Vector Geometry – Dot Product and Projections

In the previous section we covered the vector operations of addition, subtraction, and scalar multiplication, which were straightforward extensions of the equivalent scalar operations. Conversely, vector multiplication is not a straightforward extension of scalar multiplication. As a matter of fact, multiplication is not uniquely defined for vectors. However, several types of vector "products" are well defined. One such, called the *dot product*, we will cover in this section. The dot product is one of the most important vector operations and will be continually used throughout our studies of vector calculus. The dot product operates on two n-dimensional vectors and results in a single scalar value as defined below.

The Dot Product
Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be n-dimensional vectors:
$\boldsymbol{a} = \langle a_1, a_2, \cdots, a_N \rangle$ $\qquad\qquad$ $\boldsymbol{b} = \langle b_1, b_2, \cdots, b_N \rangle$
The dot product, $\boldsymbol{a} \cdot \boldsymbol{b}$, is defined as follows:
$\boldsymbol{a} \cdot \boldsymbol{b} = (a_1 b_1 + a_2 b_2 + \cdots + a_N b_N)$

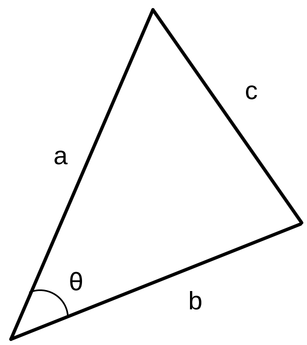
As an example, let $\boldsymbol{a} = \langle 2, -3, 1 \rangle$ and $\boldsymbol{b} = \langle 4, 2, 5 \rangle$. The dot product is computed as

$$\boldsymbol{a} \cdot \boldsymbol{b} = (2 \cdot 4 + (-3) \cdot 2 + 1 \cdot 5) = \mathbf{7}$$

Next, we state some of the fundamental properties of the dot product.

Properties of the Dot Product	
1. Commutative Property:	$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
2. Zero Property:	$\mathbf{a} \cdot 0 = \mathbf{0}$
3. Scalar Multiplication Property:	$\lambda(\mathbf{a} \cdot \mathbf{b}) = (\lambda \mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\lambda \mathbf{b})$
4. Distributive Property:	$\mathbf{v} \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{v} \cdot \mathbf{b} + \mathbf{v} \cdot \mathbf{b}$
5. Relation to Length:	$\boldsymbol{v} \cdot \boldsymbol{v} = \lVert \boldsymbol{v} \rVert^2$

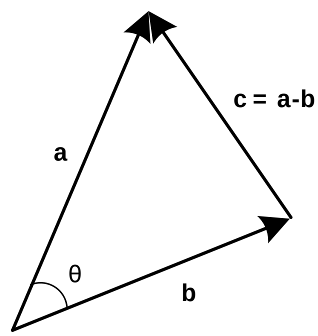
Our studies will generally be limited to interpreting vectors as space vectors that exist in either two- or three-dimensional space. For this case we will see that the dot product is closely related to the angle between two vectors. We can show this by deriving an alternate form of the dot product using the Law of Cosines as shown below.



In the triangle above the Law of Cosines is stated as follows:

$$c^2 = a^2 + b^2 - 2ab\,cos(\theta)$$

Now let's interpret the sides of the triangle using vectors as follows shown below.



Rewriting the Law of Cosines, we have

$$\|\boldsymbol{a}-\boldsymbol{b}\|^2 = \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 - 2\|\boldsymbol{a}\|\|\boldsymbol{b}\|\,cos(\theta)$$

Using the properties from above we can expand the left-hand side as follows:

$$\|\boldsymbol{a}-\boldsymbol{b}\|^2 = (\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b}) = (\boldsymbol{a}\cdot\boldsymbol{a}) - (\boldsymbol{2a}\cdot\boldsymbol{b}) + (\boldsymbol{b}\cdot\boldsymbol{b}) = \|\boldsymbol{a}\|^2 - \boldsymbol{2}(\boldsymbol{a}\cdot\boldsymbol{b}) + \|\boldsymbol{b}\|^2$$

Substituting back, we have

$$\cancel{\|\boldsymbol{a}\|^2 - \boldsymbol{2}}(\boldsymbol{a}\cdot\boldsymbol{b}) + \cancel{\|\boldsymbol{b}\|^2} = \cancel{\|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 - 2}\|\boldsymbol{a}\|\|\boldsymbol{b}\|\,cos(\theta)$$

$$(\boldsymbol{a}\cdot\boldsymbol{b}) = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\,cos(\theta)$$

In this form the dot product can be interpreted as the product of the lengths of the two vectors multiplied by the angle between them. The angle, however, is not unique since both θ and $2\pi n - \theta$ can serve as the angle between the two vectors. To avoid confusion, we adopt the following convention.

The angle between two vectors is chosen to satisfy $0 \le \theta \le \pi$

Therefore, using either 2 or 3 dimensions, we can extend our definition of the dot product as shown below.

The Dot Product in 2 and 3 Dimensions

Given two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$**,** as well as the angle, θ, between the two vectors.

The dot product can be equivalently be defined in the following two ways:

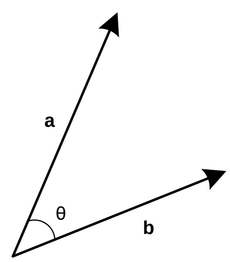
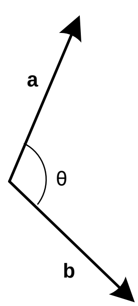
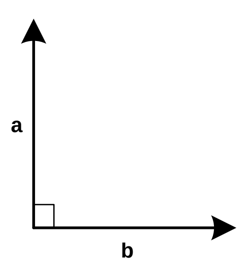
$$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \, cos(\theta)$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = \left(a_x b_x + a_y b_y + a_z b_z\right)$$

Furthermore, if the angle is unknown it may be found as follows:

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\|\|\boldsymbol{b}\|}\right) = \cos^{-1}\left(\frac{a_x b_x + a_y b_y + a_z b_z}{\|\boldsymbol{a}\|\|\boldsymbol{b}\|}\right)$$

Using the geometric definition of the dot product and making note of the properties of the cosine function between 0° and 180°, we can make the following important observations.

$\boldsymbol{a} \cdot \boldsymbol{b} > \boldsymbol{0}$		The angle between the two vectors is acute, i.e. $0^{\circ} \leq \theta < 90^{\circ}$
$\boldsymbol{a} \cdot \boldsymbol{b} < \boldsymbol{0}$		The angle between the two vectors is obtuse, i.e. $90^{\circ} < \theta \leq 180^{\circ}$
$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{0}$		The angle between the two vectors is 90°. **Note: We use the word *orthogonal* to refer to vectors that form a 90° angle.**

Before moving to projections, lets do some examples to get more familiar with the dot product.

Example 1: Compute the dot product of the following vectors.

i. $\boldsymbol{a} = \langle 3, -1 \rangle,\ \boldsymbol{b} = \langle 2, -5 \rangle$

ii. $\boldsymbol{a} = \langle 4, -5, 1 \rangle,\ \boldsymbol{b} = \langle 3, 6, -1 \rangle$

iii. $\|\boldsymbol{a}\| = \mathbf{3},\ \|\boldsymbol{b}\| = \mathbf{4}$; the angle between is $\frac{\pi}{4}$

Solution:

i. $\boldsymbol{a} \cdot \boldsymbol{b} = \big(3 \cdot 2 + (-1) \cdot (-5)\big) = 12$

ii. $\boldsymbol{a} \cdot \boldsymbol{b} = \big(4 \cdot 3 + (-5) \cdot 6 + 1 \cdot (-1)\big) = -19$

iii. $\boldsymbol{a} \cdot \boldsymbol{b} = 3 \cdot 4\, cos\left(\frac{\pi}{4}\right) = 12\frac{\sqrt{2}}{2} = 6\sqrt{2}$

Example 2: Determine whether the angle between the following vectors is acute, obtuse, or right.

i. $\boldsymbol{a} = \langle 1, 2, 1 \rangle,\ \boldsymbol{b} = \langle 7, -3, -1 \rangle$

ii. $\boldsymbol{a} = \langle \frac{12}{5}, \frac{-4}{5} \rangle,\ \boldsymbol{b} = \langle \frac{1}{2}, \frac{-7}{4} \rangle$

iii. $\boldsymbol{a} = \langle 0, 3, -2 \rangle,\ \boldsymbol{b} = \langle 1, -1, 0 \rangle$

Solution:

i. $\boldsymbol{a} \cdot \boldsymbol{b} = \big(1 \cdot 7 + 2 \cdot (-3) + 1 \cdot (-1)\big) = 0$; The vectors are orthogonal, i.e. $\theta = 90^{\circ}$.

ii. $\boldsymbol{a} \cdot \boldsymbol{b} = \left(\frac{12}{5} \cdot \frac{1}{2} + \left(\frac{-4}{5}\right) \cdot \left(\frac{-7}{4}\right)\right) > 0$; The angle is acute, i.e. $0^{\circ} \leq \theta < 90^{\circ}$

iii. $\boldsymbol{a} \cdot \boldsymbol{b} = (0 \cdot 1 + 3 \cdot (-1) + (-2) \cdot 0) = -3$; The angle is obtuse. $90^{\circ} < \theta \leq 180^{\circ}$.

Example 3: Find a vector that is orthogonal to both $\boldsymbol{a} = \langle 1, 1, 1\rangle$ and $\boldsymbol{b} = \langle 2, 0, 4\rangle$.

Solution:

If the vector, $\boldsymbol{v}$, is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$ the following two equations must be true.

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{a} &= \boldsymbol{0} \\ \langle v_x, v_y, v_z, \rangle \cdot \langle 1, 1, 1\rangle &= 0 \\ v_x + v_y + v_z &= 0 \end{aligned}$$

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{b} &= \boldsymbol{0} \\ \langle v_x, v_y, v_z, \rangle \cdot \langle 2, 0, 4\rangle &= 0 \\ 2v_x + 4v_z &= 0 \end{aligned}$$

This system of equations forms an underdetermined system with an infinite number of solutions. Staring with the equation on the right side we have

$$v_x = -\,2v_z$$

Substituting this into the equation on the left we find

$$\begin{aligned} -\,2v_z + v_y + v_z &= 0 \\ -v_z + v_y &= 0 \\ v_z &= v_y \end{aligned}$$

Setting v_z to a constant, c, the general solution can be written as:

$$\boldsymbol{v} = \langle -2c, c, c\rangle$$

Letting $c = 1$, a particular solution is

$$\boldsymbol{v} = \langle -2, 1, 1\rangle$$

Example 4: Find all angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ if $\boldsymbol{a} \cdot \boldsymbol{b} = -\frac{1}{2}\|\boldsymbol{a}\|\|\boldsymbol{b}\|$.

Solution:

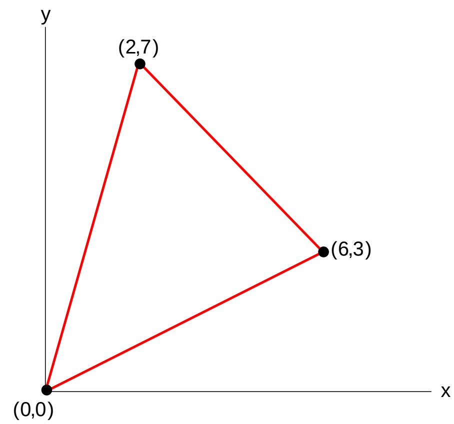
Using the geometrical definition of the dot product we can write

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= -\frac{1}{2}\|\boldsymbol{a}\|\|\boldsymbol{b}\| \\ \cancel{\|\boldsymbol{a}\|\|\boldsymbol{b}\|}\, cos(\theta) &= -\frac{1}{2}\cancel{\|\boldsymbol{a}\|\|\boldsymbol{b}\|} \end{aligned}$$

Therefore

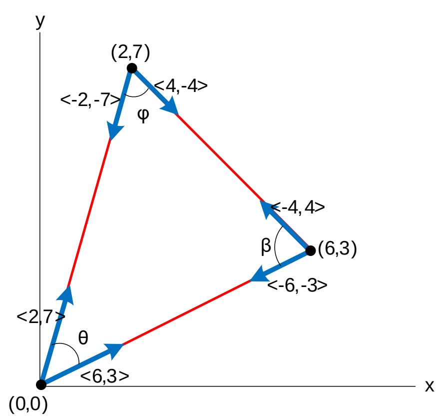
$$\begin{aligned} cos(\theta) &= -\frac{1}{2} \\ \theta &= 120^{\circ} \end{aligned}$$

Example 5: Find all three angles in the triangle below.



Solution:

To find the angles we redraw the triangle using vectors as shown below.



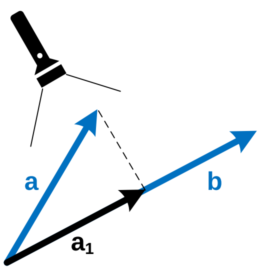
Note when creating the vectors, we need to make sure they point in the direction such that the desired angle is the angle between the two vectors. The angles can then be found as follows:

$$\begin{aligned}
\theta &= cos^{-1}\left(\frac{\langle 2,7\rangle \cdot \langle 6,3\rangle}{\|\langle 2,7\rangle\|\|\langle 6,3\rangle\|}\right) \\
&= cos^{-1}\left(\frac{2\cdot 6+7\cdot 3}{\left(\sqrt{2^2+7^2}\right)\left(\sqrt{6^2+3^2}\right)}\right) \\
&= cos^{-1}\left(\frac{33}{\left(\sqrt{53}\right)\left(\sqrt{45}\right)}\right) \\
&\cong 47.5^{\circ}
\end{aligned}$$

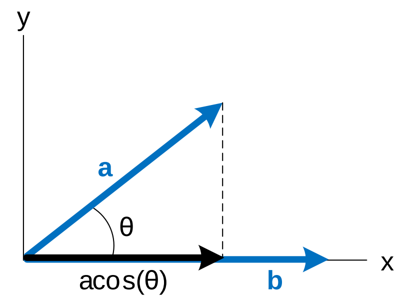
$$\begin{aligned}\varphi &= \cos^{-1}\left(\frac{\langle -2,-7\rangle \cdot \langle 4,-4\rangle}{\|\langle -2,-7\rangle\|\|\langle 4,-4\rangle\|}\right)\\ &= \cos^{-1}\left(\frac{(-2)\cdot 4+(-7)\cdot(-4)}{\left(\sqrt{2^2+7^2}\right)\left(\sqrt{4^2+(-4)^2}\right)}\right)\\ &= \cos^{-1}\left(\frac{20}{(\sqrt{53})(\sqrt{32})}\right)\\ &\cong 60.9^{\circ}\end{aligned}$$
$$\begin{aligned}\beta &= cos^{-1}\left(\frac{\langle -6,-3\rangle \cdot \langle -4,4\rangle}{\|\langle -2,-7\rangle\|\|\langle 4,-4\rangle\|}\right)\\ &= cos^{-1}\left(\frac{(-6)\cdot(-4)+(-3)\cdot 4}{\left(\sqrt{(-6)^2+(-3)^2}\right)\left(\sqrt{(-4)^2+4^2}\right)}\right)\\ &= cos^{-1}\left(\frac{12}{(\sqrt{45})(\sqrt{32})}\right)\\ &\cong 71.6^{\circ}\end{aligned}$$ Note: We can also find β as: $\beta = 180^{\circ} - (\theta+\varphi) = 180^{\circ} - (47.5^{\circ}+60.9^{\circ}) = 71.6^{\circ}$

Projections

The figure below illustrates a vector projection. The projection of the vector $\boldsymbol{a}$ onto the vector $\boldsymbol{b}$ is shown as a new vector $\boldsymbol{a_1}$, which can be envisioned as the "shadow" vector obtained from shining a light directly down onto the vector $\boldsymbol{b}$.



This projection vector, which we refer to as "the projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ $\boldsymbol{a}_{||\boldsymbol{b}}$", can be computed using the dot product. By placing the vector $\boldsymbol{b}$ along the x-axis we can more easily illustrate the computation of the projection vector.



With figure above we can use basic right triangle trigonometry and see that the length of the projection vector is equal to $\|\boldsymbol{a}\|\, cos(\theta)$. We also notice that the direction of the projection vector is in the same direction as $\boldsymbol{b}$. Therefore, we can write the projection vector using a unit vector in the direction of $\boldsymbol{b}$ multiplied by the scalar length we found above. Recall, a unit vector in the direction of $\boldsymbol{b}$ can be computed as

$$\widehat{\boldsymbol{b}} = \frac{\boldsymbol{b}}{\|\boldsymbol{b}\|}$$

With this we can write the projection vector as

$$\boldsymbol{a}_{||\boldsymbol{b}} = \left(\frac{\|\boldsymbol{a}\|\, cos(\theta)}{\|\boldsymbol{b}\|}\right)\boldsymbol{b}$$

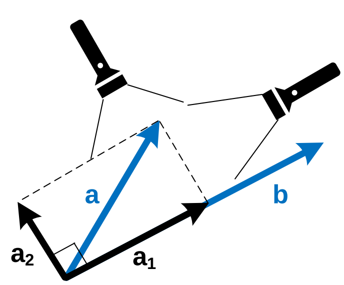
To see how this can be computed using the dot product we multiply through by $\frac{\|\boldsymbol{b}\|}{\|\boldsymbol{b}\|}$.

$$\begin{aligned}\boldsymbol{a}_{||\boldsymbol{b}} &= \frac{\|\boldsymbol{b}\|}{\|\boldsymbol{b}\|}\left(\frac{\|\boldsymbol{a}\|\, cos(\theta)}{\|\boldsymbol{b}\|}\right)\boldsymbol{b} \\ &= \left(\frac{\|\boldsymbol{a}\|\|\boldsymbol{b}\|\, cos(\theta)}{\|\boldsymbol{b}\|^2}\right)\boldsymbol{b} \\ &= \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\cdot\boldsymbol{b}}\right)\boldsymbol{b}\end{aligned}$$

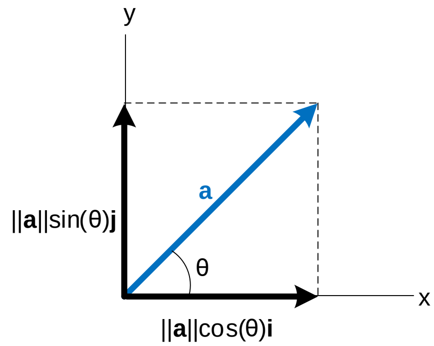
The projection vector is formally stated below.

Projection Vector
The projection of a vector $\boldsymbol{a}$ onto the vector $\boldsymbol{b}$ is the vector, $\boldsymbol{a}_{\|\boldsymbol{b}}$ given by $$\boldsymbol{a}_{\|\boldsymbol{b}} = \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{b}\cdot\boldsymbol{b}}\right)\boldsymbol{b} = \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{b}\|^2}\right)\boldsymbol{b} = \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{b}\|}\right)\widehat{\boldsymbol{b}}$$ The scalar, $\left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{b}\|}\right) = \|\boldsymbol{a}\|\, cos(\theta)$, is called the component of $\boldsymbol{a}$ along $\boldsymbol{b}$.

Recall, the projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ was initially envisioned as a "shadow" vector obtained by shining a light directly downward on $\boldsymbol{b}$. We can similarly shine a light in a direction that is perpendicular to $\boldsymbol{b}$ and obtain an additional "shadow" vector. In the figure below, we refer to $\boldsymbol{a_1}$ as a parallel projection and $\boldsymbol{a_2}$ as a perpendicular projection.



Note that the sum of these projections is equal to the original vector itself. The process described is referred to as the *decomposition of a vector.* We are likely already familiar with decomposing a vector into its orthogonal components with respect to the x- and y-axis as shown below.



However, we can more generally decompose a vector, $\boldsymbol{a}$, with respect to another vector, $\boldsymbol{b}$, using projections as follows.

$$\boldsymbol{a} = \boldsymbol{a}_{\|\boldsymbol{b}} + \boldsymbol{a}_{\perp\boldsymbol{b}}$$

Where we start by computing the parallel projection.

$$\boldsymbol{a}_{\|\boldsymbol{b}} = \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{b}\|^2}\right)\boldsymbol{b}$$

We can then find the perpendicular projection as follows

$$\boldsymbol{a}_{\perp\boldsymbol{b}} = \boldsymbol{a} - \boldsymbol{a}_{\|\boldsymbol{b}}$$

Example 6: Find the projection of $\boldsymbol{a}$ on $\boldsymbol{b}$.

i.	$\boldsymbol{a} = \langle 2,5\rangle,\ \boldsymbol{b} = \langle 1,1\rangle$
ii.	$\boldsymbol{a} = \langle 1,2,3\rangle,\ \boldsymbol{b} = \langle -2,4,-1\rangle$

Solution:

i.

$$\boldsymbol{a}_{||\boldsymbol{b}} = \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{b}\|^2}\right)\boldsymbol{b}$$

$$= \left(\frac{\langle 2,5\rangle\cdot\langle 1,1\rangle}{\|\langle 1,1\rangle\|^2}\right)\langle 1,1\rangle$$

$$= \left(\frac{2+5}{1^2+1^2}\right)\langle 1,1\rangle = \left(\frac{7}{2}\right)\langle 1,1\rangle = \langle\frac{7}{2},\frac{7}{2}\rangle$$

ii.

$$\begin{aligned}\boldsymbol{a}_{||\boldsymbol{b}} &= \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{||\boldsymbol{b}||^2}\right)\boldsymbol{b}\\ &= \left(\frac{\langle 1,2,3\ \rangle\cdot\langle -2,4,-1\rangle}{||\langle -2,4,-1\rangle||^2}\right)\langle -2,4,-1\rangle\\ &= \left(\frac{(-2)+8-3}{(-2)^2+(4)^2+(-1)^2}\right)\langle -2,4,-1\rangle\\ &= \left(\frac{1}{7}\right)\langle -2,4,-1\rangle = \langle\frac{-2}{7},\frac{4}{7},\frac{-1}{7}\rangle\end{aligned}$$

Example 7: Compute the vector components of $\boldsymbol{a}$ with respect to $\boldsymbol{b}$.

i.	$\boldsymbol{a} = \langle 2,-3\rangle,\ \boldsymbol{b} = \langle 1,2\rangle$
ii.	$\boldsymbol{a} = \langle 3,2,1\rangle,\ \boldsymbol{b} = \langle 1,0,1\rangle$

Solution:

i. The component parallel to $\boldsymbol{b}$ is computed as

$$\begin{aligned}\boldsymbol{a}_{||\boldsymbol{b}} &= \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{||\boldsymbol{b}||^2}\right)\boldsymbol{b}\\ &= \left(\frac{\langle 2,-3\rangle\cdot\langle 1,2\rangle}{||\langle 1,2\rangle||^2}\right)\langle 1,2\rangle\\ &= \left(\frac{2-6}{(1)^2+(2)^2}\right)\langle 1,2\rangle = \langle\frac{-4}{5},\frac{-8}{5}\rangle\end{aligned}$$

The perpendicular component is then found as follows:

$$\begin{aligned}\boldsymbol{a}_{\perp\boldsymbol{b}} &= \boldsymbol{a} - \boldsymbol{a}_{\|\boldsymbol{b}}\\ &= \langle 2,-3\rangle - \langle\frac{-4}{5},\frac{-8}{5}\rangle = \langle\frac{14}{5},\frac{-7}{5}\rangle\end{aligned}$$

Therefore, we can now express the vector $\boldsymbol{a}$ as the sum of orthogonal components parallel and perpendicular to $\boldsymbol{b}$.

$$\boldsymbol{a} = \boldsymbol{a}_{\|\boldsymbol{b}} + \boldsymbol{a}_{\perp\boldsymbol{b}}$$

$$\langle 2,-3\rangle = \langle\frac{-4}{5},\frac{-8}{5}\rangle + \langle\frac{14}{5},\frac{-7}{5}\rangle$$

ii. The component parallel to $\boldsymbol{b}$ is computed as

$$\begin{aligned} \boldsymbol{a}_{||\boldsymbol{b}} &= \left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{||\boldsymbol{b}||^2}\right)\boldsymbol{b} \\ &= \left(\frac{\langle 3,2,1\rangle\cdot\langle 1,0,1\rangle}{||\langle 1,0,1\rangle||^2}\right)\langle 1,0,1\rangle \\ &= \left(\frac{4}{2}\right)\langle 1,-1,0\rangle = \langle 2,-2,0\rangle \end{aligned}$$

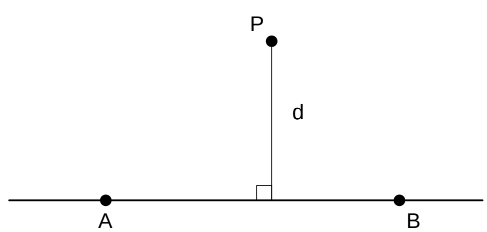
The perpendicular component is then found as follows:

$$\begin{aligned} \boldsymbol{a}_{\perp\boldsymbol{b}} &= \boldsymbol{a} - \boldsymbol{a}_{\|\boldsymbol{b}} \\ &= \langle 3,2,1\rangle - \langle 2,-2,0\rangle \\ &= \langle 1,4,1\rangle \end{aligned}$$

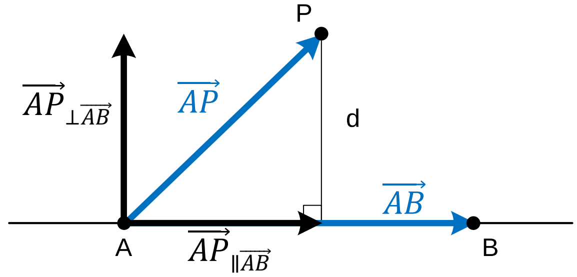
Therefore, we can now express the vector $\boldsymbol{a}$ as the sum of orthogonal components parallel and perpendicular to $\boldsymbol{b}$.

$$\begin{aligned} \mathbf{a} &= \mathbf{a}_{\|\mathbf{b}} + \mathbf{a}_{\perp\mathbf{b}} \\ \langle 3,2,1\rangle &= \langle 2,-2,0\rangle + \langle 1,4,1\rangle \end{aligned}$$

Example 8: Given three points A, B, and P as shown below, describe two different ways that you could use the dot product to calculate the distance, d, between point P and the line that connects A and B. Then compute the distance using the points $A = (1,0,1)$, B $= (2,3,1)$, and $P = (5,3,0)$.



We start by creating the vector $\overrightarrow{AB}$ and $\overrightarrow{AP}$ as shown in the figure below. We also show the decomposition of the vector $\overrightarrow{AP}$ with respect to $\overrightarrow{AB}$.



With this we can use the following two methods to compute the distance, d.

Method 1:

1. Compute the projection of $\overrightarrow{AP}$ onto $\overrightarrow{AB}$.

$$\overrightarrow{AP}_{\parallel\overrightarrow{AB}} = \left(\frac{\overrightarrow{AP} \cdot \overrightarrow{AB}}{\left\|\overrightarrow{AB}\right\|^2}\right)\overrightarrow{AB}$$

2. Compute the perpendicular component of $\overrightarrow{AP}$ with respect to $\overrightarrow{AB}$.

$$\overrightarrow{AP}_{\perp\overrightarrow{AB}} = \overrightarrow{AP} - \overrightarrow{AP}_{\parallel\overrightarrow{AB}}$$

3. The distance, d, is the equal to the magnitude of $\overrightarrow{AP}_{\perp\overrightarrow{AB}}$.

$$d = \left\|\overrightarrow{AP}_{\perp\overrightarrow{AB}}\right\|$$

Method 2:

1. Compute the magnitude of the projection of $\overrightarrow{AP}$ onto $\overrightarrow{AB}$.

$$\left\|\overrightarrow{AP}_{\parallel\overrightarrow{AB}}\right\| = \left(\frac{\overrightarrow{AP} \cdot \overrightarrow{AB}}{\left\|\overrightarrow{AB}\right\|}\right)$$

2. Use the Pythagorean theorem to compute the distance, d.

$$d = \sqrt{\left\|\overrightarrow{AP}\right\|^2 - \left\|\overrightarrow{AP}_{\parallel\overrightarrow{AB}}\right\|^2}$$

Now, let's verify both methods using the values given. We start by computing the vectors, $\overrightarrow{AP}$ and $\overrightarrow{AB}$.

$$\overrightarrow{AP} = \langle(5-1),(3-0),(0-1)\rangle \qquad \overrightarrow{AB} = \langle(2-1),(3-0),(1-1)\rangle$$
$$\overrightarrow{AP} = \langle 4,3,-1\rangle \qquad \overrightarrow{AB} = \langle 1,3,0\rangle$$

Method 1:

1. Compute the projection of $\overrightarrow{AP}$ onto $\overrightarrow{AB}$.

$$\overrightarrow{AP}_{\parallel\overrightarrow{AB}} = \left(\frac{\overrightarrow{AP}\cdot\overrightarrow{AB}}{\left\|\overrightarrow{AB}\right\|^2}\right)\overrightarrow{AB}$$

$$= \left(\frac{\langle 4,3,-1\rangle\cdot\langle 1,3,0\rangle}{\|\langle 1,3,0\rangle\|^2}\right)\langle 1,3,0\rangle$$

$$= \left(\frac{13}{10}\right)\langle 1,3,0\rangle = \frac{1}{10}\langle 13,39,0\rangle$$

2. Compute the perpendicular component of $\overrightarrow{AP}$ with respect to $\overrightarrow{AB}$.

$$\overrightarrow{AP}_{\perp\overrightarrow{AB}} = \overrightarrow{AP} - \overrightarrow{AP}_{\parallel\overrightarrow{AB}}$$

$$= \langle 4,3,-1\rangle - \frac{1}{10}\langle 13,39,0\rangle = \frac{1}{10}\langle 27,-9,-10\rangle$$

3. The distance, d, is the equal to the magnitude of $\overrightarrow{AP}_{\perp\overrightarrow{AB}}$.

$$d = \left\|\overrightarrow{AP}_{\perp\overrightarrow{AB}}\right\|$$

$$= \sqrt{\frac{27^2+(-9)^2+(-10)^2}{10^2}} = \sqrt{\frac{91}{10}}$$

Method 2:

1. Compute the magnitude of the projection of $\overrightarrow{AP}$ onto $\overrightarrow{AB}$.

$$\left\|\overrightarrow{AP}_{\parallel\overrightarrow{AB}}\right\| = \left(\frac{\overrightarrow{AP}\cdot\overrightarrow{AB}}{\left\|\overrightarrow{AB}\right\|}\right)$$

$$\left(\frac{\langle 4,3,-1\rangle\cdot\langle 1,3,0\rangle}{\|\langle 1,3,0\rangle\|}\right) = \frac{13}{\sqrt{10}}$$

2. Use the Pythagorean theorem to compute the distance, d.

$$d = \sqrt{\left\|\overrightarrow{AP}\right\|^2 - \left\|\overrightarrow{AP}_{\parallel\overrightarrow{AB}}\right\|^2}$$

$$= \sqrt{(4^2+3^2+(-1)^2) - \left(\frac{13}{\sqrt{10}}\right)^2}$$

$$= \sqrt{\frac{260}{10} - \left(\frac{13}{\sqrt{10}}\right)^2}$$

$$= \sqrt{\frac{260}{10} - \frac{169}{10}} = \sqrt{\frac{91}{10}}$$

Final Summary for Vector Geometry – Dot Products and Projections

The Dot Product

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be n-dimensional vectors:

$$\boldsymbol{a} = \langle a_1, a_2, \cdots, a_N \rangle \qquad \boldsymbol{b} = \langle b_1, b_2, \cdots, b_N \rangle$$

The dot product, $\boldsymbol{a} \cdot \boldsymbol{b}$, is defined as follows:

$$\boldsymbol{a} \cdot \boldsymbol{b} = (a_1 b_1 + a_2 b_2 + \cdots + a_N b_N)$$

Properties of the Dot Product

1. *Commutative Property:* $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
2. *Zero Property:* $\boldsymbol{a} \cdot 0 = \boldsymbol{0}$
3. *Scalar Multiplication Property:* $\lambda(\boldsymbol{a} \cdot \boldsymbol{b}) = (\lambda \boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (\lambda \boldsymbol{b})$
4. *Distributive Property:* $\boldsymbol{v} \cdot (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{v} \cdot \boldsymbol{b} + \boldsymbol{v} \cdot \boldsymbol{b}$
5. *Relation to Length:* $\boldsymbol{v} \cdot \boldsymbol{v} = \|\boldsymbol{v}\|^2$

The Dot Product in 2 and 3 Dimensions

Given two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$**,** as well as the angle, θ, between the two vectors.

The dot product can be equivalently be defined in the following two ways:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \, cos(\theta)$$

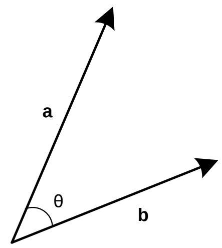
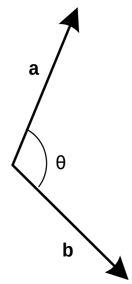
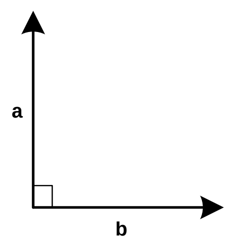
$$\boldsymbol{a} \cdot \boldsymbol{b} = \left(a_x b_x + a_y b_y + a_z b_z\right)$$

Furthermore, if the angle is unknown it may be found as follows:

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\|\|\boldsymbol{b}\|}\right) = \cos^{-1}\left(\frac{a_x b_x + a_y b_y + a_z b_z}{\|\boldsymbol{a}\|\|\boldsymbol{b}\|}\right)$$

The angle between two vectors is chosen to satisfy $0 \leq \theta \leq \pi$

Geometric Properties of The Dot Product

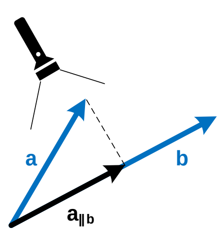
$\boldsymbol{a} \cdot \boldsymbol{b} > \mathbf{0}$		The angle between the two vectors is acute, i.e. $0^{\circ} \leq \theta < 90^{\circ}$
$\boldsymbol{a} \cdot \boldsymbol{b} < \mathbf{0}$		The angle between the two vectors is obtuse, i.e. $90^{\circ} < \theta \leq 180^{\circ}$
$\boldsymbol{a} \cdot \boldsymbol{b} = \mathbf{0}$		The angle between the two vectors is 90°. **Note: We use the word *orthogonal* to refer to vectors that form a $\mathbf{90}^{\circ}$ angle.**

Projection Vector

The projection of a vector $\boldsymbol{a}$ onto the vector $\boldsymbol{b}$ is the vector, $\boldsymbol{a}_{||\boldsymbol{b}}$ given by

$$\boldsymbol{a}_{||\boldsymbol{b}} = \left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\right)\boldsymbol{b} = \left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{||\boldsymbol{b}||^2}\right)\boldsymbol{b} = \left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{||\boldsymbol{b}||}\right)\widehat{\boldsymbol{b}}$$

The scalar, $\left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{||\boldsymbol{b}||}\right) = ||\boldsymbol{a}||\, cos(\theta)$, is called the component of $\boldsymbol{a}$ along $\boldsymbol{b}$.



Vector Decomposition

Any vector, $\boldsymbol{a}$ can be decomposed into two orthogonal component vectors with respect to another vector, $\boldsymbol{b}$ as:

$$\boldsymbol{a} = \boldsymbol{a}_{\|\boldsymbol{b}} + \boldsymbol{a}_{\perp\boldsymbol{b}}$$

Where the parallel projection is given above, and the perpendicular projection is found as:

$$\boldsymbol{a}_{\perp\boldsymbol{b}} = \boldsymbol{a} - \boldsymbol{a}_{\|\boldsymbol{b}}$$

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