Infinite Series – Power Series

In this section we'll focus on a specific type of series referred to as a power series. The applications of power series are practically innumerable. For example, many functions such as trigonometric, exponential, and logarithm functions can be expressed as a power series. Even further, power series show up in many signal analysis type applications.

The general form of a power series is written as shown below.

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

Where, x is a variable and c is a constant. We refer to this as a power series with center c. The series can also be expressed in expanded form as shown below.

$$F(x) = a_0 + a_1(x-c)^1 + a_2(x-c)^2 + a_3(x-c)^3 + \ldots$$

As we have previously learned an infinite series may or may not converge. Because the power series has a variable, x, it may converge for some values of x and diverge for others. Evaluating power series for convergence is generally done using the ratio test. We illustrate with an example below.

Example 1: For what values of x does $F(x) = \sum_{n=0}^{\infty} 2^n x^n$ converge?

Using the ratio test we have:

$$L = \lim_{n\to\infty} \left| \frac{2^{n+1}x^{n+1}}{2^n x^n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{2^n} 2 \cancel{x^n} x}{\cancel{2^n} \cancel{x^n}} \right| = \lim_{n\to\infty} |2x| = 2|x|$$

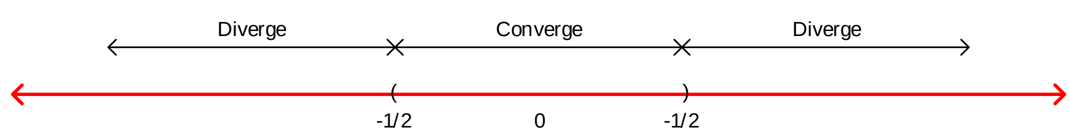
The power series converges when $L < 1$, and therefore we have

$$2|x| < 1 \rightarrow |x| < \frac{1}{2}$$

Note that the ratio test is inconclusive for the endpoints of the region found above. Therefore, we must test these points separately as follows.

$x = -\frac{1}{2}$	$x = \frac{1}{2}$
$\sum_{n=0}^{\infty} 2^n \left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n$ The limit DNE.	$\sum_{n=0}^{\infty} 2^n \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} (1)^n$ The series diverges.

The result is a so-called *region of convergence*, ROC, shown below.



Interestingly enough all power series will display similar behavior as the example above. We summarize the possible scenarios below.

Radius of Convergence for Power Series

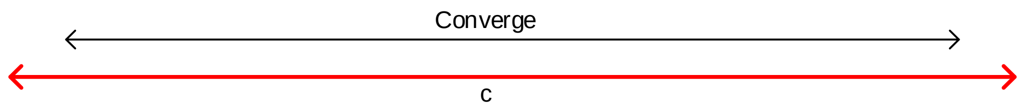
For every power series of the form

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

There are three possible cases for the radius of convergence, R, as given below.

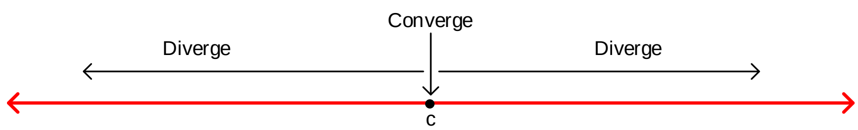
Case 1:

The series converges for all values of x, i.e. $R = \infty$



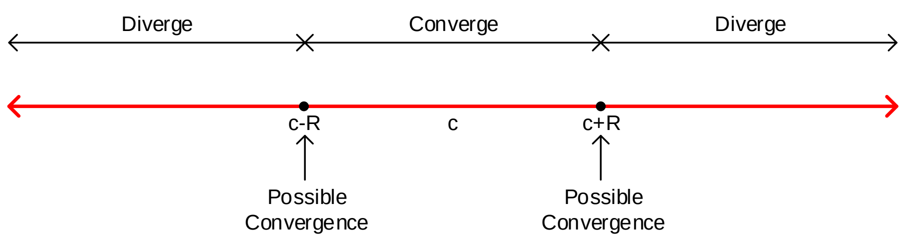
Case 2:

The series diverges for all values of x except $x = c$, i.e. $R = 0$



Case 3:

The series converges for $|x - c| < R$, i.e. $R > 0$



Based on the above summary there are two steps required to find the interval over which a power series converges.

1. Find the radius of convergence, R, using the ratio test.
2. Check endpoints when $R \neq 0 \; or \; \infty$

Let's practice with some examples.

Example 2: Find the interval over with the following power series converge.

a.

$$F(x) = \sum_{n=4}^{\infty} \frac{x^n}{n^5}$$

b.

$$F(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^5}$$

c.

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2+1}}$$

d.

$$F(x) = \sum_{n=1}^{\infty} n(x-3)^n$$

e.

$$F(x) = \sum_{n=1}^{\infty} \frac{2^n}{3n}(x+3)^n$$

f.

$$F(x) = \sum_{n=12}^{\infty} e^n(x-2)^n$$

Solutions:

1.

$$F(x) = \sum_{n=4}^{\infty} \frac{x^n}{n^5}$$

$$L = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^5} \cdot \frac{n^5}{x^n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{x^n} x n^5}{\cancel{x^n}(n+1)^5} \right| = |x| \lim_{n\to\infty} \left| \frac{n^5}{(n+1)^5} \right| = |x|$$

The power series converges when $L < 1$, and therefore we have

$$|x| < 1$$

Next, we check the endpoints.

$x = -1$	$x = 1$
$\sum_{n=4}^{\infty} \frac{(-1)^n}{n^5}$ Converges based on the alternating series test.	$\sum_{n=4}^{\infty} \frac{1^n}{n^5} = \sum_{n=4}^{\infty} \frac{1}{n^5}$ Which is a convergent p-series.

The region of convergence can be defined using interval notation as follows:

$$x = [-1, 1]$$

2.

$$F(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^5}$$

$$L = \lim_{n\to\infty} \left| \frac{x^{\cancel{n}+1}}{\left((n+1)!\right)^5} \cdot \frac{(n!)^5}{\cancel{x^n}} \right| = \lim_{n\to\infty} \left| x \left(\frac{n!}{(n+1)!} \right)^5 \right| = |x| \lim_{n\to\infty} \left| \frac{1}{(n+1)^5} \right| = |x| \cdot 0 = 0$$

Therefore, the power series converges for all values of x.

$$x = (-\infty, \infty)$$

3.

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2+1}}$$

$$L = \lim_{n\to\infty} \left| \frac{x^{\cancel{n}+1}}{\sqrt{(n+1)^2+1}} \cdot \frac{\sqrt{n^2+1}}{\cancel{x^n}} \right| = \lim_{n\to\infty} \left| x \left(\sqrt{\frac{n^2+1}{n^2+2n+2}} \right) \right| = |x| \sqrt{\lim_{n\to\infty} \left| \frac{n^2+1}{n^2+2n+2} \right|}$$

$$= |x| \cdot \sqrt{1}$$

$$= |x|$$

The power series converges when $L < 1$, and therefore we have

$$|x| < 1$$

Next, we check the endpoints.

$x = -1$	$x = 1$
$\sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n}{\sqrt{n^2+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$ You can find that this series diverges by using the limit comparison test with the harmonic series $\sum_{n=0}^{\infty} \frac{1}{n}$.	$\sum_{n=0}^{\infty} \frac{(-1)^n(1)^n}{\sqrt{n^2+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ Converges based on the alternating series test.

The region of convergence can be defined using interval notation as follows:

$$x = (-1, 1]$$

4.

$$F(x) = \sum_{n=1}^{\infty} n(x-3)^n$$

$$L = \lim_{n\to\infty}\left|\frac{(n+1)\cancel{(x-3)^n}(x-3)}{n\cancel{(x-3)^n}}\right| = |x-3|\lim_{n\to\infty}\left|\frac{(n+1)}{n}\right| |x-3|\cdot 1$$

Therefore, we have

$$|x-3| < 1 \to x = (2,4)$$

Next, we check the endpoints.

$x = 2$	$x = 4$
$\sum_{n=1}^{\infty} n(-1)^n$ Which diverges.	$\sum_{n=1}^{\infty} n(1)^n = \sum_{n=n}^{\infty} n$ Which diverges.

The region of convergence can be defined using interval notation as follows:

$$x = (2,4)$$

5.

$$F(x) = \sum_{n=1}^{\infty} \frac{2^n}{3n}(x+3)^n$$

$$L = \lim_{n\to\infty}\left|\frac{2^{\cancel{n}+1}(x+3)^{\cancel{n}+1}}{3(n+1)}\cdot\frac{3n}{\cancel{2^n(x+3)^n}}\right| = \lim_{n\to\infty}\left|\frac{2(x+3)3n}{3n+3}\right|$$

$$= 2|(x+3)|\lim_{n\to\infty}\left|\frac{3n}{3n+3}\right| = 2|(x+3)|\cdot 1$$

Therefore, we have

$$|(x+3)| < \tfrac{1}{2}$$

For which we can find the interval as shown below.

$$(x+3) < \frac{1}{2} \quad \textit{and} \quad (x+3) > -\frac{1}{2}$$

$$x < \frac{1}{2} - 3 \qquad x > -\frac{1}{2} - 3$$

$$x < -2.5 \qquad x > -3.5$$

$$-3.5 < x < -2.5$$

Finally, we check the endpoints.

$x = -3.5$	$x = -2.5$
$\sum_{n=1}^{\infty} \frac{2^n}{3n}\left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{3n} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ Which converges based on the alternating series test.	$\sum_{n=1}^{\infty} \frac{2^n}{3n}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1^n}{3n} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n}$ Which is a divergent harmonic series.

The region of convergence can be defined using interval notation as follows:

$$x = [-3.5, -2.5)$$

6.

$$F(x) = \sum_{n=12}^{\infty} e^n (x-2)^n$$

$$L = \lim_{n\to\infty} \left| \frac{e^{\cancel{n}+1}(x-2)^{\cancel{n}+1}}{\cancel{e^n (x-2)^n}} \right| = \lim_{n\to\infty} |e^1 (x-2)| = e^1 |(x-2)|$$

Therefore, we have

$$|(x-2)| < \frac{1}{e^1} \rightarrow x = \left(2 - \frac{1}{e^1}, 2 + \frac{1}{e^1}\right)$$

Finally, we check the endpoints.

$x = 2 - \frac{1}{e^1}$	$x = 2 + \frac{1}{e^1}$
$\sum_{n=12}^{\infty} e^n \left(-\frac{1}{e^1}\right)^n = \sum_{n=12}^{\infty} (-1)^n$ Which diverges.	$\sum_{n=12}^{\infty} e^n \left(\frac{1}{e^1}\right)^n = \sum_{n=12}^{\infty} (1)^n$ Which diverges.

The region of convergence can be defined using interval notation as follows:

$$x = \left(2 - \frac{1}{e^1}, 2 + \frac{1}{e^1}\right)$$

Important to the practical applications of power series is the ability to represents functions with power series. One function in particular that we are already aware of is related to the geometric series. As shown below a geometric series is a form of a power series.

Geometric Series as a Power Series
$$F(x) = \sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$$ For $\lvert x \rvert < 1$

As shown above, the power series, $F(x) = \sum_{n=0}^{\infty} ax^n$, can be used to represent the function, $F(x) = \frac{a}{1-x}$, for $-1 < x < 1$.

Of course, this is not the only function that can be represented with a power series. Below we discuss three tools that can be used to derive additional power series representations for various functions. They are: 1. Substitution, 2. Differentiation, and 3. Integration.

Substitution

Substitution is a powerful means to find power series representations of functions by using previously known functions. Let's see how this works with the following example.

Example 3: Find the power series representation of the following function.

$$F(x) = \frac{1}{1-3x}$$

We start by rewriting the function using the substitution, $z = 3x$.

$$F(z) = \frac{1}{1-z}$$

Which, as shown below, is a function for which we already know the power series representation

$$F(z) = \sum_{n=0}^{\infty} z^n$$

For $\lvert z \rvert < 1$.

Finally, we resubstitute $z = 3x$ into the known power series as follows:

$$F(x) = F(z)\big|_{z=3x} = \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} 3^n x^n$$

For $\lvert 3x \rvert < 1 \;\rightarrow \lvert x \rvert < \frac{1}{3}$

Let's practice this technique with additional functions below.

Example 4: Find the power series representation of the following functions.

a. $$F(x) = \frac{1}{4+3x}$$

b. $$F(x) = \frac{1}{1+x^2}$$

c. $$F(x) = \frac{1}{16+2x^3}$$

Solutions:

a.
$$F(x) = \frac{1}{4+3x}$$

We start by rewriting as follows:

$$F(x) = \frac{1}{4+3x} = \frac{1/4}{1+3/4\,x} = \frac{1/4}{1-(-3/4\,x)}$$

Therefore,

$$F(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} 1/4\,(z)^n$$

Where, $z = -3/4\,x$

Finally, we have:

$$F(x) = F(z)|_{z=-3/4x} = \sum_{n=0}^{\infty} 1/4\,(-3/4\,x)^n$$
$$F(x) = \sum_{n=0}^{\infty} 1/4\,(-1)^n \left(\frac{3}{4}\right)^n x^n$$

For $|-3/4\,x| < 1 \rightarrow |x| < \frac{4}{3}$

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b.

$$F(x) = \frac{1}{1+x^2}$$

$$F(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

Therefore,

$$F(x) = F(z)|_{z=-x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

For $|-x^2| < 1 \rightarrow |x| < 1$

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c.

$$F(x) = \frac{1}{16+2x^3}$$

$$F(x) = \frac{1}{16+2x^3} = \frac{1/16}{1-\left(-\frac{1}{8}x^3\right)}$$

Therefore,

$$F(x) = F(z)|_{z=-\frac{1}{8}x^3} = \sum_{n=0}^{\infty} 1/16\left(-\frac{1}{8}x^3\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{8^n}$$

For $\left|-\frac{1}{8}x^3\right| < 1 \rightarrow |x| < 8$

Power Series Differentiation and Integration

Since differentiation and integration are linear operators, we can use them to operate on a power series term by term. Doing this will also allow us to develop power series representations of additional functions. Before we do some examples, we state the theorem below.

Power Series Differentiation and Integration
Assume that the power series $$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$ Has a radius of convergence, $R > 0$. Then $F(x)$ can be differentiated and integrated for $c - R < x < c + R$. The differentiation and integration is done term by term and can be expressed as follows:
Differentiation
$$\frac{d}{dx}\big(F(x)\big) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n (x-c)^n\right) = \sum_{n=0}^{\infty} a_n \frac{d}{dx}(x-c)^n$$ $$\boldsymbol{\frac{d}{dx}\big(F(x)\big) = \sum_{n=1}^{\infty} a_n n (x-c)^{n-1}}$$
Integration
$$\int F(x)dx = \int\left(\sum_{n=0}^{\infty} a_n (x-c)^n\right)dx = \sum_{n=0}^{\infty}\left(\int a_n (x-c)^n\right)dx$$ $$\boldsymbol{\int F(x)dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1} + C}$$
The resulting series have the same radius of convergence, R.

Let's demonstrate the above theorem with some examples.

Example 5: Use differentiation to find the power series representation of the following function.

$$f(x) = \frac{1}{(1-x)^2} \quad , -1 < x < 1$$

We start by noticing the following:

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$$

Furthermore, since the left-hand side function is the geometric power series, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we can write the following:

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{1}{(1-x)^2}$$

Finally, differentiating the power series we have

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{1}{(1-x)^2}$$

$$\sum_{n=0}^{\infty} \frac{d}{dx}(x^n) = \frac{1}{(1-x)^2}$$

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$$

Therefore,

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}, -1 < x < 1$$

Example 6: Use integration to find the power series representation of the following function.

$$f(x) = ln(1+x), \qquad -1 < x < 1$$

We start by noticing the following:

$$\int \left(\frac{1}{1+x}\right) dx = ln(1+x)$$

We can find the power series of the integrand on the left-hand side using substitution with the geometric power series as follows:

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

With this we can write the following:

$$\int \left(\sum_{n=0}^{\infty} (-1)^n x^n \right) dx = ln(1+x)$$

$$\sum_{n=0}^{\infty} (-1)^n \int x^n \, dx = ln(1+x)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + C = ln(1+x)$$

To solve for C we let $x = 0$ and find $C = 0$.

Which is more commonly written as,

$$ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n, \qquad -1 < x < 1$$

Example 7: Find the power series representation of the following functions.

a.

$$\frac{x^2}{(1-2x)^2}$$

b.

$$tan^{-1}(2x)$$

Solution:

a. We start by rewriting the function as, $x^2 \cdot \frac{1}{(1-2x)^2}$, and noticing that

$$\frac{d}{dx}\left(\frac{1}{(1-2x)}\right) = \frac{2}{(1-2x)^2}$$

$$\left(\frac{1}{2}\right)\frac{d}{dx}\left(\frac{1}{(1-2x)}\right) = \frac{1}{(1-2x)^2}$$

Next, the power series of the function on the left-had side can be found using substitution with the geometric power series as follows:

$$\frac{1}{(1-(2x))} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$$

Using this result, we have

$$\frac{1}{(1-2x)^2} = \frac{1}{2}\frac{d}{dx}\left(\frac{1}{(1-2x)}\right)$$

$$= \frac{1}{2}\frac{d}{dx}\left(\sum_{n=0}^{\infty} 2^n\, x^n\right)$$

$$= \frac{1}{2}\left(\sum_{n=1}^{\infty} 2^n\, nx^{n-1}\right)$$

$$= \sum_{n=1}^{\infty} 2^{n-1}\, nx^{n-1}$$

Finally, we can multiply both sides by x^2.

$$\frac{x^2}{(1-2x)^2} = x^2 \sum_{n=1}^{\infty} 2^{n-1}\, nx^{n-1}$$

$$= \sum_{n=1}^{\infty} 2^{n-1}\, nx^{n-1}x^2$$

$$= \sum_{n=1}^{\infty} 2^{n-1}\, nx^{n+1}$$

$$\frac{x^2}{(1-2x)^2} = \sum_{n=0}^{\infty} 2^n\,(n+1)x^{n+2}$$

Where, in the last step we let the series start at $n = 0$ and adjusted the terms inside the summation appropriately.

b. In this case we start by noticing that

$$\frac{d}{dx}(tan^{-1}(2x)) = \frac{2}{1+4x^2}$$

Or

$$tan^{-1}(2x) + C = \int \frac{2}{1+4x^2}\,dx$$

Next, the power series of the function of the integrand can be found using substitution with the geometric power series as follows:

$$\frac{2}{1-(-4x^2)} = \sum_{n=0}^{\infty} 2(-4x^2)^n = \sum_{n=0}^{\infty}(-1)^n 2\cdot 4^n \cdot x^{2n} = \sum_{n=0}^{\infty}(-1)^n 2^{2n+1}x^{2n}$$

Using this result, we have

$$tan^{-1}(2x) + C = \int \left(\sum_{n=0}^{\infty} (-1)^n 2^{2n+1} x^{2n} \right) dx$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{2n+1} x^{2n+1}$$

If we let, $x = 0$ we find that $C = 0$. Therefore, we can finally write the series as

$$tan^{-1}(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{2n+1} x^{2n+1}$$

Example 8: Use what we learned about power series to approximate the following integral.

$$\int_0^{1/2} \frac{1}{1+x^4} dx$$

Solution: We will start by finding the power series representation of the integrand.

$$\frac{1}{1-(-x^4)} = \sum_{n=0}^{\infty} (-x^4)^n = \sum_{n=0}^{\infty} (-1)^n x^{4n}$$

With this, along with the fact that the interval of integration is within the region of convergence, we can integrate as follows:

$$\int_0^{1/2} \left(\sum_{n=0}^{\infty} (-1)^n x^{4n} \right) dx = \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{4n+1} \right) \Bigg|_0^{1/2}$$

We can estimate the integral using the first 3 terms of the series.

$$= \frac{(-1)^0 \left(\frac{1}{2}\right)^{4\cdot 0+1}}{4\cdot 0+1} + \frac{(-1)^1 \left(\frac{1}{2}\right)^{4\cdot 1+1}}{4\cdot 1+1} + \frac{(-1)^2 \left(\frac{1}{2}\right)^{4\cdot 2+1}}{4\cdot 2+1}$$
$$= \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^5}{5} + \frac{\left(\frac{1}{2}\right)^9}{9}$$
$$= \frac{1}{2} - \frac{1}{5\cdot 2^5} + \frac{1}{9\cdot 2^9} \cong 0.49397$$

Final Summary for Infinite Series – Power Series

General Form of a Power Series

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

Where, x is a variable and c is a constant. We refer to this as a power series with center c.

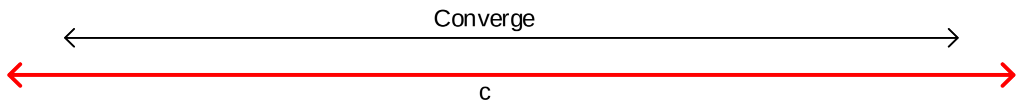
The series can also be expressed in expanded form as shown below.

$$F(x) = a_0 + a_1(x-c)^1 + a_2(x-c)^2 + a_3(x-c)^3 + ..$$

Radius of Convergence for Power Series

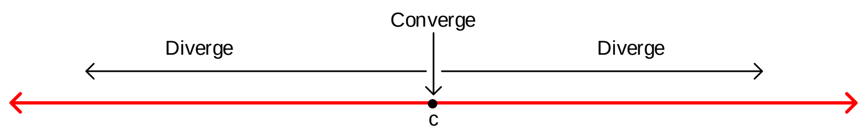
Case 1:

The series converges for all values of x, i.e. $R = \infty$



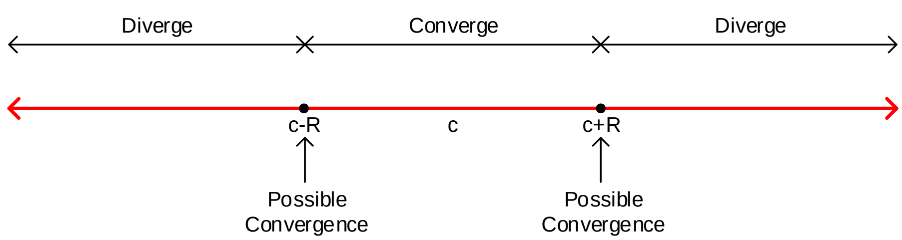
Case 2:

The series diverges for all values of x except $x = c$, i.e. $R = 0$



Case 3:

The series converges for $|x - c| < R$, i.e. $R > 0$



Geometric Series as a Power Series

$$F(x) = \sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}$$

For $|x| < 1$

Note: Power series for other functions can be found with substitution using this series as a abase.

Power Series Differentiation and Integration
Assume that the power series $$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$ Has a radius of convergence, $R > 0$. Then $F(x)$ can be differentiated and integrated for $c - R < x < c + R$. The differentiation and integration is done term by term and can be expressed as follows:
Differentiation
$$\frac{d}{dx}\big(F(x)\big) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n (x-c)^n\right) = \sum_{n=0}^{\infty} a_n \frac{d}{dx}(x-c)^n$$ $$\boldsymbol{\frac{d}{dx}\big(F(x)\big) = \sum_{n=1}^{\infty} a_n n (x-c)^{n-1}}$$
Integration
$$\int F(x)dx = \int \left(\sum_{n=0}^{\infty} a_n (x-c)^n\right) dx = \sum_{n=0}^{\infty} \left(\int a_n (x-c)^n\right) dx$$ $$\boldsymbol{\int F(x)dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-c)^{n+1} + C}$$
The resulting series have the same radius of convergence, R.

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