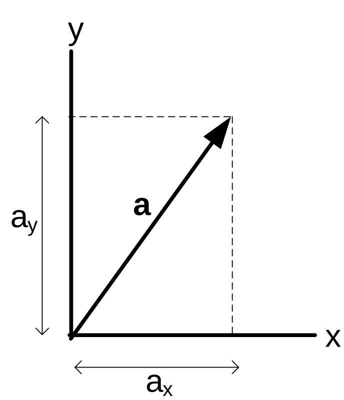
Vector Operations Using Components		
If $\boldsymbol{a} = \langle a_x,\ a_y \rangle$ and **b**$= \langle b_x,\ b_y \rangle$ then:		
i.	*Addition*	$\boldsymbol{a} + \boldsymbol{b} = \langle a_x + b_x,\ a_y + b_y \rangle$
ii.	*Subtraction*	$\boldsymbol{a} - \boldsymbol{b} = \langle a_x - b_x, a_y - b_y \rangle$
iii.	*Scalar Multiplication*	$\lambda\boldsymbol{a} = \langle \lambda a_x, \lambda a_y \rangle$
iv.	*Addition Identity*	$\boldsymbol{a} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{a} = \boldsymbol{a}$

Vectors also enjoy some of the basic properties of real number algebra as summarized below.

Basic Properties of Vector Algebra		
For all vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and for all scalars, λ		
i.	Commutative Law	$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$
ii.	Associative Law	$\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{b} + \boldsymbol{a}) + \boldsymbol{c}$
iii.	Distributive law for Scalars	$\lambda(\boldsymbol{a} + \boldsymbol{b}) = \lambda\boldsymbol{b} + \lambda\boldsymbol{a}$

Vector Norm and Unit Vectors

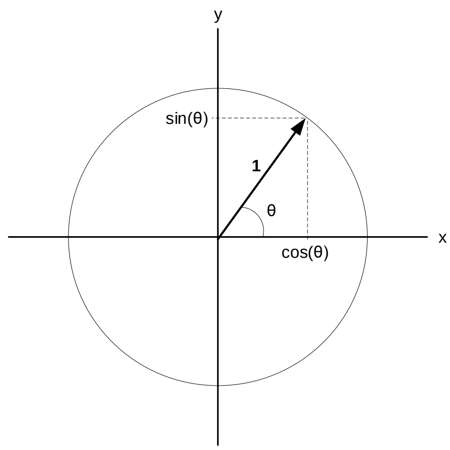
We refer to the length or magnitude of a vector as the *norm* of a vector and we write it as $\|\boldsymbol{a}\|$. In 2 dimensions it is computed using the familiar Pythagorean theorem, although the formula holds for n-dimensions.



$$\|\boldsymbol{a}\| = \sqrt{{a_x}^2 + {a_y}^2}$$

A vector with a length of 1 is called a *unit vector,* usually written as $\hat{\boldsymbol{v}}$. Using the unit circle studied in trigonometry we can specify a unit vector according to the angle it makes with the positive x-axis as follows.

$$\hat{\boldsymbol{v}} = \langle \cos\theta\,, \sin(\theta) \rangle$$



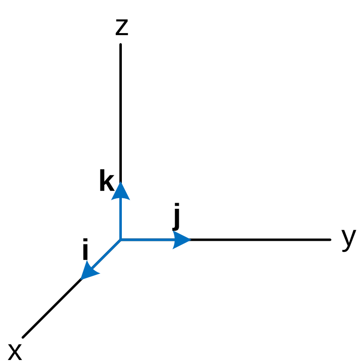
Furthermore, any vector, $\boldsymbol{v}$**,** can be scaled to be a unit vector as follows:

$$\hat{\boldsymbol{v}} = \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v}$$

Multiplying through by $\|\boldsymbol{v}\|$ we see that a vector can also be expressed as follows:

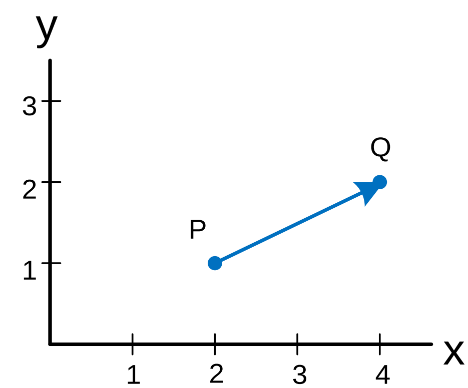
$$\boldsymbol{v} = \|\boldsymbol{v}\|\hat{\boldsymbol{v}}$$
$$\boldsymbol{v} = \|\boldsymbol{v}\|\langle cos\,\theta\,, sin(\theta) \rangle$$

Finally, we introduce what are referred to as basis vectors. Basis vectors are vectors that can be linearly combined so that they are able to represent any vector in the particular vector space, e.g. the 2 or 3-dimensional space. In 3-dimensional space the **standard basis vectors** are as shown below.

Standard Basis Vectors for Rectangular Coordinate System in 3-Dimensions	
$\boldsymbol{i} = \langle 1, 0, 0 \rangle$ $\boldsymbol{j} = \langle 0, 1, 0 \rangle$ $\boldsymbol{k} = \langle 0, 0, 1 \rangle$	

Finally, let's do some examples to practice the above concepts.

Example 1: Find the components of and sketch a vector, $\boldsymbol{v}$, in standard position that represents the vector, $\overrightarrow{\boldsymbol{PQ}}$, shown in the figure below.

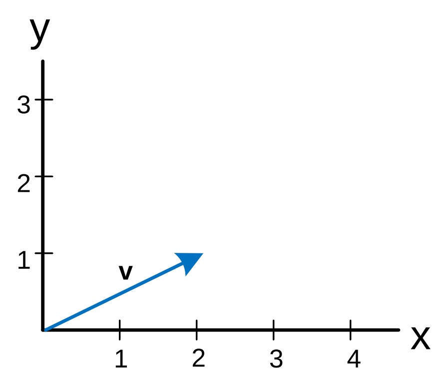


The endpoints of the given vector are

$$P = (2, 1) \qquad\qquad Q = (4, 2)$$

Therefore, we have:

$$\begin{aligned} \boldsymbol{v} &= \overrightarrow{\boldsymbol{PQ}} \\ &= \langle (Q_x - P_x), (Q_y - P_y) \rangle \\ &= \langle (4-2), (2-1) \rangle \\ &= \langle 2, 1 \rangle \end{aligned}$$



Example 2: For $\boldsymbol{a} = \langle 3, 2 \rangle$ and $\boldsymbol{b} = \langle 1, 4 \rangle$, compute the following:

a. $2\boldsymbol{a} + \boldsymbol{b}$	*b.* $\boldsymbol{a} - 5\boldsymbol{b}$	*c.* $-2\boldsymbol{a} - 4\boldsymbol{b}$
d. $\frac{1}{\lVert\boldsymbol{a}\rVert}\boldsymbol{a}$	*e.* $\lVert\boldsymbol{a} + \boldsymbol{b}\rVert$	*f.* $\lVert\boldsymbol{a}\rVert + \lVert\boldsymbol{b}\rVert$

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a)

$$\begin{aligned} 2\boldsymbol{a} + \boldsymbol{b} &= 2\langle 3, 2\rangle + \langle 1, 4\rangle \\ &= \langle 6, 4\rangle + \langle 1, 4\rangle \\ &= \langle 7, 8\rangle \end{aligned}$$

b)

$$\begin{aligned} \boldsymbol{a} - 5\boldsymbol{b} &= \langle 3, 2\rangle - 5\langle 1, 4\rangle \\ &= \langle 3, 2\rangle - \langle 4,20\rangle \\ &= \langle -1, -18\rangle \end{aligned}$$

c)

$$\begin{aligned} -2\boldsymbol{a} - \boldsymbol{4b} &= -2\langle 3, 2\rangle - 4\langle 1, 4\rangle \\ &= \langle -6, -4\rangle + \langle -4, -16\rangle \\ &= \langle -10, -20\rangle \end{aligned}$$

d)

$$\begin{aligned} \frac{1}{\|\boldsymbol{a}\|}\boldsymbol{a} &= \frac{\mathbf{1}}{\sqrt{13}}\langle 3, 2\rangle \\ &= \langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\rangle \end{aligned}$$

Note, the result is a unit vector since $\sqrt{\left(\frac{3}{\sqrt{13}}\right)^2 + \left(\frac{2}{\sqrt{13}}\right)^2} = \sqrt{\frac{9}{13} + \frac{4}{13}} = \sqrt{\frac{13}{13}} = 1.$

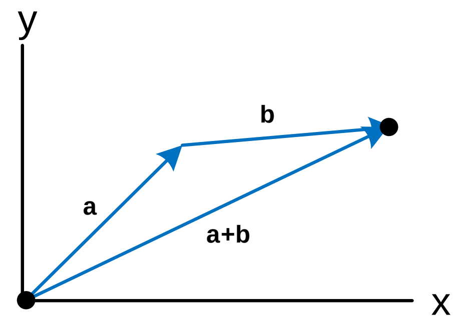
e)

$$\begin{aligned} \|\boldsymbol{a} + \boldsymbol{b}\| &= \|\langle 3, 2\rangle + \langle 1, 4\rangle\| \\ &= \|\langle 4, 6\rangle\| \\ &= \sqrt{4^2 + 6^2} \\ &= \sqrt{52} \cong 7.2 \end{aligned}$$

f)

$$\begin{aligned} \|\boldsymbol{a}\| + \|\boldsymbol{b}\| &= \|3, 2\| - \|\langle 1, 4\rangle\| \\ &= \sqrt{3^2 + 2^2} + \sqrt{1^2 + 5^2} \\ &= \sqrt{13} + \sqrt{26} \cong 8.7 \end{aligned}$$

The last two problems highlight a theorem referred to as the Triangle Inequality. Intuitively it can be understood using the familiar saying: "The shortest distance between two points is a straight line" and is illustrated in the figure below.



The theorem is formally stated below.

Trainable Inequality

For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\|\boldsymbol{a}+\boldsymbol{b}\| \le \|\boldsymbol{a}\| + \|\boldsymbol{b}\|$$

The equality holds when $\boldsymbol{a} = \boldsymbol{0}$ or $\boldsymbol{b} = \boldsymbol{0}$, or if $\boldsymbol{b} = \lambda \boldsymbol{a}$, where $\lambda > 0$.

Example 3: For each of the following, find a vector that satisfies the given condition.

a) A vector in the opposite direction of $\boldsymbol{v} = \langle \mathbf{1}, \mathbf{2}, \mathbf{3} \rangle$ and whose magnitude is half of $\boldsymbol{v}$.

b) A unit vector in the same direction as the vector from $P_1 = (1,0,5)$ to $P_2 = (3,-1,2)$.

c) A vector in 2 dimensions that makes an angle of 30° with the positive x-axis and has a magnitude of 2.

d) A vector $\boldsymbol{v}$ that satisfies $3\boldsymbol{v} + \langle \mathbf{5}, \mathbf{20} \rangle = \langle \mathbf{11}, \mathbf{17} \rangle$.

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a) As pointed out when subtracting two vectors if we negate each component of a vector, we get a new vector pointing in the opposite direction. Furthermore, we can scale a vector with a constant, λ. Therefore, the new vector, $\boldsymbol{u}$, that satisfies the given conditions is found as follows:

$$\boldsymbol{u} = -2\boldsymbol{v} = -2\langle 1, 2, 3 \rangle = \langle -2, -4, -6 \rangle$$

b) In this case we find the standard vector first.

$$
\begin{aligned}
\boldsymbol{v} &= \overrightarrow{P_1P_2} \\
&= \langle (3-1), (-1-0), (2-5) \rangle \\
&= \langle 2, -1, -3 \rangle
\end{aligned}
$$

Next, we divide by the magnitude to find the equivalent unit vector, $\widehat{\boldsymbol{v}}$.

$$
\begin{aligned}
\widehat{\boldsymbol{v}} &= \frac{1}{\|v\|}\boldsymbol{v} \\
&= \frac{1}{\sqrt{(2)^2 + (-1)^2 + (-3)^2}} \langle 2, -1, -3 \rangle \\
&= \frac{1}{\sqrt{14}} \langle 2, -1, -3 \rangle \\
&= \langle \frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{-3}{\sqrt{14}} \rangle
\end{aligned}
$$

c) We start by writing the unit vector that makes an angle of 30° with the positive x-axis as

$$
\begin{aligned}
\widehat{\boldsymbol{v}} &= \langle cos(30), sin(30) \rangle \\
&= \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle
\end{aligned}
$$

Next, we multiple by 2 to create the vector, $\boldsymbol{v}$, with the required properties.

$$
\begin{aligned}
\boldsymbol{v} &= \boldsymbol{2}\widehat{\boldsymbol{v}} \\
&= \boldsymbol{2} \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle = \langle \sqrt{3}, 1 \rangle
\end{aligned}
$$

d) In this case we need to find, $\boldsymbol{v}$, which we can do using the vector algebra we learned above.

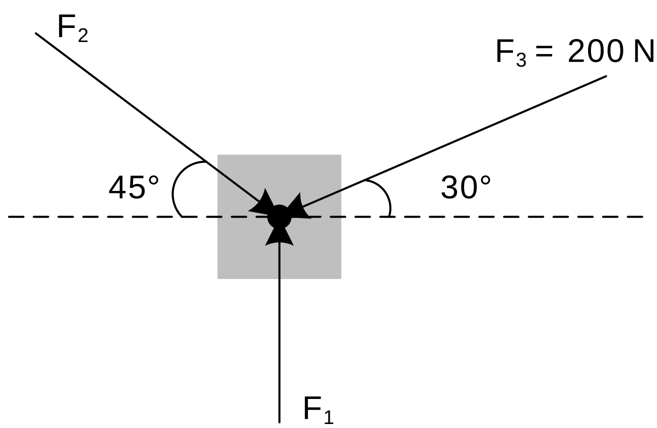
$$
\begin{aligned}
3\boldsymbol{v} + \langle 5, 20 \rangle &= \langle 11, 17 \rangle \\
3\boldsymbol{v} &= \langle 11, 17 \rangle - \langle 5, 20 \rangle \\
3\boldsymbol{v} &= \langle 6, -3 \rangle \\
\boldsymbol{v} &= \frac{1}{3} \langle 6, -3 \rangle \\
\boldsymbol{v} &= \langle 2, -1 \rangle
\end{aligned}
$$

Example 4: Write the vector $\boldsymbol{v} = \langle 4,\ -6,\ 8\rangle$ as a linear combination of the standard basis vectors.

Since the standard basis vectors are $\boldsymbol{i} = \langle 1,\ 0,\ 0\rangle$, $j = \langle 0,\ 1,\ 0\rangle$, and $k = \langle 0,\ 0,\ 1\rangle$, the linear combination is given as.

$$\begin{aligned} \boldsymbol{v} &= 4\boldsymbol{i} - 6\boldsymbol{j} + 8\boldsymbol{k} \\ &= 4\langle 1,0,0\rangle - 6\langle 0,1,0\rangle + 8\langle 0,0,1\rangle \\ &= \langle 4,0,0\rangle + \langle 0,-6,0\rangle + \langle 0,0,8\rangle \\ &= \langle 4,-6,8\rangle \end{aligned}$$

Example 5: Determine the magnitude of the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ in the figure below, assuming that there is no net force on the object.



To ensure the net force is zero we write the following vector equation.

$$\boldsymbol{F_1} + \boldsymbol{F_2} + \boldsymbol{F_3} = 0$$

Next, we write the vectors in component form as follows:

$$F_1\langle 0,1\rangle + F_2\langle cos(45)\,, -sin(45)\rangle + 200\langle -cos(30)\,, -sin(30)\rangle = \langle 0,0\rangle$$

$$\langle 0, F_1\rangle + \langle F_2 cos(45)\,, -F_2 sin(45)\rangle + \langle -200cos(30)\,, -200\,sin(30)\rangle = \langle 0,0\rangle$$

With this we can write two simultaneous equations, one for the x-component and one for the y-component, to solve for F_1 and F_2.

$$F_2 cos(45) - 200 cos(30) = 0$$

$$F_2 = \frac{200 cos(30)}{cos(45)}$$

$$F_2 = 200\sqrt{\frac{3}{2}}$$

$$F_2 \cong 245\ N$$

$$F_1 - F_2 sin(45) - 200\ sin(30) = 0$$

$$F_1 = 200\sqrt{\frac{3}{2}} sin(45) + 200\ sin(30)$$

$$F_1 = 200\left(\sqrt{\frac{3}{2}} \cdot \frac{\sqrt{2}}{2} + 200\frac{1}{2}\right)$$

$$F_1 = 100\left(\sqrt{3} + 1\right)$$

$$F_1 \cong 273.2\ N$$

Final Summary for Vector Geometry – Vector Introduction

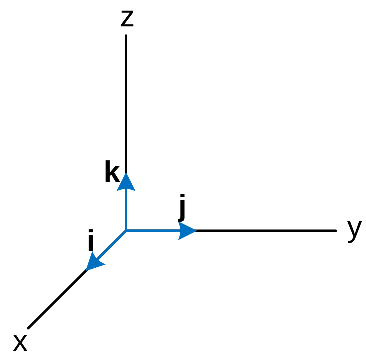
Basic Vector Definitions
A vector is used to express a quantity with both a magnitude and direction. A 2 or 3-dimensional vector, $\boldsymbol{v}$, is determined by a basepoint P (the "tail") and a terminal point Q (the "head") as follows: $$\boldsymbol{v} = \overrightarrow{PQ} = \langle\, (Q_x - P_x), \left(\, Q_y - P_y\right)\rangle = \langle\, v_x, v_y\rangle$$ Where, v_x, v_y are called the components of the vector. The magnitude, i.e. length, of a vector, $\boldsymbol{v}$, is referred to as the *norm* and is given by $$\|\boldsymbol{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$$ A unit-vector is a vector that has a magnitude of one and can be expressed as follows: $$\hat{\boldsymbol{v}} = \langle\cos\theta\,, \sin(\theta)\rangle$$ Furthermore, any vector, $\boldsymbol{v}$**,** can be scaled to be a unit vector as follows: $$\hat{\boldsymbol{v}} = \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v}$$
Vector Operations Using Components
If $\boldsymbol{a} = \langle a_x,\ a_y\rangle$ and **b**$= \langle b_x,\ b_y\rangle$ then: *i.* *Addition* $\boldsymbol{a} + \boldsymbol{b} = \langle a_x + b_x, a_y + b_y\rangle$ *ii.* *Subtraction* $\boldsymbol{a} - \boldsymbol{b} = \langle a_x - b_x, a_y - b_y\rangle$ *iii.* *Scalar Multiplication* $\lambda\boldsymbol{a} = \langle\lambda a_x, \lambda a_y\,\rangle$ *iv.* *Addition Identity* $\boldsymbol{a} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{a} = \boldsymbol{a}$ ***Note: The operations are shown in 2 dimensions but apply equally in 3 dimensions.*
Basic Properties of Vector Algebra
For all vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and for all scalars, λ i. Commutative Law $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$ ii. Associative Law $\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{b} + \boldsymbol{a}) + \boldsymbol{c}$ iii. Distributive law for Scalars $\lambda(\boldsymbol{a} + \boldsymbol{b}) = \lambda\boldsymbol{b} + \lambda\boldsymbol{a}$

Standard Basis Vectors for Rectangular Coordinate System in 3-Dimensions

$$\boldsymbol{i} = \langle 1, 0, 0 \rangle$$

$$\boldsymbol{j} = \langle 0, 1, 0 \rangle$$

$$\boldsymbol{k} = \langle 0, 0, 1 \rangle$$



All vectors in a particular vector space, e.g. 2D or 3D, can be written as a linear combination of the basic vectors.

$$\boldsymbol{v} = \langle a, b \rangle = a\boldsymbol{i} + b\boldsymbol{j}$$

Triangle Inequality

For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\|\boldsymbol{a} + \boldsymbol{b}\| \leq \|\boldsymbol{a}\| + \|\boldsymbol{b}\|$$

The equality holds when $\boldsymbol{a} = \boldsymbol{0}$ or $\boldsymbol{b} = \boldsymbol{0}$, or if $\boldsymbol{b} = \lambda\boldsymbol{a}$, where $\lambda > 0$.

By: ferrantetutoring