Vector Geometry – Cross Product

As mentioned in the previous section there are several types of vector products. The last section covered the dot product. In this section we study another vector product called the *cross product*. Unlike the dot product, which resulted in a scalar, the cross product results in another vector. Another key difference is that the cross product if defined only for vectors in three dimensions. The computations involved in generating a cross product are more cumbersome than those used for the dot product. The computation can be defined using a determinant, which we introduce below.

2X2 and 3X3 Matrix Determinant

The determinant of a matrix is a scalar value and, for a $2X2$ matrix, is computed as the difference of the product of the diagonal elements as shown below.

$$det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \stackrel{\text{def}}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (a \cdot d) - (c \cdot b)$$

The determinant of a $3X3$ matrix can be defined by constructing three $2X2$ matrices as shown below.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Note that that $2X2$ matrices are produced by crossing out the first row and the i^{th} column. For example, the second $2X2$ matrix is obtained as follows:

$$\begin{vmatrix} \cancel{a_1} & \cancel{b_1} & \cancel{c_1} \\ a_2 & \cancel{b_2} & c_2 \\ a_3 & \cancel{b_3} & c_3 \end{vmatrix}$$

Let's do an example to demonstrate.

Example 1: Compute the determinant of the following 3X3 matrix.

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & -7 \\ -1 & 5 & 3 \end{bmatrix}$$

Solution:

$$\begin{vmatrix} 2 & 4 & 3 \\ 0 & 1 & -7 \\ -1 & 5 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & -7 \\ 5 & 3 \end{vmatrix} - 4\begin{vmatrix} 0 & -7 \\ -1 & 3 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ -1 & 5 \end{vmatrix}$$

$$= 2(3 + 35) - 4(0 - 7) + 3(0 + 1) \;=\; 107$$

Now that we have reviewed the determinant, we show how this concept is used to compute the cross product.

The Cross Product
The cross product of two vectors, $\boldsymbol{a} = \langle a_x, a_y, a_z\rangle$ and $\boldsymbol{b} = \langle b_x, b_y, b_z\rangle$ is a new vector $\boldsymbol{v}$, given as $$\boldsymbol{v} = \boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{\boldsymbol{\imath}}\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{\boldsymbol{\jmath}}\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \widehat{\boldsymbol{k}}\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$$ $$= (a_yb_z - b_ya_z)\hat{\boldsymbol{\imath}} - (a_xb_z - b_xa_z)\hat{\boldsymbol{\jmath}} + (a_xb_y - b_xa_y)\widehat{\boldsymbol{k}}$$

Example 1: Compute $\boldsymbol{a}\times\boldsymbol{b}$ where $\boldsymbol{a} = \langle -2,1,4\rangle$ and $\boldsymbol{b} = \langle 3,2,5\rangle$

Solution:

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ -2 & 1 & 4 \\ 3 & 2 & 5 \end{vmatrix} = \hat{\boldsymbol{\imath}}\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} - \hat{\boldsymbol{\jmath}}\begin{vmatrix} -2 & 4 \\ 3 & 5 \end{vmatrix} + \widehat{\boldsymbol{k}}\begin{vmatrix} -2 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= (5-8)\hat{\boldsymbol{\imath}} - (-10-12)\hat{\boldsymbol{\jmath}} + (-4-3)\widehat{\boldsymbol{k}}$$

$$= -3\hat{\boldsymbol{\imath}} + 22\hat{\boldsymbol{\jmath}} - 7\widehat{\boldsymbol{k}}$$

$$= \langle -3, 22, -7\rangle$$

Geometric Interpretation

In the last section we saw that the dot product is related to the angle between the two vectors as follows.

$$\boldsymbol{a}\cdot\boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\, cos(\theta)$$

As it turns out the cross product is also related to the angle between the two vectors as follows.

$$\|\boldsymbol{a}\times\boldsymbol{b}\| = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\sin(\theta)$$

Note the equation above gives us only the magnitude of the cross product, however the result of the cross product, unlike the dot product, is a vector. Fortunately, there is a simple “rule” that can be used to determine the direction of the resulting vector. Let’s see if we can get a hint of the rule using the following two simple examples shown below. In the first example we let $\boldsymbol{a} = \langle 1,0,0\rangle$ and $\boldsymbol{b} = \langle 0,1,0\rangle$, and in the second $\boldsymbol{a} = \langle 0,1,0\rangle$ and $\boldsymbol{b} = \langle 1,0,0\rangle$.

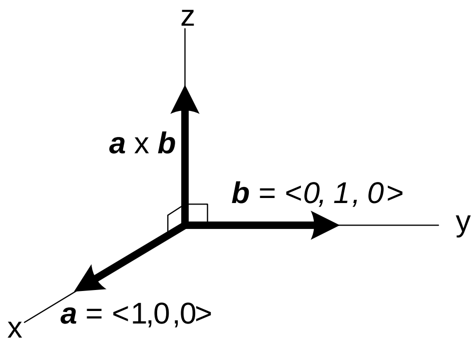
$$\langle 1,0,0\rangle \times \langle 0,1,0\rangle = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \hat{\boldsymbol{\imath}} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - \hat{\boldsymbol{\jmath}} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \widehat{\boldsymbol{k}} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= (0-0)\hat{\boldsymbol{\imath}} - (0-0)\hat{\boldsymbol{\jmath}} + (1-0)\widehat{\boldsymbol{k}}$$

$$= 0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 1\widehat{\boldsymbol{k}}$$

$$= \langle 0,0,1\rangle$$



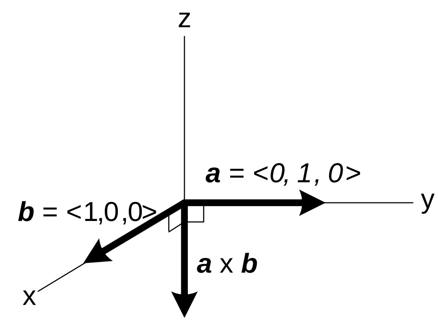
$$\langle 0,1,0\rangle \times \langle 1,0,0\rangle = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \hat{\boldsymbol{\imath}} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - \hat{\boldsymbol{\jmath}} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + \widehat{\boldsymbol{k}} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

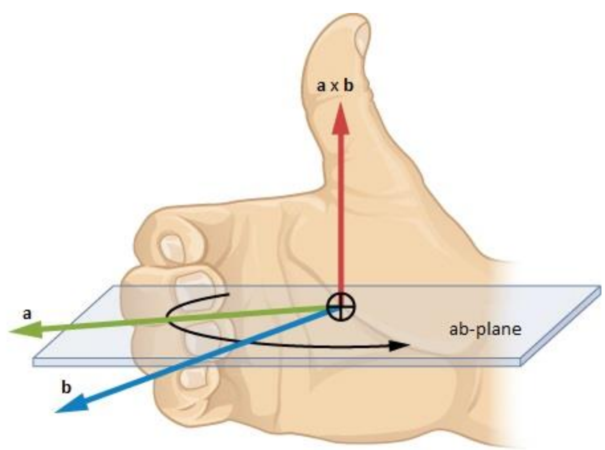
$$= (0-0)\hat{\boldsymbol{\imath}} - (0-0)\hat{\boldsymbol{\jmath}} + (0-1)\widehat{\boldsymbol{k}}$$

$$= 0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} - 1\widehat{\boldsymbol{k}}$$

$$= \langle 0,0,-1\rangle$$



In both examples the resulting vector is orthogonal to both of the vectors being crossed. As it turns out this fact is true in general. The rule that is used to determine the direction of $\boldsymbol{a} \times \boldsymbol{b}$ is called the *right-hand rule* and is illustrated below. The vector



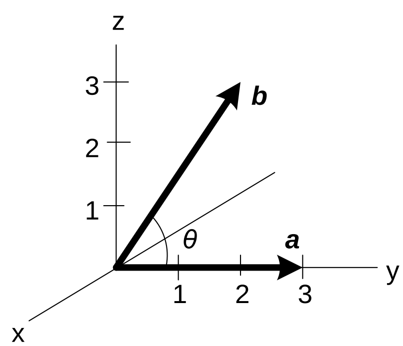
The rule can be stated as: *The vector, $\boldsymbol{a} \times \boldsymbol{b}$, is orthogonal to a plane that is parallel to $\boldsymbol{a}$ and $\boldsymbol{b}$. Furthermore, when the fingers of your right hand curl from $\boldsymbol{a}$ to $\boldsymbol{b}$, your thumb points to the side of the plane for which the resulting vector points.*

A geometric interpretation of the cross product, which is summarized below, can be inferred based on the discussion above.

Geometric Interpretation of the Cross Product
Given two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, the cross product, $\boldsymbol{a} \times \boldsymbol{b}$ is a unique vector with the following properties. i. $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to $\boldsymbol{a}$ and $\boldsymbol{b}$. ii. The length of $\boldsymbol{a} \times \boldsymbol{b}$ is $\lVert\boldsymbol{a}\rVert\lVert\boldsymbol{b}\rVert\sin(\theta)$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ and is chosen to satisfy $0 \le \theta \le \pi$.

Example 2: Let $\boldsymbol{a} = \langle 0,3,0\rangle$ and $\boldsymbol{b} = \langle 0,2,3\rangle$. Determine $\boldsymbol{a} \times \boldsymbol{b}$ using the geometric properties discussed above.

Solution: The figure below illustrate the fact that the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ exist in the y-z plane.



Using the right-hand rule, we see that $\boldsymbol{a} \times \boldsymbol{b}$ points away from the y-z plane in the positive x direction. To find the magnitude we need to find the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. Since $\boldsymbol{a}$ is on the x-axis we can use simple right triangle trigonometry to find the angle as follows.

$$\theta = tan^{-1}\left(\frac{3}{2}\right) \cong 56.3^{\circ}$$

The magnitude is then found as

$$\lVert\boldsymbol{a} \times \boldsymbol{b}\rVert = \lVert\boldsymbol{a}\rVert\lVert\boldsymbol{b}\rVert\, sin(\theta) = 3\sqrt{13}\, sin(56.3) = 9$$

Therefore, we can write cross product as follows.

$$\boldsymbol{a} \times \boldsymbol{b} = \langle 9,0,0\rangle$$

Finally, let's verify this result using the matrix computation first introduced.

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ 0 & 3 & 0 \\ 0 & 2 & 3 \end{vmatrix} = \hat{\boldsymbol{\imath}}\begin{vmatrix} 3 & 0 \\ 2 & 3 \end{vmatrix} - \hat{\boldsymbol{\jmath}}\begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} + \widehat{\boldsymbol{k}}\begin{vmatrix} 0 & 3 \\ 0 & 2 \end{vmatrix}$$

$$= 9\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 0\widehat{\boldsymbol{k}}$$

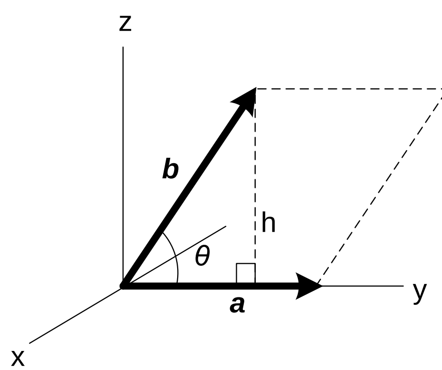
$$= \langle 9,0,0\rangle$$

Before moving to some additional application of the cross product we list some common properties below.

Properties of the Cross Product	
i.	$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$
ii.	$\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$
iii.	$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$ is and only if $\boldsymbol{a} = \lambda \boldsymbol{b}$ for some scalar λ or $\boldsymbol{b} = \boldsymbol{0}$
iv.	$\lambda(\boldsymbol{a} \times \boldsymbol{b}) = (\lambda \boldsymbol{a}) \times \boldsymbol{b} = \boldsymbol{a} \times (\lambda \boldsymbol{b})$
v.	$\boldsymbol{c} \times (\boldsymbol{a} + \boldsymbol{b}) = (\boldsymbol{c} \times \boldsymbol{a}) + (\boldsymbol{c} \times \boldsymbol{b})$, $(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a} \times \boldsymbol{c}) + (\boldsymbol{b} \times \boldsymbol{c})$

Area and Volume

The cross product is also related to the area of a parallelogram. We can investigate this relationship by drawing a parallelogram formed by two vectors in the y-z plane as shown below.



The base of the parallelogram is equal to the length of the vector $\boldsymbol{a}$, and the height is labeled h. From basic geometry we can compute the area of the parallelogram as

$$A_{\mathcal{P}} = \|\boldsymbol{a}\| h$$

Furthermore, the height of the parallelogram can be found using the right triangle formed from h and the vector $\boldsymbol{b}$.

$$h = \|\boldsymbol{b}\| \, sin(\theta)$$

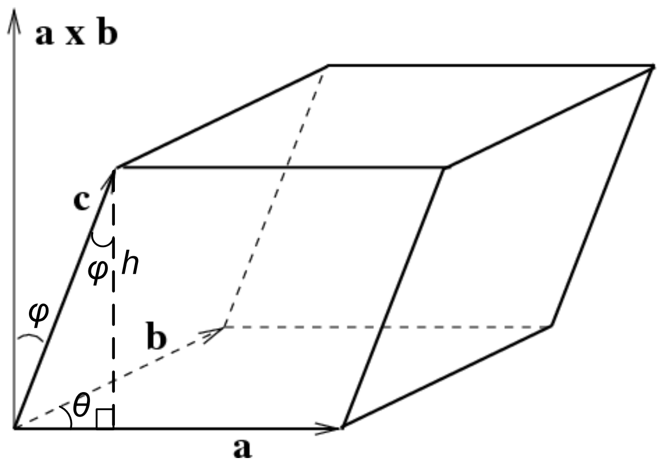
Substituting we can express the area in terms of the $\boldsymbol{a}$ and $\boldsymbol{b}$ as

$$A_{\mathcal{P}} = \|\boldsymbol{a}\| \|\boldsymbol{b}\| \, sin(\theta)$$

Which, as we now learned, also represents the cross product of $\boldsymbol{a}$ and $\boldsymbol{b}$. This result is true in general for any two vectors as stated formally below.

Area and the Cross Product
If $\mathcal{P}$ is the parallelogram formed by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the area, $A_{\mathcal{P}}$, can be found as $$A_{\mathcal{P}} = \|\boldsymbol{a} \times \boldsymbol{b}\|$$

A parallelepiped is a solid body where each face is a parallelogram. As we illustrate below, the volume of parallelepiped can be found using a combination of the dot product and the cross product.



The volume of a parallelepiped is given as

$$V_{\mathcal{P}} = (area\ of\ base) \cdot (height) = Bh$$

As we have already learned, the magnitude of the cross product can be used to find the area of the base.

$$B = \|\boldsymbol{a} \times \boldsymbol{b}\|$$

Furthermore, from the figure we can find the height using the right triangle with the hypotenuse represented by the vector, $\boldsymbol{c}$.

$$h = \|\boldsymbol{c}\| \, cos(\varphi)$$

Therefore, the volume can now be represented as

$$\begin{aligned} V_{\mathcal{P}} &= (B)(h) \\ &= (\boldsymbol{a} \times \boldsymbol{b}) \cdot (\|\boldsymbol{c}\| \, cos(\varphi)) \end{aligned}$$

Which we should notice is equivalent to the dot product of the vector, $\boldsymbol{a} \times \boldsymbol{b}$ with the vector, $\boldsymbol{c}$.

$$V_{\mathcal{P}} = |(\boldsymbol{c}) \cdot (\boldsymbol{a} \times \boldsymbol{b})|$$

Where, we use the absolute value in case the cross product points away from the parallelepiped.

Volume and the Cross Product
If $\mathcal{P}$ is the parallelepiped formed by the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, then the volume, $V_{\mathcal{P}}$, can be found as $$V_{\mathcal{P}} = |(\boldsymbol{c}) \cdot (\boldsymbol{a} \times \boldsymbol{b})|$$ Where, $(\boldsymbol{c}) \cdot (\boldsymbol{a} \times \boldsymbol{b})$ is referred to as the *vector triple product*.

Example 3: Use the geometric interpretation of the cross product along with the properties of the cross product to compute the following.

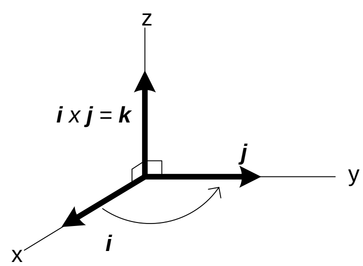
i. $2\hat{\boldsymbol{\imath}} \times 3\hat{\boldsymbol{\jmath}}$

ii. $(\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}) \times (\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})$

iii. $(\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}) \times (\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}})$

iv. $(2\hat{\imath} - 3\hat{\boldsymbol{\jmath}} + 4\hat{\boldsymbol{k}}) \times (\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - 7\hat{\boldsymbol{k}})$

Solution: After using various cross product properties each of the four questions above will require us to compute the cross product between a pair of standard basis vectors, $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$. The cross product between any two standard basis vectors is equal to the third, with the possibility of a minus sign. This can be illustrated by drawing a diagram and using the right-hand rule. For example, the figure below illustrates the fact that $\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}$.



For reference we list all of the basic cross products below.

$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}$	$\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}} = \hat{\boldsymbol{\imath}}$	$\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}}$
$\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}} = -\hat{\boldsymbol{k}}$	$\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\jmath}} = -\hat{\boldsymbol{\imath}}$	$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{k}} = -\hat{\boldsymbol{\jmath}}$
$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}} = \mathbf{0}$	$\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}} = \mathbf{0}$	$\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} = \mathbf{0}$

i.

$$2\hat{\boldsymbol{\imath}} \times 3\hat{\boldsymbol{\jmath}} = 2 \cdot 3(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}}) = 6\hat{\boldsymbol{k}} = \langle 0, 0, 6\rangle$$

ii.

$$\begin{aligned}
&(\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}) \times (\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}) \\
&= (\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}}) + (-3\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}}) + (2\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\jmath}}) + (\hat{\boldsymbol{\imath}} \times -\hat{\boldsymbol{k}}) + (-3\hat{\boldsymbol{\jmath}} \times -\hat{\boldsymbol{k}}) + (2\hat{\boldsymbol{k}} \times -\hat{\boldsymbol{k}}) \\
&= (\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}}) - 3(\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}}) + 2(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\jmath}}) - 1(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{k}}) + 3(\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}}) - 2(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}}) \\
&= (\hat{\boldsymbol{k}}) - 3(\mathbf{0}) + 2(-\hat{\boldsymbol{\imath}}) - 1(-\hat{\boldsymbol{\jmath}}) + 3(\hat{\boldsymbol{\imath}}) - 2(\mathbf{0}) \\
&= (\hat{\boldsymbol{k}}) - 2(\hat{\boldsymbol{\imath}}) + 1(\hat{\boldsymbol{\jmath}}) + 3(\hat{\boldsymbol{\imath}}) \\
&= \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}} = \langle 1,1,1\rangle
\end{aligned}$$

iii.

$$\begin{aligned}
(\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}) \times (\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) &= (\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}}) + (\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}}) - (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\jmath}}) - (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}}) \\
&= \mathbf{0} + (\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}}) + (\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}}) - \mathbf{0} \\
&= 2(\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}}) \\
&= 2\hat{\boldsymbol{\imath}} = \langle 2, 0, 0\rangle
\end{aligned}$$

iv.

$$
\begin{aligned}
&\left(2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}\right) \times \left(\hat{\imath} + \hat{\jmath} - 7\hat{k}\right) \\
&= \mathbf{0} + 2(\hat{\imath} \times \hat{\jmath}) - 14(\hat{\imath} \times \hat{k}) - 3(\hat{\jmath} \times \hat{\imath}) - \mathbf{0} + 21(\hat{\jmath} \times \hat{k}) + 4(\hat{k} \times \hat{\imath}) + 4(\hat{k} \times \hat{\jmath}) - \mathbf{0} \\
&= 2(\hat{k}) - 14(-\hat{\jmath}) - 3(-\hat{k}) + 21(\hat{\imath}) + 4(\hat{\jmath}) + 4(-\hat{\imath}) \\
&= 2(\hat{k}) + 14(\hat{\jmath}) + 3(\hat{k}) + 21(\hat{\imath}) + 4(\hat{\jmath}) - 4(\hat{\imath}) \\
&= 17\hat{\imath} + 18\hat{\jmath} + 5\hat{k} = \langle 17, 18, 5 \rangle
\end{aligned}
$$

Example 4: For each of the following compute $\boldsymbol{a} \times \boldsymbol{b}$ and verify that it is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

i. $\boldsymbol{a} = \langle 3, -4, 1 \rangle,\ \boldsymbol{b} = \langle 2, -2, 3 \rangle$ *ii.* $\boldsymbol{a} = \langle 2, -2, 6 \rangle,\ \boldsymbol{b} = \langle -1, 2, -1 \rangle$

Solution:

i.

$$
\begin{aligned}
\langle 3, -4, 1 \rangle \times \langle 2, -2, 3 \rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -4 & 1 \\ 2 & -2 & 3 \end{vmatrix} &= \hat{\imath}\begin{vmatrix} -4 & 1 \\ -2 & 3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & -4 \\ 2 & -2 \end{vmatrix} \\
&= \hat{\imath}(-12 + 2) - \hat{\jmath}(9 - 2) + \hat{k}(-6 + 8) \\
&= -10\hat{\imath} - 7\hat{\jmath} + 2\hat{k} = \langle -10, -7, 2 \rangle
\end{aligned}
$$

Next, to verify orthogonality we check if the dot product is zero.

$$
\begin{aligned}
\boldsymbol{a} \cdot (\boldsymbol{a} \times \boldsymbol{b}) &= \langle 3, -4, 1 \rangle \cdot \langle -10, -7, 2 \rangle \\
&= -30 + 28 + 2 = 0
\end{aligned}
\qquad
\begin{aligned}
\boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{b}) &= \langle 2, -2, 3 \rangle \cdot \langle -10, -7, 2 \rangle \\
&= -20 + 14 + 6 = 0
\end{aligned}
$$

ii.

$$
\begin{aligned}
\langle 2, -2, 6 \rangle \times \langle -1, 2, -1 \rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -2 & 6 \\ -1 & 2 & -1 \end{vmatrix} &= \hat{\imath}\begin{vmatrix} -2 & 6 \\ 2 & -1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 2 & 6 \\ -1 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -2 \\ -1 & 2 \end{vmatrix} \\
&= \hat{\imath}(2 - 12) - \hat{\jmath}(-2 + 6) + \hat{k}(4 - 2) \\
&= -10\hat{\imath} - 4\hat{\jmath} + 2\hat{k} = \langle -10, -4, 2 \rangle
\end{aligned}
$$

We again verify orthogonality with the dot product.

$$
\begin{aligned}
\boldsymbol{a} \cdot (\boldsymbol{a} \times \boldsymbol{b}) &= \langle 2, -2, 6 \rangle \cdot \langle -10, -4, 2 \rangle \\
&= -20 + 8 + 12 = 0
\end{aligned}
\qquad
\begin{aligned}
\boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{b}) &= \langle -1, 2, -1 \rangle \cdot \langle -10, -4, 2 \rangle \\
&= 10 - 8 - 2 = 0
\end{aligned}
$$

Example 5: Calculate the cross product below assuming that $\boldsymbol{a} = \langle 0, 3, -1\rangle$ and $\boldsymbol{b} = \langle 5, 2, 4\rangle$

$$(\boldsymbol{a} - 2\boldsymbol{b}) \times (\boldsymbol{a} + 2\boldsymbol{b})$$

Solution:

$$\begin{aligned}
(\boldsymbol{a} - 2\boldsymbol{b}) \times (\boldsymbol{a} + 2\boldsymbol{b}) &= \cancel{(\boldsymbol{a} \times \boldsymbol{a})} + 2(\boldsymbol{a} \times \boldsymbol{b}) - 2(\boldsymbol{b} \times \boldsymbol{a}) - 4\cancel{(\boldsymbol{b} \times \boldsymbol{b})} \\
&= 4(\boldsymbol{a} \times \boldsymbol{b}) \\
&= 4\begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ 0 & 3 & -1 \\ 5 & 2 & 4 \end{vmatrix} \\
&= 4\left(\hat{\boldsymbol{\imath}}\begin{vmatrix} 3 & -1 \\ 2 & 4 \end{vmatrix} - \hat{\boldsymbol{\jmath}}\begin{vmatrix} 0 & -1 \\ 5 & 4 \end{vmatrix} + \widehat{\boldsymbol{k}}\begin{vmatrix} 0 & 3 \\ 5 & 2 \end{vmatrix}\right) \\
&= 4\left(\hat{\boldsymbol{\imath}}(12 + 2) - \hat{\boldsymbol{\jmath}}(0 + 5) + \widehat{\boldsymbol{k}}(0 - 15)\right) \\
&= 56\hat{\boldsymbol{\imath}} - 20\hat{\boldsymbol{\jmath}} + -60\widehat{\boldsymbol{k}} = \langle 56, -20, -60\rangle
\end{aligned}$$

Example 6: Find two unit vectors orthogonal to both $\boldsymbol{a} = \langle 3, 1, 1\rangle$ and $\boldsymbol{b} = \langle -1, 2, 1\rangle$.

Solution: We know the cross product results in a vector that is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$. We can generate another orthogonal vector by simply negating, (i.e. rotating by 180°), this vector.

$$\langle 3, 1, 1\rangle \times \langle -1, 2, 1\rangle = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ 3 & 1 & 1 \\ -1 & 2 & 1 \end{vmatrix} = \hat{\boldsymbol{\imath}}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - \hat{\boldsymbol{\jmath}}\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} + \widehat{\boldsymbol{k}}\begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix}$$

$$= -1\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}} + 7\widehat{\boldsymbol{k}} = \langle -1, -4, 7\rangle$$

To create a unit vector, we divide by the magnitude.

$$||\langle -1, -4, 7\rangle|| = \sqrt{1 + 16 + 49} = \sqrt{66}$$

Therefore, the two unit vectors are

$$\frac{1}{\sqrt{66}}\langle -1, -4, 7\rangle \qquad \frac{1}{\sqrt{66}}\langle 1, 4, -7\rangle$$

Example 7: Compute the volume of the parallelepiped with adjacent edges of $\boldsymbol{a} = \langle 1, 2, 3\rangle$, $\boldsymbol{b} = \langle 3, 4, 0\rangle$ and $\boldsymbol{c} = \langle -1, 3, -2\rangle$.

Solution: To compute the volume we use the vector triple product as described above. We can derive a convenient form of the vector triple product as shown below.

$$(\boldsymbol{c})\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \langle c_x, c_y, c_z\rangle \cdot \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= \langle c_x, c_y, c_z\rangle \cdot \left(\hat{\boldsymbol{\imath}}\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{\boldsymbol{\jmath}}\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \widehat{\boldsymbol{k}}\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}\right)$$

$$= \left(c_x\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - c_y\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + c_z\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}\right)$$

Which is equal to the determinant of the $3x3$ matrix formed from $\boldsymbol{a}$**,** $\boldsymbol{b}$ and $\boldsymbol{c}$ as shown below.

$$(\boldsymbol{c})\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \begin{vmatrix} c_x & c_y & c_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \boldsymbol{det}\begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}$$

Let's use this expression to compute the volume.

$$(\boldsymbol{c})\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \begin{vmatrix} -1 & 3 & -2 \\ 1 & 2 & 3 \\ 3 & 4 & 0 \end{vmatrix} = \left(-1\begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 3 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}\right)$$

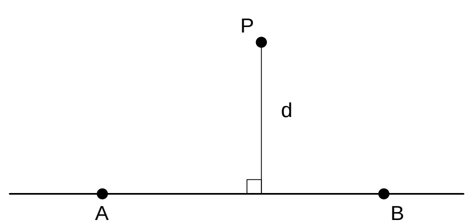
$$= \big(-1(0-12) - 3(0-9) - 2(4-6)\big)$$

$$= 43$$

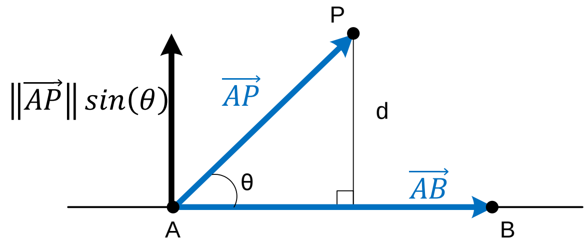
Finally, the volume of the parallelepiped is given as

$$V_{\mathcal{P}} = |(\boldsymbol{c})\cdot(\boldsymbol{a}\times\boldsymbol{b})| = |43| = 43$$

Example 8: Given three points A, B, and P as shown below, describe how you can use the cross product to calculate the distance, d, between point P and the line that connects A and B. Then compute the distance using the points $A = (1,0,1)$, B $= (2,3,1)$, and $P = (5,3,0)$.



We start by creating the vector $\overrightarrow{AB}$ and $\overrightarrow{AP}$ as shown in the figure below.



The figure above shows that the distance, d, is equal to the length of the orthogonal component of the vector $\overrightarrow{AP}$, i.e. $\|\overrightarrow{AP}\|\, sin(\theta)$.

Next, we notice that the magnitude of the cross product of the vectors $\overrightarrow{AP}$ and $\overrightarrow{AB}$ can be written as

$$\|\overrightarrow{AP} \times \overrightarrow{AB}\| = \|\overrightarrow{AP}\|\|\overrightarrow{AB}\|\, sin(\theta)$$

This too will give us the distance, d, if we divide through by $\|\overrightarrow{AB}\|$.

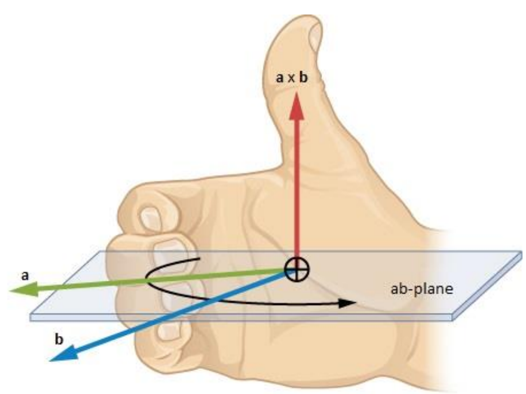
$$d = \frac{\|\overrightarrow{AP}\|\|\overrightarrow{AB}\|\, sin(\theta)}{\|\overrightarrow{AB}\|} = \frac{\|\overrightarrow{AP} \times \overrightarrow{AB}\|}{\|\overrightarrow{AB}\|}$$

Using the values given we write the vectors, $\overrightarrow{AP}$ and $\overrightarrow{AB}$, and use them to compute the distance, d, as follows.

$$\overrightarrow{AP} = \langle(5-1),(3-0),(0-1)\rangle \qquad \overrightarrow{AB} = \langle(2-1),(3-0),(1-1)\rangle$$
$$\overrightarrow{AP} = \langle 4,3,-1\rangle \qquad \overrightarrow{AB} = \langle 1,3,0\rangle$$

$$\begin{aligned} d = \frac{\|\overrightarrow{AP} \times \overrightarrow{AB}\|}{\|\overrightarrow{AB}\|} &= \frac{1}{\sqrt{10}}\left\|\begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ 4 & 3 & -1 \\ 1 & 3 & 0 \end{vmatrix}\right\| \\ &= \frac{1}{\sqrt{10}}\left\|\hat{\boldsymbol{\imath}}\begin{vmatrix} 3 & -1 \\ 3 & 0 \end{vmatrix} - \hat{\boldsymbol{\jmath}}\begin{vmatrix} 4 & -1 \\ 1 & 0 \end{vmatrix} + \widehat{\boldsymbol{k}}\begin{vmatrix} 4 & 3 \\ 1 & 3 \end{vmatrix}\right\| \\ &= \frac{1}{\sqrt{10}}\|\mathbf{3}\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + 9\widehat{\boldsymbol{k}}\| \\ &= \frac{\sqrt{9+1+81}}{\sqrt{10}} = \sqrt{\frac{91}{10}} \end{aligned}$$

Final Summary for Vector Geometry – Cross Product

The Cross Product
The cross product of two vectors, $\boldsymbol{a} = \langle a_x, a_y, a_z \rangle$ and $\boldsymbol{b} = \langle b_x, b_y, b_z \rangle$ is a new vector $\boldsymbol{v}$, given as $$\boldsymbol{v} = \boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{\boldsymbol{\imath}} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{\boldsymbol{\jmath}} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{\boldsymbol{k}} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$$ $$= (a_y b_z - b_y a_z)\hat{\boldsymbol{\imath}} - (a_x b_z - b_x a_z)\hat{\boldsymbol{\jmath}} + (a_x b_y - b_x a_y)\hat{\boldsymbol{k}}$$
Geometric Interpretation of the Cross Product
Given two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, the cross product, $\boldsymbol{a} \times \boldsymbol{b}$ is a unique vector with the following properties. i. $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to $\boldsymbol{a}$ and $\boldsymbol{b}$. ii. The length of $\boldsymbol{a} \times \boldsymbol{b}$ is $\|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin(\theta)$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ and is chosen to satisfy $0 \leq \theta \leq \pi$.
The Right-Hand Rule
The right-hand rule can be stated as: *The vector, $\boldsymbol{a} \times \boldsymbol{b}$, is orthogonal to a plane that is parallel to $\boldsymbol{a}$ and $\boldsymbol{b}$. Furthermore, when the fingers of your right hand curl from $\boldsymbol{a}$ to $\boldsymbol{b}$, your thumb points to the side of the plane for which the resulting vector points.* 

Properties of the Cross Product

i. $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$

ii. $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$

iii. $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$ is and only if $\boldsymbol{a} = \lambda \boldsymbol{b}$ for some scalar λ or $\boldsymbol{b} = \boldsymbol{0}$

iv. $\lambda(\boldsymbol{a} \times \boldsymbol{b}) = (\lambda \boldsymbol{a}) \times \boldsymbol{b} = \boldsymbol{a} \times (\lambda \boldsymbol{b})$

v. $\boldsymbol{c} \times (\boldsymbol{a} + \boldsymbol{b}) = (\boldsymbol{c} \times \boldsymbol{a}) + (\boldsymbol{c} \times \boldsymbol{b})$, $(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a} \times \boldsymbol{c}) + (\boldsymbol{b} \times \boldsymbol{c})$

Cross Product of The Standard Basis Vectors

$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}$	$\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}} = \hat{\boldsymbol{\imath}}$	$\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}}$
$\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}} = -\hat{\boldsymbol{k}}$	$\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\jmath}} = -\hat{\boldsymbol{\imath}}$	$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{k}} = -\hat{\boldsymbol{\jmath}}$
$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}} = \boldsymbol{0}$	$\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}} = \boldsymbol{0}$	$\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} = \boldsymbol{0}$

Area and Cross Product

If $\mathcal{P}$ is the parallelogram formed by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the area, $A_{\mathcal{P}}$, can be found as

$$A_{\mathcal{P}} = \|\boldsymbol{a} \times \boldsymbol{b}\|$$

Volume and Cross Product

If $\mathcal{P}$ is the parallelepiped formed by the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, then the volume, $V_{\mathcal{P}}$, can be found as

$$V_{\mathcal{P}} = |(\boldsymbol{c}) \cdot (\boldsymbol{a} \times \boldsymbol{b})|$$

Where, $(\boldsymbol{c}) \cdot (\boldsymbol{a} \times \boldsymbol{b})$ is referred to as the *vector triple product* and can be represented as

$$(\boldsymbol{c}) \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \begin{vmatrix} c_x & c_y & c_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \boldsymbol{det}\begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}$$

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