Infinite Series – Ratio and Root Tests for Convergence

In this section we'll learn two additional convergence tests; the ratio test and the root test. After learning these tests, we will summarize all of the tests we have learned and provide strategies for determining which test to apply to a given series.

Ratio Test

The ratio test can be useful for series containing factorials and constants to the power n. The test is formally stated below.

Ratio Test
Given the series $\sum a_n$ we define the following $$L = \lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$$ Then, • If $L < 1$ then $\sum a_n$ converges absolutely. • If $L > 1$ then $\sum a_n$ diverges. • If $L = 1$ the test is inconclusive.

Let's do a few examples.

Example 1: Apply the ratio test to the following series.

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

We start by defining the following.

$$a_n = \frac{2^n}{n!} \qquad\qquad a_{n+1} = \frac{2^{n+1}}{(n+1)!}$$

Next, we apply the ratio test

$$\begin{aligned} L &= \lim_{n\to\infty} \left\lvert \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right\rvert \\ &= \lim_{n\to\infty} \left\lvert \frac{\cancel{2^n} 2^1}{(n+1)\cdot \cancel{(n)\cdot(n-1)\cdot(n-2)\ldots(2)\cdot(1)}} \cdot \frac{\cancel{(n)\cdot(n-1)\cdot(n-2)\ldots(2)\cdot(1)}}{\cancel{2^n}} \right\rvert \\ &= \lim_{n\to\infty} \left\lvert \frac{2}{(n+1)} \right\rvert = 0 \end{aligned}$$

Since $L < 0$, the series converges.

Example 2: Apply the ratio test to the following alternating series.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{1000^n}$$

Directly applying the ratio test we have

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{(n+1)!}{1000^{n+1}} \cdot \frac{1000^n}{n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)\cdot \cancel{(n)\cdot(n-1)\cdot(n-2)\ldots(2)\cdot(1)}}{\cancel{1000^n}1000^1} \cdot \frac{\cancel{1000^n}}{\cancel{(n)\cdot(n-1)\cdot(n-2)\ldots(2)\cdot(1)}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)}{1000} \right| = \infty
\end{aligned}$$

In this case $L > 0$, and therefore the series diverges.

Example 3: Evaluate whether the following two series converge by first using the ratio test.

$$\sum_{n=1}^{\infty} n^2 \qquad\qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

We apply the ration test to both series below.

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \right| \\
&= \lim_{n\to\infty} \left| \frac{n^2+2n+1}{n^2} \right| \\
&= \lim_{n\to\infty} \left| \frac{n^2}{n^2} \right| = 1
\end{aligned}
\qquad\qquad
\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{n^2}{(n+1)^2} \right| \\
&= \lim_{n\to\infty} \left| \frac{n^2}{n^2+2n+1} \right| \\
&= \lim_{n\to\infty} \left| \frac{n^2}{n^2} \right| = 1
\end{aligned}$$

Unfortunately, the ratio test in inconclusive in both cases since $L = 1$. However, we can easily determine from the nth term divergence test that the first series diverges and that the second series in a convergent p-series.

As you can see from this example a series can either diverge or converge when the ratio test results in $L = 1$.

Root Test

The root test can be useful for series containing terms for the form $f(n)^{g(n)}$. The test is formally stated below.

Root Test
Given the series $\sum a_n$ we define the following $$L = \lim_{n\to\infty}\left(\sqrt[n]{\lvert a_n\rvert}\right)$$ Then, • If $L < 1$ then $\sum a_n$ converges absolutely. • If $L > 1$ then $\sum a_n$ diverges. • If $L = 1$ the test is inconclusive.

Let's do a few examples.

Example 4: Apply the root test to the following series.

$$\sum_{n=1}^{\infty}\left(\frac{n}{2n+3}\right)^n$$

We identify the sequence, a_n, has the form $f(n)^{g(n)}$, where $f(n) = \frac{n}{2n+3}$ and $g(n) = n$. Therefore, we can directly apply the root test.

$$\begin{aligned} L &= \lim_{n\to\infty}\left(\sqrt[n]{\lvert a_n\rvert}\right) \\ &= \lim_{n\to\infty}\left(\sqrt[n]{\left\lvert\left(\frac{n}{2n+3}\right)^n\right\rvert}\right) \\ &= \lim_{n\to\infty}\left(\frac{n}{2n+3}\right) = \frac{1}{2} \end{aligned}$$

Since $L < 1$ the series converges.

Example 5: Apply the root test to the following series.

$$\sum_{n=1}^{\infty} \frac{2^n}{n^{2n}}$$

In this case we start by rewriting the series as follows

$$\sum_{n=1}^{\infty} \frac{2^n}{n^{2n}} = \sum_{n=1}^{\infty} \left(\frac{2}{n^2}\right)^n$$

We can now directly apply the root test.

$$L = \lim_{n\to\infty} \left(\sqrt[n]{\left| \left(\frac{2}{n^2}\right)^n \right|} \right)$$
$$= \lim_{n\to\infty} \left(\frac{2}{n^2}\right) = 0$$

Again, since $L < 1$ the series converges.

The summary at the end of this section includes general guidelines for determining which test to apply when first looking at an arbitrary series. You should read over that before attempting the next set of examples,

Example 6: Evaluate the following series using any test we learned so far.

a. $$\sum_{n=1}^{\infty} \frac{n}{2n+1}$$

b. $$\sum_{n=1}^{\infty} \frac{sin(n)}{n^2}$$

c. $$\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$$

d. $$\sum_{n=1}^{\infty} \frac{n^3}{5^n}$$

e. $$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - n^2}}$$

f. $$\sum_{n=1}^{\infty} (0.8)^{-n} n^{-0.8}$$

g. $$\sum_{n=1}^{\infty} sin\left(\frac{1}{n^2}\right)$$

h. $$\sum_{n=1}^{\infty} \frac{(2n)!}{5n+1}$$

i. $$\sum_{n=2}^{\infty} \frac{e^{4n}}{(n-2)!}$$

$$a. \qquad \sum_{n=1}^{\infty} \frac{n}{2n+1}$$

We start by checking the n^{th} term divergence test, which states that if $\lim_{n\to\infty}(a_n) \neq 0$ then the series diverges.

$$\lim_{n\to\infty}\left(\frac{n}{2n+1}\right) = \lim_{n\to\infty}\left(\frac{\cancel{n}}{2\cancel{n}}\right) = \frac{1}{2} \neq 0$$

Therefore, the series diverges.

$$b. \qquad \sum_{n=1}^{\infty} \frac{sin(n)}{n^2}$$

Since this series in non-positive we check for absolute convergence using the direct comparison test with $b_n = \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty} b_n$ is a convergent p-series, we need to show that $a_n < b_n$ for all $n > M$.

$$0 \leq |sin(n)| \leq 1$$
$$0 \leq \frac{|sin(n)|}{n^2} \leq \frac{1}{n^2}$$

Therefore since $\frac{|sin(n)|}{n^2} \leq \frac{1}{n^2}$ the series converges absolutely. Furthermore, since the series converges absolutely the original no-positive series also converges.

$$c. \qquad \sum_{n=1}^{\infty} \frac{n!}{(2n)!}$$

In this case we directly apply the ratio test.

$$\begin{aligned}
L &= \lim_{n\to\infty}\left|\frac{(n+1)!}{\big(2(n+1)\big)!}\cdot\frac{(2n)!}{n!}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)\cancel{(n)(n-1)\dots(1)}}{(2n+2)(2n+1)\cancel{(2n)(2n-1)(2n-2)\dots(1)}}\cdot\frac{\cancel{(2n)(2n-1)(2n-1)\dots(1)}}{\cancel{(n)(n-1)\dots(1)}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)}{(2n+2)(2n+1)}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)}{(4n^2+5n+2)}\right| \\
&= \lim_{n\to\infty}\left|\frac{\cancel{n}}{4n^{\cancel{2}}}\right| = 0 < 1
\end{aligned}$$

Therefore, the series converges.

$$d. \qquad \sum_{n=1}^{\infty} \frac{n^3}{5^n}$$

We again apply the ratio test.

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(n+1)^3}{5^{n+1}} \cdot \frac{5^n}{n^3} \right| \\ &= \lim_{n\to\infty} \left| \frac{n^3 + 3n^2 + 3n + 1}{5 \cdot \cancel{5^n}} \cdot \frac{\cancel{5^n}}{n^3} \right| \\ &= \lim_{n\to\infty} \left| \frac{n^3 + 3n^2 + 3n + 1}{5n^3} \right| \\ &= \lim_{n\to\infty} \left| \frac{\cancel{n^3}}{5\cancel{n^3}} \right| = \frac{1}{2} < 1 \end{aligned}$$

Therefore, the series converges.

$$e. \qquad \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - n^2}}$$

In this case we use the limit comparison test with $b_n = \frac{1}{\sqrt{n^3}}$. Since $\sum_{n=1}^{\infty} b_n$.

$$\begin{aligned} L &= \lim_{n\to\infty} \left(\frac{1}{\sqrt{n^3 - n^2}} \cdot \frac{\sqrt{n^3}}{1} \right) \\ &= \lim_{n\to\infty} \left(\sqrt{\frac{n^3}{n^3 - n^2}} \right) \\ &= \sqrt{\lim_{n\to\infty} \left(\frac{n^3}{n^3 - n^2} \right)} \\ &= \sqrt{\lim_{n\to\infty} (1)} = \sqrt{1} = 1 \end{aligned}$$

Therefore, the series exhibit the same behavior. Since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3}} = \sum_{n=2}^{\infty} \frac{1}{n^{1.5}}$ is a convergent p-series, the original series converges.

$$f. \quad \sum_{n=1}^{\infty}(0.8)^{-n}n^{-0.8}$$

We'll use the ratio test, but first we'll rearrange the sequence as follows:

$$\sum_{n=1}^{\infty}(0.8)^{-n}n^{-0.8}=\sum_{n=1}^{\infty}\frac{10^n}{8^n(n^{4/5})}$$

Now applying the ratio test we have:

$$L=\lim_{n\to\infty}\left|\frac{10^{n+1}}{8^{n+1}((n+1)^{4/5})}\cdot\frac{8^n\left(n^{4/5}\right)}{10^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{10}{8((n+1)^{4/5})}\cdot\frac{\left(n^{4/5}\right)}{1}\right|$$

$$=\left(\frac{5}{4}\right)\lim_{n\to\infty}\left|\sqrt[5]{\left(\frac{n}{n+1}\right)^4}\right|$$

$$=\left(\frac{5}{4}\right)\sqrt[5]{\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^4}$$

$$=\left(\frac{5}{4}\right)\sqrt[5]{1}=\frac{5}{4}>1$$

In this case since $L>1$ the series diverges.

$$g. \quad \sum_{n=1}^{\infty}\sin\left(\frac{1}{n^2}\right)$$

In this case we test for absolute convergence using the limit comparison test with $b_n=\frac{1}{n^2}$.

$$L=\lim_{n\to\infty}\left|\frac{\sin\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}}\right|$$

To evaluate this limit, we apply Hopital's rule as shown.

$$L = \lim_{n\to\infty}\left|\frac{cos\left(\frac{1}{n^2}\right)\cdot \cancel{(-2n^{-3})}}{\cancel{(-2n^{-3})}}\right|$$
$$= \lim_{n\to\infty}\left|cos\left(\frac{1}{n^2}\right)\right|$$
$$= \lim_{n\to\infty}|cos(0)| = 1$$

The series have similar behavior and since $\sum_{n=1}^{\infty} b_n$ converges, the original series converges.

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h. $$\sum_{n=1}^{\infty}\frac{(2n)!}{5n+1}$$

We'll use the ratio test.

$$L = \lim_{n\to\infty}\left|\frac{(2n+2)!}{5n+6}\cdot\frac{5n+1}{(2n)!}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)\cancel{(2n)(2n-1)\dots(2)(1)}}{5n+6}\cdot\frac{5n+1}{\cancel{(2n)(2n-1)\dots(2)(1)}}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)(5n+1)}{5n+6}\right| = \infty$$

Therefore, the series diverges.

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i. $$\sum_{n=1}^{\infty}\frac{e^{4n}}{(n-2)!}$$

We'll use the ratio test.

$$L = \lim_{n\to\infty}\left|\frac{e^{4n+4}}{(n-1)!}\cdot\frac{(n-2)!}{e^{4n}}\right|$$
$$= \lim_{n\to\infty}\left|\frac{e^4\cancel{e^{4n}}}{(n-1)\cancel{(n-2)(n-3)(n-4)\dots(2)(1)}}\cdot\frac{\cancel{(n-2)(n-3)(n-4)\dots(2)(1)}}{\cancel{e^{4n}}}\right|$$
$$= \lim_{n\to\infty}\left|\frac{e^4}{(n-1)}\right| = 0$$

Therefore, the series converges.

Final Summary for Infinite Series

Summary of Tests
nth Term Divergence Test
If $\lim_{n\to\infty}(a_n) \neq 0$ then the series $\sum_{n=1}^{\infty} a_n$, diverges If $\lim_{n\to\infty}(a_n) = 0$ then the test is inconclusive.
Direct Comparison Test
Assume that there exists $M > 0$ such that $0 \leq a_n \leq b_n$ for all $n \geq M$. *i.* If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges. *ii.* If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.
Limit Comparison Test
Let a_n and b_n be positive sequences and let the following limit exist. $L = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)$ i. If $L > 0$ then $a_n \cong Lb_n$ for large n, and both series either converge or diverge. ii. If $L = 0$ then $b_n \gg a_n$ for large n, and if $\sum_{n=1}^{\infty} b_n$ converge, so does $\sum_{n=1}^{\infty} a_n$ iii. If $L = \infty$ then $a_n \gg b_n$ for large n, and if $\sum_{n=1}^{\infty} a_n$ converge, so does $\sum_{n=1}^{\infty} b_n$.
Integral Test
Let $a_n = f(n)$, where f is a positive, decreasing, and continuous function of x for $x \geq 1$. • If $\int_1^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\int_1^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.
Ratio Test
Given the series $\sum a_n$ we define the following $L = \lim_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\|$ Then, • If $L < 1$ then $\sum a_n$ converges absolutely. • If $L > 1$ then $\sum a_n$ diverges. • If $L = 1$ the test is inconclusive.
Root Test
Given the series $\sum a_n$ we define the following $L = \lim_{n\to\infty}\left(\sqrt[n]{\|a_n\|}\right)$ Then, • If $L < 1$ then $\sum a_n$ converges absolutely. • If $L > 1$ then $\sum a_n$ diverges. • If $L = 1$ the test is inconclusive.

Alternating Series Test
An alternating series of the form $$\sum_{n=1}^{\infty}(-1)^{n-1}b_n$$ Converges if • $b_n \geq 0$, Non-negative • $b_{n+1} < b_n$, Decreasing $$\lim_{n\to\infty}(b_n) = 0$$
Absolute Convergence
The series $\sum a_n$ ***converges absolutely*** if $\sum\lvert a_n\rvert$ converges.
Absolute Convergence Implies Convergence
• If $\sum\lvert a_n\rvert$ converges, then $\sum a_n$ also converges. • If $\sum\lvert a_n\rvert$ diverges, then the behavior of $\sum a_n$ is inconclusive.
Conditional Convergence
A series $\sum a_n$ ***converges conditionally*** if $\sum a_n$ converges but $\sum\lvert a_n\rvert$ diverges.
Convergence of p-Series
The infinite series $$\sum_{n=1}^{\infty}\frac{1}{n^p}$$ Converges for $p > 1$, and diverges otherwise.
Geometric Series
A geometric series, with $C \neq 0$ has the general form $$S = \sum_{n=M}^{\infty} Cr^n$$ If $\lvert r\rvert < 1$ the geometric series converges and $$S = \sum_{n=M}^{\infty} Cr^n = \frac{Cr^M}{(1-r)}$$ Note: if $M = 0$ we can write $$S = \sum_{n=0}^{\infty} Cr^n = \frac{C}{(1-r)}$$ If $\lvert r\rvert \geq 1$ the geometric series diverges.

Test Determination Strategies
It's not always obvious which test to apply to a particular series. Below are some general guidelines for determining which test to apply when first looking at an arbitrary series.
1. The n^{th} term Divergence Test
One should always check this test first, as it is usually relatively east to check. If the series diverges, i.e. $\lim_{n\to\infty}(a_n) \neq 0$, no other test are required.
2. Positive Series
a. The Direct Comparison Test   For this test we should consider whether dropping terms in the numerator or denominator of the sequence results in a series that we know either converges for diverges.   If by dropping terms we create a series with a sequence, b_n, that converges, we can prove the original series with the sequence, a_n, also converges if $a_n < b_n$ for all $n > M$.   For example, if given $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2+n}$, we can create a new series, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$. Then since $\sum_{n=1}^{\infty} b_n$ is a convergent p-series and $a_n < b_n$ we can conclude that $\sum_{n=1}^{\infty} a_n$ converges.   Conversely, if by dropping terms we create a series with a sequence, b_n, that diverges, we can prove the original series with the sequence, a_n, also diverges if $a_n > b_n$ for all $n > M$ For example, if given $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$, we can create a new series, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$. Then since $\sum_{n=1}^{\infty} b_n$ is a divergent p-series and $a_n > b_n$ we can conclude that $\sum_{n=1}^{\infty} a_n$ diverges.   An example of a series where this test doesn't work is $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2-n}$. In this case we can use the limit comparison test, which we explain next.
a. The Limit Comparison Test   For this test we should consider the dominant term in the numerator and denominator as our comparison sequence, b_n. Take the example from the previous test.   Given $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2-n}$, we would create a new series as $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$.   Performing the limit comparison test we have, $L = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{n^2}{n^2-n}\right) = 1$, and since $L > 0$ the series behave the same, and therefore $\sum_{n=1}^{\infty} a_n$ converges.
b. The Ratio Test   The ratio test can be useful for series containing factorials and constants to the power n. When forming the ratio, the power n cancels and there is also quite a bit of cancellation with the factorials. For example, when we apply the ratio test to the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{5^n}{n!}$, the result is as follows: $$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{5^{n+1}}{(n+!)!}\cdot\frac{n!}{5^{n+1}}\right) = \lim_{n\to\infty}\left(\frac{5}{n+1}\right) = 0$$ And since $L < 1$ the series converges.

c. *The Root Test*

The root test can be useful for series containing a term of the form $f(n)^{g(n)}$. For example, applying the root test to the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\left(\frac{3n^3+5n}{7n^3+2}\right)^n$ yields the following.

$$L = \lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}\left(\left(\frac{3n^3+5n}{7n^3+2}\right)^n\right)^{1/n} = \lim_{n\to\infty}\left(\frac{3n^3+5n}{7n^3+2}\right) = \lim_{n\to\infty}\left(\frac{3\cancel{n^3}}{7\cancel{n^3}}\right) = \frac{3}{7}$$

And since $L < 1$ the series converges

f. *The Integral Test*

When other tests fail and the series is positive and decreasing, we can consider the integral test. With the integral test the series converges if $\int_1^\infty f(x)dx$, where we map the discrete sequence a_n to the continuous function, $f(x)$. For example, the following series is difficult to evaluate using any of the above tests but can be easily evaluated using the integral test as shown below. Given the series

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty}\frac{1}{n\ln n}$$

We find

$$\int_2^\infty \frac{1}{x\ln x}dx = \ln(\ln x)|_2^\infty = \infty$$

Therefore, the series diverges.

3. Non-Positive Series

a. *The Alternating Series Test*

An alternating series is of the form $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$, and converges if:

- $b_n \geq 0$, Non-negative
- $b_{n+1} < b_n$, Decreasing
- $\lim_{n\to\infty}(b_n) = 0$

For example, the alternating harmonic series shown below.

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

Converges since

- $\frac{1}{n} \geq 0$, Non-negative
- $\frac{1}{n+1} < \frac{1}{n}$, Decreasing
- $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$

b. *Absolute Convergence*

If the series is not alternating, but nonetheless non-positive we may compute the absolute value of the sequence and check if this series converges. If so, the original series converges as well.