Infinite Series – Taylor Series

In the previous section we were able to generate new power series from known power series using substitution, differentiation, integration. In this section we will develop a more general method for finding power series of various functions. The power series we generate is generally referred to as a Taylor series. In the special case when the series is centered at $x = 0$, the series is sometimes referred to as a Maclaurin series. We begin by deriving the general form of a Taylor series.

Taylor Series

To derive the so-called Taylor series, we start with the general expression for a power series centered at $x = c$ and that is valid on the interval $(c - R, c + R)$.

$$f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$$

Next, we express the series in expanded form and compute its derivates as shown below.

$$\begin{array}{llllll} f(x) = & a_0 + & a_1(x-c) + & a_2(x-c)^2 + & a_3(x-c)^3 + & \cdots \\ f^1(x) = & a_1 + & 2\cdot a_2(x-c) + & 3a_3(x-c)^2 + & 4a_4(x-c)^3 + & \cdots \\ f^2(x) = & 2a_2 + & 2\cdot 3a_3(x-c) + & 3\cdot 4a_4(x-c)^2 + & 4\cdot 5a_5(x-c)^3 + & \cdots \\ f^3(x) = & 2\cdot 3a_3 + & 2\cdot 3\cdot 4a_4(x-c) + & 3\cdot 4\cdot 5a_5(x-c)^2 + & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

Now, we let $x = c$ and find the following:

$$\begin{aligned} f(c) &= a_0 \\ f^1(c) &= a_1 \\ f^2(c) &= 2a_2 \\ f^3(c) &= 2\cdot 3a_3 \\ f^4(c) &= 2\cdot 3\cdot 4a_4 \\ &\vdots \\ f^k(c) &= k!\, a_k \end{aligned}$$

Solving the general expression for the coefficients, a_k, we find

$$a_k = \frac{f^k(c)}{k!}$$

We can now use the coefficient expression above to rewrite the power series from above. The result is referred to as a Taylor series, $T(x)$, of the function $f(x)$ centered at $x = c$.

$$T(x) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n$$

The results are formally stated below with the following theorem.

Taylor Series Expansion

If $f(x)$ is represented as a power series centered at c in an interval $|x-c| < R$ with $R > 0$, then the power series is the **Taylor series**

$$T(x) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n$$

In the special case where $c = 0$, $T(x)$ is also called the **Maclaurin series**

$$T(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n$$

It's important to note that the theorem above tells us that if we want to represent a function, f, by a power series centered at c, the *only* candidate for this is the Taylor series. Unfortunately, there is no general method to determine whether $T(x)$ will converge to $f(x)$, (it is not guaranteed that $T(x)$ even converges), however we may use the following theorem in some special cases.

Taylor Series Convergence

Suppose there exists a $K > 0$ such that all derivatives of $f(x)$ are bounded by K on an interval, $I = (c-R, c+R)$, i.e.

$$|f^k(x)| \leq K\ , for\ all\ k > 0\ and\ x \in I$$

Then $f(x)$ can be represented by the Taylor series, $T(x)$ on I

$$T(x) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n\ \ , for\ all\ x \in I$$

Let's start with a straightforward example where the function we want to represent by a Taylor Series is a polynomial.

Example 1: Find the Taylor series for $f(x)$ centered at $x = 0$, i.e. the Maclaurin series.

$$f(x) = 4x^3 + 8x^2 + 6x + 7$$

Solution: The general form of the Taylor series centered at zero is

$$T(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

Therefore, we start by finding the coefficients, $\frac{f^n(0)}{n!}$.

$$\frac{f^0(0)}{0!} = \frac{7}{1} = 7$$

$$\frac{f^1(0)}{1!} = \frac{3 \cdot 4x^2 + 2 \cdot 8x + 6}{1}\bigg|_{x=0} = 6$$

$$\frac{f^2(0)}{2!} = \frac{2 \cdot 3 \cdot 4x + 2 \cdot 8}{2}\bigg|_{x=0} = \frac{16}{2} = 8$$

$$\frac{f^3(0)}{3!} = \frac{2 \cdot 3 \cdot 4}{6} = \frac{24}{6} = 4$$

Where, we stop at the third derivative since all higher derivatives are zero. For this case, and any polynomial, the series is finite.

$$T(x) = \frac{f^0(0)}{0!}x^0 + \frac{f^1(0)}{1!}x^1 + \frac{f^2(0)}{2!}x^2 + \frac{f^3(0)}{3!}x^3$$

$$T(x) = 7 + 6x^1 + 8x^2 + 4x^3 + 0x^4$$

$$T(x) = 4x^3 + 8x^2 + 6x + 7$$

The Taylor series is the polynomial itself and of course if valid for all x!

Next, we will look at few non-polynomial functions.

Example 2: Find the Taylor series representation of the function below centered at $x = 1$.

$$f(x) = \frac{1}{1-2x}$$

Solution: Using the substitution method from the previous section we can easily find the series expansion centered at zero, i.e. the Maclaurin series. However, we are asked to find the series expansion centered at $x = 1$, therefore we use the general Taylor series formula.

$$T(x) = \sum_{n=0}^{\infty} \frac{f^n(1)}{n!}(x-1)^n$$

In this case it is often useful to create a table to see if one can find a pattern for the coefficients.

n	$n!$	$f^n(x)$	$\frac{f^n(1)}{n!}$
0	1	$\frac{1}{1-2x}$	-1
1	1	$\frac{2}{(1-2x)^2}$	2
2	2	$\frac{8}{(1-2x)^3}$	-4
3	6	$\frac{48}{(1-2x)^4}$	8
4	24	$\frac{384}{(1-2x)^5}$	-16
…	…	…	…

In the last column we see two patterns: 1.) The coefficient flips sign each time starting with a negative value at $n = 0$, hence the term: $(-1)^{n+1}$ and 2.) The magnitude of the coefficients increase by a power of two, hence the term 2^n. The Taylor series is then written as follows:

$$T(x) = \sum_{n=0}^{\infty} (-1)^{n+1} 2^n (x-1)^n$$

Finally, we use the ratio test to determine where the series converges and find the following.

$$L = \lim_{n\to\infty} \left| \frac{\cancel{2^n(x-1)^n}\, 2^1 (x-1)^1}{\cancel{2^n(x-1)^n}} \right| = 2|x-1|$$

The series converges when $L < 1$, and therefore we have

$$2|x-1| < 1 \rightarrow \frac{1}{2} < |x| < \frac{3}{2}$$

Example 3: Find the Taylor series representation of the function below centered at $x = 1$.

$$f(x) = x^{-2}$$

Solution: Again, we can construct a table to see if we can find a pattern for the coefficients.

n	$f^n(x)$	$\frac{f^n(1)}{n!}$
0	$1x^{-2}$	$\frac{1}{1} = 1$
1	$-2x^{-3}$	$\frac{-2}{1} = -2$
2	$3 \cdot 2x^{-4}$	$\frac{6}{2} = 3$
3	$-4 \cdot 3 \cdot 2x^{-5}$	$\frac{-24}{6} = -4$
…	…	…
…	…	…
	$(-1)^n(n+1)!\,x^{-(n+2)}$	$\frac{(-1)^n(n+1)!}{n!} = (-1)^n(n+1)$

The Taylor series is then written as follows:

$$T(x) = \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n$$

We can again use the ratio test to determine where the series converges. In this case we find:

$$\frac{1}{2} < |x| < \frac{3}{2}$$

Example 4: Find the Maclaurin series representation of the following well known functions.

a. $f(x) = e^x$

b. $f(x) = sin(x)$

Solution:

a. Note that all derivatives of e^x evaluated at $x = 0$ are $e^0 = 1$. Therefore, the Maclaurin series for $f(x) = e^x$ is:

$$e^x = \sum_{n=0}^{\infty}\frac{(x)^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}, \qquad for\ \ all\ x$$

b. In this case we start by creating a table for the coefficients.

n	$f^n(x)$	$f^n(0)$
0	$sin(x)$	0
1	$cos(x)$	1
2	$-sin(x)$	0
3	$-cos(x)$	-1
4	$sin(x)$	0
…	…	…
…	…	…

After $n = 3$ the pattern repeats. Additionally, the series contains odd terms only which alternate in sign. With this, the series expansion of $sin(x)$ is

$$sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{(x)^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots, \qquad for\ \ all\ x$$

Furthermore, if we were to repeat this procedure for $cos(x)$ we would find a similar result with the exception that the series will contain only even terms.

$$cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{(x)^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots, \qquad for\ \ all\ x$$

In the previous section we used the substitution method to find series representations for functions that were similar to the geometric power series. We can use the series we learned in the previous example to find additional series in a similar fashion.

Example 5: Find the Maclaurin series representation of the following functions.

a. $f(x) = x^3\, sin(x)$

b. $f(x) = x^4 e^{2x^2}$

Solution:

a. We use the fact that we know the series representation of $sin(x)$.

$$x^3\, sin(x) = x^3\left(\sum_{n=0}^{\infty}(-1)^n \frac{(x)^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{(x)^{2n+1}\cdot x^3}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{(x)^{2n+4}}{(2n+1)!}$$

b. In this case, we additionally use the substitution method.

$$x^4 e^{(2x^2)} = x^4 \sum_{n=0}^{\infty} \frac{(2x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n \cdot x^{2n} \cdot x^4}{n!} = \sum_{n=0}^{\infty} \frac{2^n \cdot x^{2n+4}}{n!}$$

Taylor Series Integration

In many cases there turns out to be no convenient general formula for the coefficients as in the examples so far. However, we can always compute as many coefficients as we desire. One application of Taylor series is to use them to approximate definite integrals that may not otherwise have an analytic solution. Let's look at an example below.

Example 6: The Normal distribution is one of the most well-known probability distribution functions in statistics. To find the probability of an event one needs to compute a definite integral with the normal distribution as the integrand. Unfortunately, the integral cannot be evaluated analytically. Use what we know about Taylor series to approximate the following probability distribution integral.

$$P = \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

Note: The integrand represents a normal distribution with a zero mean and unity standard deviation.

Solution: We start by finding the Maclaurin series representation of the integrand.

$$e^{\left(-\frac{x^2}{2}\right)} = \sum_{n=0}^{\infty} \frac{\left(-\frac{x^2}{2}\right)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n \cdot n!}$$

Substituting this expression into the integral we have:

$$P = \int_0^1 \left(\frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n \cdot n!} \right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n \cdot n!} \int_0^1 x^{2n}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n \cdot n!} \cdot \left. \frac{x^{2n+1}}{2n+1} \right|_0^1$$

Since the lower limit of the integral is zero, we only need to evaluate the sum for $x = 1$. We can use as many terms of the sum as we like to improve the approximation. We will use the first four terms.

$$= \frac{1}{\sqrt{2\pi}}\left[\left(\frac{1}{2^0 \cdot 0! \cdot (1)}\right) + \left(\frac{-1}{2^1 \cdot 1! \cdot (2+1)}\right) + \left(\frac{1}{2^2 \cdot 2! \cdot (4+1)}\right) + \left(\frac{-1}{2^3 \cdot 3! \cdot (6+1)}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[(1) - \left(\frac{1}{6}\right) + \left(\frac{1}{40}\right) - \left(\frac{1}{336}\right)\right] \cong 0.341238$$

Before ending this section, we will derive one more important series, the Binomial Series.

We start by recalling the Binomial Theorem, stated below.

Binomial Theorem

If a is any positive integer then,

$$(x+y)^a = \sum_{k=0}^{a} \binom{a}{k} x^{a-k} y^k$$

$$(x+y)^a = \binom{a}{0} x^a y^0 + \binom{a}{1} x^{a-1} y^1 + \binom{a}{2} x^{a-2} y^2 + \cdots + \binom{a}{a-1} x^1 y^{n-1} + \binom{a}{a} x^0 y^a$$

Where, the binomial coefficient, $\binom{a}{k}$, is defined as

$$\binom{a}{k} = \frac{a!}{k!\,(a-k)!}$$

or equivalently as

$$\binom{a}{k} = \frac{a(a-1)(a-2)\cdots\big(a-(k-1)\big)}{k!} \qquad \binom{a}{0} \stackrel{\text{def}}{=} 1$$

This theorem is useful when n is large as shown in the example below.

Example 7: Use the Binomial Theorem to expand the following

$$f(x) = (x+3)^4$$

$$(x+3)^4 = \sum_{k=0}^{4} \binom{4}{k} a^{4-k} b^k$$

$$= \binom{4}{0} x^4 3^0 + \binom{4}{1} x^3 3^1 + \binom{4}{2} x^2 3^2 + \binom{4}{3} x^1 3^3 + \binom{4}{4} x^0 3^4$$

$$= x^4 + 12x^3 + 54x^2 + 108x^1 + 81$$

In the binomial series the above theorem is generalized so that n can be any number. We can derive this generalization by computing the Maclaurin series of $f(x) = (1 + x)^a$, where a is any number. We start by constructing a table of the coefficients as follows:

n	$f^n(x)$	$\frac{f^n(0)}{n!}$
0	$(1 + x)^a$	1
1	$a(1 + x)^{a-1}$	$\frac{a}{n!}$
2	$a(a - 1)(1 + x)^{a-2}$	$\frac{a(a-1)}{n!}$
3	$a(a - 1)(a - 2)(1 + x)^{a-3}$	$\frac{a(a-1)(a-2)}{n!}$
…	…	…
…	…	…
n	$a(a - 1)(a - 2) \cdots \big(a - (n - 1)\big)(1 + x)^{a-n}$	$\frac{a(a-1)(a-2)\cdots\big(a-(n-1)\big)}{n!}$

Note the Maclaurin coefficients are of the exact form of the binominal coefficients from above! In this case we replace n with k and note that a can be any number.

$$\binom{a}{n} = \frac{a(a-1)(a-2)\cdots\big(a-(n-1)\big)}{n!}$$

The binomial series is then defined below

Binomial Series
For any exponent a and for $\lvert x \rvert < 1$. $$(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$$ Where, the binomial coefficient, $\binom{a}{n}$, is defined as $$\binom{a}{n} = \frac{a(a-1)(a-2)\cdots\big(a-(n-1)\big)}{n!} \qquad \binom{a}{0} \stackrel{\text{def}}{=} 1$$

Let’s do an example to illustrate.

Example 8: Write down the first four terms of the binomial series for the following function

$$f(x) = \sqrt{9 - x}$$

Solution: Rewrite the function first to get into the form needed for the binomial series formula.

$$\begin{aligned} f(x) &= \frac{\sqrt{9}}{\sqrt{9}}\sqrt{9-x} \\ &= 3\left(1 + \left(-\frac{1}{9}x\right)\right)^{1/2} \end{aligned}$$

The binomial series is then

$$\begin{aligned} \sqrt{9-x} &= \sum_{n=0}^{\infty} 3\binom{1/2}{n}\left(-\frac{1}{9}x\right)^n \\ &= 3\sum_{n=0}^{\infty}(-1)^n\binom{1/2}{n}\left(\frac{x}{9}\right)^n \end{aligned}$$

Using the first four terms we have

$$\begin{aligned} \sqrt{9-x} &\cong 3\left(\binom{1/2}{0}\left(\frac{x}{9}\right)^0 - \binom{1/2}{1}\left(\frac{x}{9}\right)^1 + \binom{1/2}{2}\left(\frac{x}{9}\right)^2 - \binom{1/2}{3}\left(\frac{x}{9}\right)^3\right) \\ &\cong 3\left(1 - \frac{1}{2}\left(\frac{x}{9}\right)^1 + \frac{\left(\frac{1}{2}\cdot -\frac{1}{2}\right)}{2}\left(\frac{x}{9}\right)^2 - \frac{\left(\frac{1}{2}\cdot -\frac{1}{2}\cdot -\frac{3}{2}\right)}{6}\left(\frac{x}{9}\right)^3\right) \\ &\cong 3\left(1 - \frac{x}{18} - \frac{x^2}{648} + \frac{3x^3}{34992}\right) \\ &\cong \left(3 - \frac{x}{6} - \frac{x^2}{216} - \frac{x^3}{3888}\right) \end{aligned}$$

Final Summary for Infinite Series – Taylor Series

Taylor Series Expansion
If $f(x)$ is represented as a power series centered at c in an interval $\lvert x-c\rvert < R$ with $R > 0$, then the power series is the **Taylor series** $$T(x) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n$$ In the special case where $c = 0$, $T(x)$ is also called the **Maclaurin series** $$T(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n$$
Taylor Series Convergence
Suppose there exists a $K > 0$ such that all derivatives of $f(x)$ are bounded by K on an interval, $I = (c-R, c+R)$, i.e. $$\lvert f^k(x)\rvert \leq K \text{ , for all } k > 0 \text{ and } x \in I$$ Then $f(x)$ can be represented by the Taylor series, $T(x)$ on I $$T(x) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n \text{ , for all } x \in I$$
Binomial Theorem
If a is any positive integer then, $$(x+y)^a = \sum_{k=0}^{a} \binom{a}{k} x^{a-k}y^k$$ $$(x+y)^a = \binom{a}{0}x^a y^0 + \binom{a}{1}x^{a-1}y^1 + \binom{a}{2}x^{a-2}y^2 + \cdots + \binom{a}{a-1}x^1 y^{n-1} + \binom{a}{a}x^0 y^a$$ Where, the binomial coefficient, $\binom{a}{k}$, is defined as $$\binom{a}{k} = \frac{a!}{k!\,(a-k)!}$$ or equivalently as $$\binom{a}{k} = \frac{a(a-1)(a-2)\cdots\big(a-(k-1)\big)}{k!} \qquad \binom{a}{0} \stackrel{\text{def}}{=} 1$$

Table of Common Maclaurin Series		
$f(x)$	**Maclaurin Series**	**Converges to $f(x)$**
e^x	$\sum_{n=0}^{\infty}\frac{(x)^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$All\ x$
$sin(x)$	$\sum_{n=0}^{\infty}(-1)^n\frac{(x)^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$All\ x$
$cos(x)$	$\sum_{n=0}^{\infty}(-1)^n\frac{(x)^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$All\ x$
$(1+x)^a$	$\sum_{n=0}^{\infty}\binom{a}{n}x^n = \binom{a}{0}x^0 + \binom{a}{1}x^1 + \binom{a}{2}x^2 + \binom{a}{3}x^3 + \cdots$ $= 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots$	$\lvert x\rvert < 1$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty}x^n = 1 + x + x^2 + x^3 + \cdots$	$\lvert x\rvert < 1$
$\frac{1}{1+x}$	$\sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$	$\lvert x\rvert < 1$
$ln(1+x)$	$\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1} = \frac{x^1}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$-1 < x \le 1$

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