Infinite Series – Series Introduction

Now that we have been introduced to sequences, we are ready to study infinite series. An infinite series is a summation of the terms of an infinite sequence. Admittedly, it sounds like an impossible task to add up infinitely many numbers, however that is indeed what we intend to do in this section. The applications of infinite series are innumerable. One example emerges from the fact that there are certain quantities, e.g. π, e^2, $sin(1)$, and many others, that do not have exact decimal representations. As it turns out many of these quantities can by represented by infinites sums. For example, $sin(1)$ can be represented as follows:

$$sin(1) = \frac{1}{1} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \ldots$$

To get an initial understanding of how this sum works we first note that numerical value obtained from a calculator.

$$sin(1) = 0.8414709848$$

Since we cannot add infinitely many terms in the sum let's start by computing the so-called partial sums, S_N, defined as the sum of the first N terms.

$$S_1 = \frac{1}{1} = 1.0$$
$$S_2 = \frac{1}{1} - \frac{1}{3!} \approx 0.833$$
$$S_3 = \frac{1}{1} - \frac{1}{3!} + \frac{1}{5!} \approx 0.841667$$
$$S_4 = \frac{1}{1} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} \approx 0.841468$$
$$S_5 = \frac{1}{1} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} \approx 0.\mathbf{841470984}6$$

As you may have surmised, the more terms we add the closer the value gets to the true value of $sin(1)$. In fact, it can be proven that

$$sin(1) = \lim_{N\to\infty} S_N$$

Furthermore, this technique of computing the limit of the partial sums is exactly what we will use in this section to enable the computation of infinite series in general. By treating the partial sums as a sequence, i.e. $S_N = S_1, S_2, S_3, \ldots S_N$, the limit shown above can be treated in the same way we treated limits of sequences in the previous section. Recall, from the previous section that the limit may or may not exist. The same holds with infinite series. If the limit of the partial sums exists then the infinite series will converge, otherwise it diverges. Let's now begin the formal treatment of infinite series below.

Convergence of Infinite Series

As mentioned, the convergence of an infinite series will be determined by taking the limit of the sequence of its partial sums. We state this formally below and then look at an example to provide a better understanding of how this is done in practice.

Convergence of an Infinite Series
An infinite series $\sum_{n=k}^{\infty} a_n$ converges to a value, S, if the sequence of its partial sums, $\{S_N\}$, converges to S. $$\lim_{N\to\infty}\left(\sum_{n=k}^{N} a_n\right) = \lim_{N\to\infty}(S_N) = S$$ Where, $$S_N = \sum_{n=k}^{N} a_n$$ • If no limit exists, we say that the infinite series diverges. • If the terms increase without bound, we say that infinite series diverges to infinity. Note: For brevity of notation we define the following notation for an infinite series. $$\sum_{n=k}^{\infty} a_n = \lim_{N\to\infty}\left(\sum_{n=k}^{N} a_n\right)$$

Determining whether an infinite series converges and finding its convergence value are two separate tasks, the latter being in general much more difficult. Fortunately, our interest will mostly be in determining whether a series converges or not. However, there are certain types of infinite series where the convergence value can be easily computed. We'll start by introducing two of these cases, the telescoping and geometric series, in order to better understand the definition from above.

Telescoping Series

A telescoping series is one in which nearly all of the terms in the series cancel leaving only a few of the initial and final terms. The general form for a telescoping series is as follows:

$$S = \sum_{n=1}^{\infty} b(n) - b(n+A)$$

Below we demonstrate the behavior of a telescoping series using $A = 1$. We start by expanding the partial sum, S_N.

$$\begin{aligned} S_N &= \sum_{n=1}^{N} b(n) - b(n+1) \\ &= [b(1) - \cancel{b(2)}] + [\cancel{b(2)} - \cancel{b(3)}] + [\cancel{b(3)} - \cancel{b(4)}] + \cdots + [\cancel{b(N)} - b(N+1)] \\ &= b(1) - b(N) \end{aligned}$$

According to the definition above, the infinite sum, S, is then

$$\begin{aligned} S &= \lim_{N\to\infty} (S_N) \\ &= \lim_{N\to\infty} \big(b(1) - b(N+1)\big) \\ &= b(1) - \lim_{N\to\infty} \big(b(N+1)\big) \end{aligned}$$

Therefore, if $\lim_{N\to\infty} \big(b(N+1)\big) = 0$, the entire series is simply equal to the first term.

Based on this example, we state the more general formula for a telescoping series below.

Telescoping Series
A telescoping has the general from $$S = \sum_{n=1}^{\infty} b(n) - b(n+A)$$ Expanding this series and cancelling terms we have $$S = [b(1) + b(2) + \ldots + b(A)] - \left[\lim_{N\to\infty} \big(b(N+1) + b(N+2) + \ldots + b(N+A)\big)\right]$$ Assuming $\lim_{N\to\infty} \big(b(N+1) + b(N+2) + \ldots + b(N+A)\big) = 0$ $$S = [b(1) + b(2) + \ldots + b(A)]$$

Let's do some examples.

Example 1: Compute S_3, S_4, and S_5 and then evaluate the infinite series.

$$S = \sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

Solution: We start by computing the partial sums.

$$\begin{aligned} S_3 &= \sum_{n=1}^{3}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) \\ &= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{5}\right) \\ &= \left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{10} = \frac{126}{420} \end{aligned}$$

$$\begin{aligned} S_4 &= \sum_{n=1}^{4}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) \\ &= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \frac{1}{6}\right) \\ &= \left(\frac{1}{2} - \frac{1}{6}\right) = \frac{1}{3} = \frac{140}{420} \end{aligned}$$

$$\begin{aligned} S_5 &= \sum_{n=1}^{5}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) \\ &= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{6}} - \frac{1}{7}\right) \\ &= \left(\frac{1}{2} - \frac{1}{7}\right) = \frac{5}{14} = \frac{150}{420} \end{aligned}$$

To evaluate the infinite sum, we recognize the telescoping nature of the series by letting

$$b(n) = \frac{1}{n+1} \qquad \text{therefore} \qquad b(n+1) = \frac{1}{n+2}$$

From the formula above we then can write

$$\begin{aligned} S &= [b(1)] - \left[\lim_{N\to\infty}\left(b(N+1)\right)\right] \\ &= \left[\frac{1}{2}\right] - \left[\lim_{N\to\infty}\left(\frac{1}{N+2}\right)\right] \\ &= \left[\frac{1}{2}\right] - [0] = \frac{1}{2} \end{aligned}$$

Finally, note the partial sums from above seem to have been converging to this value since the second term was continuing to get smaller while the first term remained at $\frac{1}{2}$.

Example 2: Evaluate the following infinite series.

$$S = \sum_{n=1}^{\infty} \left(\frac{1}{2n+3} - \frac{1}{2n+7} \right)$$

Solution: We start by manipulating the second term in order to identify the series as telescoping.

$$\frac{1}{2n+7} = \frac{1}{2n+4+3} = \frac{1}{2(n+2)+3}$$

Its now clear that we have a telescoping series where

$$b(n) = \frac{1}{2n+3} \qquad \textit{and} \qquad b(n+2) = \frac{1}{2(n+2)+3}$$

We can now evaluate the series using the general formula.

$$S = [b(1) + b(2)] - \left[\lim_{N\to\infty} \left(b(N+1) + b(N+1) \right) \right]$$
$$= \left[\frac{1}{2\cdot 1+3} + \frac{1}{2\cdot 2+3} \right] - \left[\lim_{N\to\infty} \left(\frac{1}{2(N+1)+3} \right) + \lim_{N\to\infty} \left(\frac{1}{2(N+2)+3} \right) \right]$$
$$= \left[\frac{1}{5} + \frac{1}{7} \right] - [0+0] = \frac{12}{35}$$

Example 3: Evaluate the following infinite series.

$$S = \sum_{n=1}^{\infty} \left(\frac{1}{4n^2 - 1} \right)$$

Solution: In this case we can use partial fraction expansion on the sequence to determine if the series is telescoping.

$$\frac{1}{4n^2-1} = \frac{1}{(2n-1)(2n+1)} = \frac{A}{(2n-1)} + \frac{B}{(2n+1)}$$

Using the partial fraction expansion technique explained in an earlier lesson we find $A = 1/2$ and $B = -1/2$. The series can be rewritten as

$$S = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right)$$

Which we can identify as a telescoping series with

$$b(n) = \frac{1}{2n-1} \qquad \textit{and} \qquad b(n+2) = \frac{1}{2n+1}$$

Finally, we can evaluate the series as we did in the previous example.

$$S = [b(1) + b(2)] - \left[\lim_{N\to\infty}\big(b(N+1) + b(N+1)\big)\right]$$
$$= \left[\frac{1}{2\cdot 1 - 1} + \frac{1}{2\cdot 2 - 1}\right] - \left[\lim_{N\to\infty}\left(\frac{1}{2N}\right) + \lim_{N\to\infty}\left(\frac{1}{2N+1}\right)\right]$$
$$= \left[\frac{1}{1} + \frac{1}{3}\right] - [0 + 0] = \frac{4}{3}$$

Geometric Series

Another interesting series that has many practical applications is a geometric series. A geometric series is one in which each term is r times the previous term.

$$S = \sum_{n=M}^{\infty} Cr^n$$

Where, $C \neq 0$

We'll explore this series by first working with a general expression of the partial sum, S_N.

$$S_N = \sum_{n=M}^{N} Cr^n$$

Which can be evaluated by employing a small "trick" of multiplying both sides by $(1-r)$.

$$(1-r)S_N = C[(1-r)(r^M + r^{M+1} + r^{M+2} + \ldots + r^N)]$$
$$(1-r)S_N = C[(r^M \cancel{+ r^{M+1}} \cancel{+ r^{M+2}} + \ldots \cancel{+ r^N} \cancel{- r^{M+1}} \cancel{- r^{M+2}} - \ldots \cancel{- r^N} - r^{N+1})]$$
$$(1-r)S_N = C[(r^M - r^{N+1})]$$
$$S_N = C\left[\frac{(r^M - r^{N+1})}{(1-r)}\right]$$

This formula can be used to compute the finite sum, which may be easier to memorize as follows:

$$S_N = \left[\frac{(Cr^M - Cr^{N+1})}{(1-r)}\right] = \left[\frac{\big((first\ term) - (last\ term\ + 1)\big)}{(1-r)}\right]$$

Assuming $r \neq 1$

Of course, we are interested in the infinite sum, S, which can be computed as follows:

$$\begin{aligned} S &= \lim_{N\to\infty}(S_N) \\ &= \lim_{N\to\infty}\left(C\left[\frac{(r^M - r^{N+1})}{(1-r)}\right]\right) \\ &= \left(\frac{C}{(1-r)}\lim_{N\to\infty}(r^M)\right) - \left(\frac{C}{(1-r)}\lim_{N\to\infty}(r^{N+1})\right) \\ &= \frac{Cr^M}{(1-r)} - \frac{C\lim_{N\to\infty}(r^{N+1})}{(1-r)} \end{aligned}$$

Therefore, if $\lim_{N\to\infty}(r^{N+1}) = 0$, which is true if $|r| < 1$, the infinite geometric series is

$$S = \frac{Cr^M}{(1-r)}$$

We state these results formally below.

Geometric Series
A geometric series, with $C \neq 0$ has the general form $$S = \sum_{n=M}^{\infty} Cr^n$$ If $\|r\| < 1$ the geometric series converges and $$S = \sum_{n=M}^{\infty} Cr^n = \frac{Cr^M}{(1-r)}$$ Note: if $M = 0$ we can write $$S = \sum_{n=0}^{\infty} Cr^n = \frac{C}{(1-r)}$$ If $\|r\| \geq 1$ the geometric series diverges.

Let's do some examples using the geometric series.

Example 4: Use the formula for geometric series to compute the sum or state that the series diverges.

a. $\displaystyle\sum_{n=3}^{\infty} 12(5^{-n})$

b. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{4}{11}\right)^{-n}$

c. $\displaystyle\sum_{n=-4}^{\infty} \left(-\frac{4}{9}\right)^{n}$

d. $\displaystyle\sum_{n=0}^{\infty} \frac{8+2^n}{5^n}$

e. $\displaystyle\sum_{n=0}^{\infty} \frac{3(-2)^n-5^n}{8^n}$

f. $\displaystyle\sum_{n=2}^{\infty} e^{3-2n}$

Solution:

a.

$$\begin{aligned}
\sum_{n=3}^{\infty} 12(5^{-n}) &= \sum_{n=3}^{\infty} 12\left(\frac{1}{5}\right)^n \\
&= 12\left(\frac{\left(\frac{1}{5}\right)^3}{1-\frac{1}{5}}\right) \\
&= 12\left(\frac{\frac{1}{\cancel{125}5}}{\frac{4}{\cancel{5}1}}\right) = \left(\frac{12}{20}\right) = \frac{3}{5}
\end{aligned}$$

b.

$$\sum_{n=0}^{\infty} \left(\frac{4}{11}\right)^{-n} = \sum_{n=0}^{\infty} \left(\frac{11}{4}\right)^{n}$$

Which diverges since $r = \frac{11}{4} \geq 1$

c.

$$\begin{aligned}
\sum_{n=-4}^{\infty} \left(-\frac{4}{9}\right)^{n} &= \left(\frac{\left(-\frac{4}{9}\right)^{-4}}{1-\left(-\frac{4}{9}\right)}\right) \\
&= \frac{9^4}{4^4}\cdot\frac{9}{13} \\
&= \frac{59049}{3328}
\end{aligned}$$

d.

$$\sum_{n=0}^{\infty}\frac{8+2^n}{5^n}=\sum_{n=0}^{\infty}\frac{8}{5^n}+\frac{2^n}{5^n}$$

$$=8\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n+\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n$$

$$=\left(8\frac{1}{1-\left(\frac{1}{5}\right)}\right)+\left(\frac{1}{1-\left(\frac{2}{5}\right)}\right)=\frac{35}{3}$$

e.

$$\sum_{n=0}^{\infty}\frac{3(-2)^n-5^n}{8^n}=3\sum_{n=0}^{\infty}\left(-\frac{2}{8}\right)^n+\sum_{n=0}^{\infty}\left(\frac{5}{8}\right)^n$$

$$=\frac{3}{1-\left(-\frac{1}{4}\right)}+\frac{1}{1-\left(\frac{5}{8}\right)}$$

$$=\frac{3}{\frac{5}{4}}+\frac{1}{\frac{3}{8}}=\frac{76}{15}$$

f.

$$\sum_{n=2}^{\infty}e^{3-2n}=\sum_{n=2}^{\infty}e^3e^{-2n}$$

$$=e^3\sum_{n=2}^{\infty}\left(\frac{1}{e^2}\right)^n$$

$$=\frac{e^3\left(\frac{1}{e^2}\right)^2}{1-\left(\frac{1}{e^2}\right)}=\frac{\frac{1}{e}}{\frac{e^2-1}{e^2}}=\frac{e}{e^2-1}$$

Example 5: Geometric series can be used to convert a repeating decimal to fraction by grouping the repeating digits and creating an infinite sum to represent that decimal. Convert the following repeating decimals to fractions.

a. $0.33333\ldots$ *b.* $0.217217217\ldots$

d. $23.141414\ldots$ *e.* $0.0833333\ldots$

Solution:

a.

$$
\begin{aligned}
0.33333\ldots &= \frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\cdots \\
&= 3\left(\frac{1}{10^1}+\frac{1}{10^2}+\frac{1}{10^3}+\cdots\right) \\
&= 3\sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^n \\
&= 3\left(\frac{\left(\frac{1}{10}\right)^1}{1-\frac{1}{10}}\right) = \frac{3}{9}
\end{aligned}
$$

b.

$$
\begin{aligned}
0.217217217\ldots &= \frac{217}{10^3}+\frac{217}{10^6}+\frac{217}{10^9}+\cdots \\
&= 217\sum_{n=1}^{\infty}\left(\frac{1}{10^3}\right)^n \\
&= 217\left(\frac{\left(\frac{1}{10^3}\right)^1}{1-\frac{1}{10^3}}\right) \\
&= 217\left(\frac{\frac{1}{10^3}}{\frac{999}{10^3}}\right) = \frac{217}{999}
\end{aligned}
$$

c.

$$
\begin{aligned}
23.141414.. &= 23+\frac{14}{10^2}+\frac{14}{10^4}+\frac{14}{10^6}+\cdots \\
&= 23+14\sum_{n=1}^{\infty}\left(\frac{1}{10^2}\right)^n \\
&= 23+14\left(\frac{\left(\frac{1}{10^2}\right)^1}{1-\frac{1}{10^2}}\right) \\
&= 23+14\left(\frac{\frac{1}{10^2}}{\frac{99}{10^2}}\right) = 23+\frac{14}{99} = \frac{2291}{99}
\end{aligned}
$$

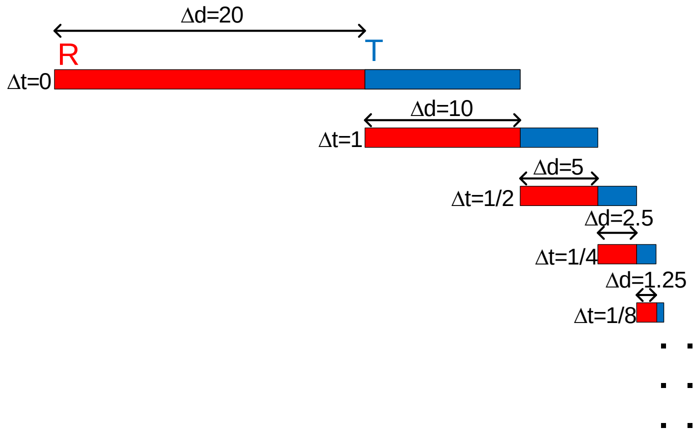
d.

$$
\begin{aligned}
0.0833333\ldots &= \left(\frac{1}{100}\right)8.33333\ldots \\
&= \left(\frac{1}{100}\right)\left(8+\ 3\left(\frac{1}{10^1}+\frac{1}{10^2}+\frac{1}{10^3}+\cdots\right)\right) \\
&= \left(\frac{1}{100}\right)\left(8+3\sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^n\right) \\
&= \left(\frac{1}{100}\right)\left(8+3\frac{\left(\frac{1}{10}\right)^1}{1-\frac{1}{10}}\right) \\
&= \left(\frac{1}{100}\right)\left(8+3\left(\frac{\frac{1}{10}}{\frac{9}{10}}\right)\right)=\frac{75}{900}
\end{aligned}
$$

Example 6: The Greek philosopher Zeno proposed a paradox commonly referred to as the "Achilles and the Tortoise" paradox. We can demonstrate the paradox using the more modern version of the "Rabbit and Turtle" as follows:

A rabbit and a turtle are to race. The rabbit allows the turtle to start the race 20 feet ahead. The rabbit runs at 20 ft/sec, while the turtle run at 10 ft/sec. According to Zeno the rabbit will never catch the turtle. He argues his point by using time intervals that are continually halved, which results in the separation distance also continually being halved, but never reaching zero. The argument can be illustrated using the table and the figure shown below.

Time interval, Δt	***Rabbit Distance, d_R***	***Turtle Distance, d_T***	***Separation Distance, Δd***
0	0	20	20
1	20	30	10
1/2	30	35	5
1/4	35	37.5	2.5
1/8	37.5	38.75	1.25
.	.	.	.
.	.	.	.
.	.	.	.



As you can see as the time intervals gets smaller so does the separation distance. However, what Zeno argued is that since we can continue to half the time interval forever it seems as though the separation distance can also continue to be halved and never reach zero. Therefore, the rabbit will never catch the turtle. Use your newfound knowledge of infinite series to resolve Zeno's paradox.

Solution: We start by treating the time intervals, Δt, and the separation distance, Δd, as infinite sequences as follows:

$$\Delta t = 0, 1, 1/2, 1/4, 1/8, \ldots \qquad\qquad \Delta d = 20, 10, 5, 2.5, 1.25, \ldots$$

Next, we attempt to sum these infinite sequences to see if they converge. If they do converge, the convergence values will represent the total time and distance it takes for the rabbit to catch the turtle.

$$T = 0 + 1 + 1/2 + 1/4 + 1/8 + \ldots \qquad D = 20 + 10 + 5 + 2.5 + 1.25, \ldots$$

$$T = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \qquad D = \sum_{n=0}^{\infty} 20\left(\frac{1}{2}\right)^n$$

Where, T is the total time in the sum. Where, D is the total distance in the sum.

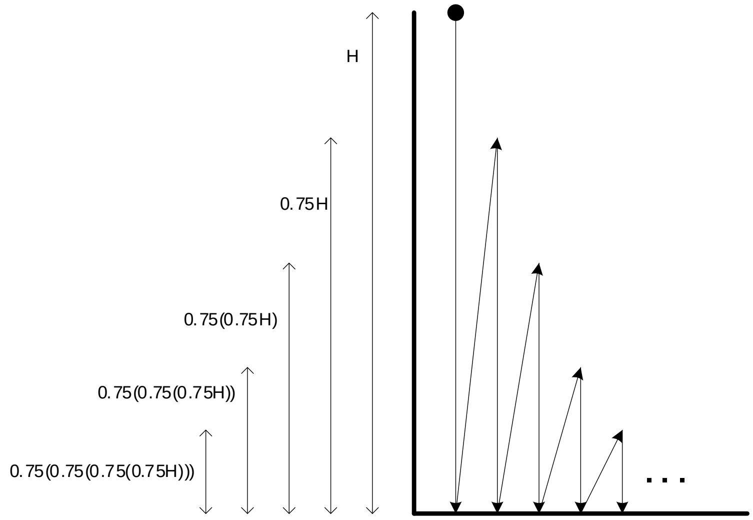
The infinite sums are both convergent geometric series for which we can evaluate as follows:

$$T = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2 \qquad D = \sum_{n=0}^{\infty} 20\left(\frac{1}{2}\right)^n = \frac{20}{1-\frac{1}{2}} = 40$$

Therefore, the rabbit will catch the turtle in 2 seconds after traversing 40 ft, and the paradox is resolved!

Example 7: A ball is dropped from a height of $H\ ft$ and begins to bounce vertically. Each time it strikes the ground, it returns to three-quarters of its previous height. What is the total vertical distance traveled by the ball if it bounces infinitely many times?

Solution: The figure below illustrates the scenario described.



The ball initially drops a distance H. For each consecutive bounce the ball travels through a distance of $\frac{3}{4}$ of the previous distance twice, once on the way up and once on the way back down. The total distance can be written as an infinite sum as follows:

$$
\begin{aligned}
D &= H_0 + 2(H_1) + 2(H_2) + 2(H_3) + \cdots \\
&= H + 2\left(\frac{3}{4}H\right) + 2\left(\frac{3}{4}\cdot\frac{3}{4}H\right) + 2\left(\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}H\right) + \cdots \\
&= H + 2H\left(\frac{3}{4}\right)^1 + 2H\left(\frac{3}{4}\right)^2 + 2H\left(\frac{3}{4}\right)^3 + \cdots
\end{aligned}
$$

Therefore, we again have a geometric series, which can easily be evaluated.

$$
\begin{aligned}
&= H + 2H\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n \\
&= H + 2H\left(\frac{\frac{3}{4}}{1-\frac{3}{4}}\right) \\
&= H + 2H(3) \\
D &= 7H
\end{aligned}
$$

Final Summary for Infinite Series – Series Introduction

Infinite Series
An infinite series is a summation of the terms of an infinite sequence, e.g. $\{a_n\}$. $$\sum_{n=k}^{\infty} a_n = \lim_{N\to\infty}\left(\sum_{n=k}^{N} a_n\right)$$
Convergence of an Infinite Series
An infinite series $\sum_{n=k}^{\infty} a_n$ converges to a value, S, if the sequence of its partial sums, $\{S_N\}$, converges to S. $$\lim_{N\to\infty}\left(\sum_{n=k}^{N} a_n\right) = \lim_{N\to\infty}(S_N) = S$$ Where, $$S_N = \sum_{n=k}^{N} a_n$$ • If no limit exists, we say that the infinite series diverges. • If the terms increase without bound, we say that infinite series diverges to infinity.

Telescoping Series
A telescoping has the general from $$S=\sum_{n=1}^{\infty} b(n)-b(n+A)$$ Expanding this series and cancelling terms we have $$S=[b(1)+b(2)+\ldots+b(A)]-\left[\lim_{N\to\infty}\left(b(N+1)+b(N+2)+\ldots+b(N+A)\right)\right]$$ Assuming $\lim_{N\to\infty}\left(b(N+1)+b(N+2)+\ldots+b(N+A)\right)=0$ $$S=[b(1)+b(2)+\ldots+b(A)]$$
Geometric Series
A geometric series, with $C\neq 0$ has the general form $$S=\sum_{n=M}^{\infty} Cr^n$$ If $\lvert r\rvert<1$ the geometric series converges and $$S=\sum_{n=M}^{\infty} Cr^n=\frac{Cr^M}{(1-r)}$$ Note: if $M=0$ we can write $$S=\sum_{n=0}^{\infty} Cr^n=\frac{C}{(1-r)}$$ If $\lvert r\rvert\geq 1$ the geometric series diverges.

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