Infinite Series – Convergence of Positive Series

As you can imagine finding the sum of an infinite series is no easy task. As a matter of fact, it is only possible in some special cases, two of which we have learned in the previous section (i.e. telescoping and geometric series). Determining whether an infinite series converges or diverges, however, is possible in a much wider range of cases. Furthermore, in many applications simply knowing if a series converges or diverges is sufficient. In the next few sections we will develop techniques that can be used to determine whether an infinite series converges or diverges. In this section we focus on positive series, i.e. where $a_n > 0$ for all n. However, we begin with the so-called *'Divergence Test'*, which is not restricted to positive series.

Divergence Test

The divergence test is typically the first test that is used to analyze the behavior of a series. The test is limited however since it can only be used to prove divergence. In other words, if the divergence test indicates divergence, we can indeed conclude that the series diverges. On the pother hand, if the test fails to indicate divergence, we may NOT conclude that the series converges, but instead must move to more sophisticated tests to analyze the series. The idea of the test is that if the individual terms of the sequence do not shrink to zero as n becomes arbitrarily large then the series will diverge. The test is formally stated below.

nth Term Divergence Test
If $\lim_{n\to\infty}(a_n) \neq 0$ then the series $\sum_{n=1}^{\infty} a_n$ diverges If $\lim_{n\to\infty}(a_n) = 0$ then the test is inconclusive.

Example 1: Us the nth term divergence test to determine of the following series diverge.

a. $$\sum_{n=1}^{\infty} \frac{n}{4n+1}$$

b. $$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$$

c. $$\sum_{n=1}^{\infty} \left(\sqrt{4n^2+1} - n\right)$$

d. $$\sum_{n=1}^{\infty} cos\left(\frac{1}{n}\right)$$

e. $$\sum_{n=1}^{\infty} \frac{1}{n}$$

f. $$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\textbf{a.}\quad \sum_{n=1}^{\infty} \frac{n}{4n+1}$$

$$\lim_{n\to\infty}\left(\frac{n}{4n+1}\right) = \lim_{n\to\infty}\left(\frac{n}{4n+1}\right)\left(\frac{1/n}{1/n}\right)$$
$$= \lim_{n\to\infty}\left(\frac{1}{4+1/n}\right) = \frac{1}{4} \neq 0$$

Therefore, the series diverges.

$$\textbf{b.}\quad \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$$

$$\lim_{n\to\infty}\left(\frac{n}{\sqrt{n^2+1}}\right) = \lim_{n\to\infty}\left(\sqrt{\frac{n^2}{n^2+1}}\right)\left(\frac{\sqrt{1/n^2}}{\sqrt{1/n^2}}\right)$$
$$= \lim_{n\to\infty}\left(\sqrt{\frac{1}{1+1/n^2}}\right) = 1 \neq 0$$

Therefore, the series diverges.

$$\textbf{c.}\quad \sum_{n=1}^{\infty}\left(\sqrt{4n^2+1}-n\right)$$

$$\lim_{n\to\infty}\left(\sqrt{4n^2+1}-n\right) = \lim_{n\to\infty}\left(\sqrt{4n^2+1}-n\right)\left(\frac{\sqrt{4n^2+1}+n}{\sqrt{4n^2+1}+n}\right)$$
$$= \lim_{n\to\infty}\left(\frac{3n^2+1}{\sqrt{4n^2+1}+n}\right)\left(\frac{1/n^2}{\sqrt{1/n^4}}\right)$$
$$= \lim_{n\to\infty}\left(\frac{3+1/n^2}{\sqrt{4/n^2+1/n^4}+1/n}\right) = \frac{3}{0} = \infty$$

Therefore, the series diverges.

$$\textbf{d.}\quad \sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty}\left(\cos\left(\frac{1}{n}\right)\right) = \cos\left(\lim_{n\to\infty}\left(\frac{1}{n}\right)\right) = \cos(0) = 1 \neq 0$$

Therefore, the series diverges.

$$\textbf{e.}\quad \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$$

In this case the divergence test is inconclusive, and therefore we are not sure whether this series diverges or converges. We will revisit this series after learning the integral test.

$$\textbf{f.}\quad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\lim_{n\to\infty} \left(\frac{1}{n^2}\right) = 0$$

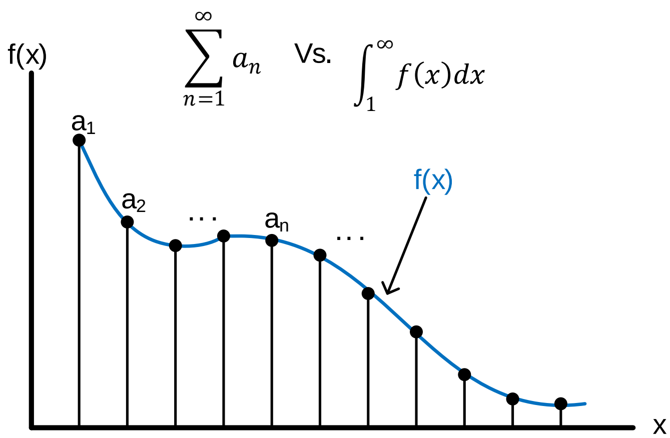
The divergence test is similarly inconclusive in this case. Interestingly, we will soon learn that these two examples, *e.* and *f.*, result in different convergence/divergence behavior even though they look similar and both evaluate to zero using the divergence test.

As we have highlighted with the last two examples, we CANNOT prove convergence using the divergence test. Therefore, when the divergence test is inconclusive, we need to further analyze the series to determine its behavior. Fortunately, there are other tests that can help us with this task. We introduce some of these tests below. As mentioned in the introduction, the tests introduced in this section apply to positive series only. The first test we introduce is based on integration.

The Integral Test

Although the integral test can indeed be formally proven, we will instead use our understanding of the integral to provide an intuitive explanation.

We start by taking the sequence, a_n, where the index n is a discrete variable, and treat it as a continuous function, $f(x)$, where we assume x is a continuous variable. Using the figure below we can see how the infinite series is similar to the integral: $\int_1^\infty f(x)dx$.



figure

With this intuitive explanation, we formally state the integral test below.

Integral Test

Let $a_n = f(n)$, where f is a positive, decreasing, and continuous function of x for $x \geq 1$.

i. If $\int_1^\infty f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

ii. If $\int_1^\infty f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Let's do some examples using the integral test, staring with the series e. and f. from example 1. Recall the divergence test was inconclusive for theses series, and therefore we will attempt to use the integral test to determine the behavior.

Example 2: Use the integral test to determine if the following series converge or diverge.

a. $$\sum_{n=1}^{\infty} \frac{1}{n}$$

b. $$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

c. $$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

a. $$\sum_{n=1}^{\infty} \frac{1}{n}$$

We start by letting $a_n = f(n) = \frac{1}{n}$. Then, since f is positive, decreasing, and continuous function of x for $x \geq 1$, we can use the integral test as shown below.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x} dx &= \lim_{R\to\infty} \left(\int_1^R \frac{1}{x} dx \right) \\
&= \lim_{R\to\infty} (ln(R) - ln(1)) \\
&= \lim_{R\to\infty} (ln(R)) = \infty
\end{aligned}$$

Therefore, the series diverges.

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b. $$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

This series can be similarly evaluated as shown below.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2} dx &= \lim_{R\to\infty} \left(\int_1^R \frac{1}{x^2} dx \right) \\
&= \lim_{R\to\infty} \left(-\frac{1}{R} + 1 \right) \\
&= -0 + 1 = 1
\end{aligned}$$

In this case the integral converges, and therefore the series also converges.

$$\textbf{c.}\quad \sum_{n=1}^{\infty} \frac{1}{n^p}$$

This series is referred to as the p-series. Assuming $p \neq 1$, in which case we already know from the example above that the series diverges, we can use the integral test as shown.

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{R\to\infty}\left(\int_1^R \frac{1}{x^p}\,dx\right)$$

$$= \lim_{R\to\infty}\left(\frac{1}{1-p}\left(R^{(1-p)} - 1^{(1-p)}\right)\right)$$

$$= \frac{1}{1-p}\left(\lim_{R\to\infty}\left(\frac{1}{R^{(p-1)}}\right) - 1\right)$$

The limit is evaluated for three different cases below.

$p \le 0$	*The quantity $(p-1)$ is negative and can be written as $-(\lvert p\rvert+1)$. In this case we have:* $\lim_{R\to\infty}\left(\frac{1}{R^{-(\lvert p\rvert+1)}}\right) = \lim_{R\to\infty}\left(R^{(\lvert p\rvert+1)}\right) = \infty$	$\int_1^{\infty} \frac{1}{x^p}\,dx$ *diverges*
$0 < p < 1$	*The quantity $(p-1)$ is negative and can be written as $-(1-p)$. In this case we have:* $\lim_{R\to\infty}\left(\frac{1}{R^{-(1-p)}}\right) = \lim_{R\to\infty}\left(R^{(1-p)}\right) = \infty$	$\int_1^{\infty} \frac{1}{x^p}\,dx$ *diverges*
$p > 1$	*The quantity $(p-1)$ is will remain positive. In this case we have:* $\lim_{R\to\infty}\left(\frac{1}{R^{(p-1)}}\right) = 0$	$\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{1-p}(-1) = \frac{1}{p-1}$ *converges*

Therefore the p-series converges only when $p > 1$. This is expressed as a theorem below.

Convergence of p-series
The infinite series $$\sum_{n=1}^{\infty} \frac{1}{n^p}$$ Converges for $p > 1$, and diverges otherwise.

Integration, as we know, is not always trivial. Therefore, using to the integral tests for all series will become difficult, and even impossible in some cases. Fortunately, there are additional tests that we may use. We introduce two of these tests below. With the first test we *directly compare* the given series with another series that we know either converges or diverges. With the second tests we *indirectly compare* two series. The tests are given without formal proof.

Direct Comparison Test

Direct Comparison Test
Assume that there exists $M > 0$ such that $0 \leq a_n \leq b_n$ for all $n \geq M$. *i.* If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges. *ii.* If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.

Let's look at some examples to demonstrate how the test is used.

Example 3: Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}3^n}$$

Letting $a_n = \frac{1}{\sqrt{n}3^n}$, we need to create another series, b_n, preferably related *and* known to converge. In this case we let $b_n = \frac{1}{3^n}$. Next, we notice that for all $n \geq 1$:

$$\frac{1}{\sqrt{n}3^n} \leq \frac{1}{3^n}$$

The table below shows the first 4 terms of the sequence for illustration.

$\boldsymbol{n}$	$\boldsymbol{a_n = \frac{1}{\sqrt{n}3^n}}$	$\boldsymbol{b_n = \frac{1}{3^n}}$
1	$\frac{1}{\sqrt{1} \cdot 3^1} = 0.3333$	$\frac{1}{3^1} = 0.3333$
2	$\frac{1}{\sqrt{2} \cdot 3^2} = 0.0786$	$\frac{1}{3^2} = 0.1111$
3	$\frac{1}{\sqrt{3} \cdot 3^3} = 0.0214$	$\frac{1}{3^3} = 0.0370$
4	$\frac{1}{\sqrt{4} \cdot 3^4} = 0.0062$	$\frac{1}{3^4} = 0.0123$

Finally, since the larger sequence, b_n, results in a geometric series with $r = \frac{1}{3}$, and therefore converges, the series with the smaller sequence, a_n, must also converge.

Example 4: Does the following series converge?

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^4 - 3n^2}{n^6 + n^2}$$

When the sequence is a rational function a technique that sometimes works is to let b_n equal the numerator along with the highest term denominator. In this case we have:

$$b_n = \frac{n^2 - 3n^2}{n^6}$$

Comparing these two sequences we see that the denominator of a_n is always greater than the denominator of b_n, and so for all $n \geq 1$.

$$\frac{n^2 - 3n^2}{n^6} > \frac{n^4 - 3n^2}{n^6 + n^2}$$

Therefore, if $\sum_{n=1}^{\infty} b_n$ converges so does the original series.

$$\begin{aligned}\sum_{n=1}^{\infty} b_n &= \sum_{n=1}^{\infty} \frac{n^4}{n^6} - \frac{-3n^2}{n^6} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} - 3\sum_{n=1}^{\infty} \frac{1}{n^4}\end{aligned}$$

Which, as we now know, are both convergent p-series. We can now state that since the larger series, $\sum_{n=1}^{\infty} \frac{n^4-3n^2}{n^6}$ converges, the original, “smaller”, series, $\sum_{n=1}^{\infty} \frac{n^4-3n^2}{n^6+n^2}$ also converges.

The next example will lead us to our final test for this section.

Example 5: Does the following series converge?

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{n^2}{n^4 - n - 1}$$

Similar to example 4, we create a new series as

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{n^2}{n^4} = \sum_{n=2}^{\infty} \frac{1}{n^2}$$

Which is again a convergent p-series. However, in this case, the denominator of the original sequence, a_n, is less that the denominator of the new sequence, b_n. Therefore, a_n is greater than the b_n for all $n \geq 2$.

$$a_n > b_n$$
$$\frac{n^2}{n^4 + n - 1} > \frac{n^2}{n^4}$$

Therefore, in this case the fact that $\sum_{n=2}^{\infty} b_n$ converges does not tell us anything about $\sum_{n=2}^{\infty} a_n$ since the sequence a_n is larger. In these situations, we can sometimes use a variation of the direct comparison test called the limit comparison test, formally stated below.

Limit Comparison Test
Let a_n and b_n be positive sequences and let the following limit exist. $$L = \lim_{n\to\infty} \left(\frac{a_n}{b_n}\right)$$ *i.* If $L > 0$ then $a_n \cong Lb_n$ for large n, and both series either converge or diverge. *ii.* If $L = 0$ then $b_n \gg a_n$ for large n, and if $\sum_{n=1}^{\infty} b_n$ converge, so does $\sum_{n=1}^{\infty} a_n$ *iii.* If $L = \infty$ then $a_n \gg b_n$ for large n, and if $\sum_{n=1}^{\infty} a_n$ converge, so does $\sum_{n=1}^{\infty} b_n$.

Let's demonstrate how this test can be used with the series from example 5.

Example 6: Does the following series converge?

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{n^2}{n^4 - n - 1}$$

Solution:

For this test the technique for choosing b_n is similar to the one used for the direct comparison test, i.e. choose the highest term in the numerator and denominator.

$$b_n = \frac{n^2}{n^4} = \frac{1}{n^2}$$

Next, we form the ratio, $\frac{a_n}{b_n}$, and compute the limit.

$$L = \lim_{n\to\infty} \left(\frac{\frac{n^2}{n^4 - n - 1}}{\frac{1}{n^2}} \right)$$

$$= \lim_{n\to\infty} \left(\frac{n^4}{n^4 - n - 1} \right)$$

Recall from calculus 1 that we can analyze the limit of a rational function by considering only the highest term in the numerator and denominator.

$$L = \lim_{n\to\infty} \left(\frac{n^4}{n^4} \right)$$

$$= \lim_{n\to\infty} (1) = 1$$

From the limit comparison test since $L > 0$, both series behave the same. And since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ is a convergent p-series, the original series, $\sum_{n=2}^{\infty} \frac{n^2}{n^4 - n - 1}$,also converges.

Finally, we end this section with additional examples to practice using the tests we have learned.

Example 7: Use the integral test to determine whether the following series converge.

a. $$\sum_{n=25}^{\infty} \frac{n^2}{(n^3+9)^{5/2}}$$

b. $$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

c. $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

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a. $$\sum_{n=25}^{\infty} \frac{n^2}{(n^3+9)^{5/2}}$$

$$\int_{25}^{\infty} \frac{x^2}{(x^3+9)^{5/2}}\,dx = \lim_{R\to\infty}\left(\int_{25}^{R} \frac{x^2}{(x^3+9)^{5/2}}\,dx\right)$$

We integrate by substitution and let $u = x^3 + 9$. Therefore $du = 3x^2dx$

$$\lim_{R\to\infty}\left(\frac{1}{3}\int_{15634}^{R} \frac{1}{(u)^{5/2}}\,du\right) = \lim_{R\to\infty}\left(\frac{1}{3}\int_{15634}^{R} u^{-5/2}du\right)$$

$$= \lim_{R\to\infty}\left(-\frac{2}{9}u^{-2/3}\Big|_{15634}^{R}\right)$$

$$= -\frac{2}{9}\lim_{R\to\infty}\left(R^{-2/3} - 15634^{-2/3}\right)$$

$$= \left(\frac{2\cdot 15634^{-2/3}}{9}\right)$$

Therefore, since the integral converges so does the original series.

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b. $$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

$$\int_{2}^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{R\to\infty}\left(\int_{2}^{R} \frac{1}{x(\ln x)^2}\,dx\right)$$

Let $u = \ln x$, therefore $du = \frac{1}{x}dx$

$$\lim_{R\to\infty}\left(\int_{\ln 2}^{R}\frac{1}{u^2}du\right) = \lim_{R\to\infty}\left(\int_{\ln 2}^{R}u^{-2}du\right)$$
$$= \lim_{R\to\infty}\left(-\frac{1}{u}\bigg|_{\ln 2}^{R}\right)$$
$$= \lim_{R\to\infty}\left(\frac{1}{\ln 2}-\frac{1}{R}\right) = \frac{1}{\ln 2}$$

Therefore, the series converges.

$$\textit{c.}\quad \sum_{\boldsymbol{n}=\mathbf{1}}^{\infty}\frac{\mathbf{1}}{\boldsymbol{n}(\boldsymbol{n}+\mathbf{1})}$$

$$\int_{1}^{\infty}\frac{1}{x(x+1)}dx = \lim_{R\to\infty}\left(\int_{1}^{R}\frac{1}{x(x+1)}dx\right)$$

Using partial fraction expansion, we can rewrite and evaluate the integral as shown.

$$\lim_{R\to\infty}\left(\int_{1}^{R}\frac{1}{x}-\frac{1}{(x+1)}dx\right) = \lim_{R\to\infty}(\ln(x)-\ln(x+1)|_1^R)$$
$$= \lim_{R\to\infty}\big((\ln(R)-\ln(R+1))-\ln(1)-\ln(2)\big)$$
$$= \lim_{R\to\infty}\left(\ln\left(\frac{R}{R+1}\right)\right)-\ln(2)$$
$$= \ln\left(\lim_{R\to\infty}\left(\frac{R}{R+1}\right)\right)-\ln(2)$$
$$= \ln(1)-\ln(2) = -\ln(2)$$

Therefore, the series converges.

Example 8: Use the direct comparison test to determine whether the following series converge.

a.
$$\sum_{n=1}^{\infty}\frac{1}{n^{1/3}+2^n}$$

b.
$$\sum_{n=1}^{\infty}\frac{\sin^2(n)}{n^2}$$

c.
$$\sum_{n=1}^{\infty}\frac{\ln n}{n^3}$$

d.
$$\sum_{n=1}^{\infty}\frac{(\ln n)^{100}}{n^{1.1}}$$

$$\boldsymbol{a.}\quad \sum_{n=1}^{\infty} \frac{\mathbf{1}}{\boldsymbol{n^{1/3} + 2^n}}$$

We let a_n be the original sequence and choose b_n as shown.

$$a_n = \frac{1}{n^{1/3} + 2^n} \qquad\qquad b_n = \frac{1}{2^n}$$

Then, since for all $n \geq 1$

$$a_n < b_n$$

$$\frac{1}{n^{1/3} + 2^n} < \frac{1}{2^n}$$

And since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

is a convergent geometric series, then the original series also converges.

$$\boldsymbol{b.}\quad \sum_{n=1}^{\infty} \frac{\boldsymbol{sin^2(n)}}{\boldsymbol{n^2}}$$

We let a_n be the original sequence and choose b_n as shown.

$$a_n = \frac{sin^2(n)}{n^2} \qquad\qquad b_n = \frac{1}{n^2}$$

Then, since for all n, $0 \leq sin^2(n) \leq 1$

$$a_n < b_n$$

$$\frac{sin^2(n)}{n^2} < \frac{1}{n^2}$$

and since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a convergent p-series, then the original series also converges.

$$\textbf{c.} \quad \sum_{n=1}^{\infty} \frac{\ln n}{n^3}$$

In this case we first rewrite the series as

$$\sum_{n=1}^{\infty} \frac{\ln n}{n} \cdot \frac{1}{n^2}$$

We then again let a_n be the original sequence and choose b_n as shown.

$$a_n = \frac{\ln n}{n} \cdot \frac{1}{n^2} \qquad\qquad b_n = \frac{1}{n^2}$$

Next, we note that if $\frac{\ln n}{n} < 1$ for all $n > M$, then

$$a_n < b_n$$
$$\frac{\ln n}{n} \cdot \frac{1}{n^2} < \frac{1}{n^2}$$

Now, $\frac{\ln n}{n}$ is less than 1 if

$$\ln n < n$$

Recall from our introduction to series we found that as n gets large

$$\ln(n) \ll n^a \ll b^n \ll n! \ll n^n$$

For $a > 0$ and $b > 1$.

Therefore, it is true that $\ln n < n$, and hence, $\frac{\ln n}{n} < 1$.

Finally, since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a convergent p-series, then the original series also converges.

$$\textbf{\textit{d.}} \quad \sum_{n=1}^{\infty} \frac{(\ln n)^{100}}{n^{1.1}}$$

In this case we use a similar “trick” as the previous example by rewriting the series as

$$\sum_{n=1}^{\infty} \frac{(ln\, n)^{100}}{n^{0.01}} \cdot \frac{1}{n^{1.09}}$$

We then again let a_n be the original sequence and choose b_n as shown.

$$a_n = \frac{(ln\, n)^{100}}{n^{0.01}} \cdot \frac{1}{n^{1.09}} \qquad\qquad b_n = \frac{1}{n^{1.09}}$$

To determine if $\frac{(ln\, n)^{100}}{n^{0.01}} < 1$, we proceed as follows:

$$((ln\, n)^{100} < n^{0.01})^{1/100}$$
$$ln\, n < n^{0.001}$$

Which is true for the reasons stated in the previous example.

And finally, since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{1.09}}$$

is a convergent p-series, then the original series also converges.

Example 9: Use the limit comparison test to determine whether the following series converge.

a.

$$\sum_{n=2}^{\infty} \frac{n^2}{n^4+1}$$

b.

$$\sum_{n=3}^{\infty} \frac{3n+5}{n(n-1)(n-2)}$$

c.

$$\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^3+1}}$$

d.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+ln(n)}$$

$$\textbf{a.}\quad \sum_{n=2}^{\infty} \frac{n^2}{n^4+1}$$

We choose a_n and b_n and then evaluate the limit as shown.

$$a_n = \frac{n^2}{n^4+1} \qquad\qquad b_n = \frac{n^2}{n^4} = \frac{1}{n^2}$$

$$L = \lim_{n\to\infty}\left(\frac{\frac{n^2}{n^4+1}}{\frac{1}{n^2}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n^4}{n^4+1}\right)$$

$$L = 1$$

Since $L > 0$, the two series have the same behavior. And since

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

is a convergent p-series, then the original series also converges.

$$\textbf{b.}\quad \sum_{n=3}^{\infty} \frac{3n+5}{n(n-1)(n-2)}$$

In this case we foil the denominator of the original sequence first.

$$a_n = \frac{3n+5}{n(n-1)(n-2)} = \frac{3n+5}{n^3-3n^2+2n}$$

Next, we choose a_n and b_n as shown.

$$a_n = \frac{3n+5}{n^3-3n^2+2n} \qquad\qquad b_n = \frac{3n}{n^3} = \frac{3}{n^2}$$

$$L = \lim_{n\to\infty}\left(\frac{3n+5}{n^3-3n^2+2n}\cdot\frac{n^2}{3}\right)$$

$$= \lim_{n\to\infty}\left(\frac{3n^3+5n^2}{3n^3-9n^2+6n}\right)$$

$$L = 1$$

Since $L > 0$, the two series have the same behavior. And since

$$\sum_{n=3}^{\infty} \frac{3}{n^2}$$

is a convergent p-series, then the original series also converges.

$$\textbf{c. } \sum_{n=2}^{\infty} \frac{\boldsymbol{n}}{\sqrt{\boldsymbol{n}^3 + \mathbf{1}}}$$

In this case we manipulate the original sequence first as follows:

$$a_n = \frac{n}{\sqrt{n^3 + 1}} = \sqrt{\frac{n^2}{n^3 + 1}}$$

Next, we choose a_n and b_n as shown.

$$a_n = \sqrt{\frac{n^2}{n^3 + 1}} \qquad\qquad b_n = \sqrt{\frac{n^2}{n^3}} = \sqrt{\frac{1}{n}}$$

$$L = \lim_{n \to \infty} \left(\sqrt{\frac{n^2}{n^3 + 1}} \cdot \sqrt{\frac{n}{1}} \right)$$

$$= \sqrt{\lim_{n \to \infty} \left(\frac{n^3}{n^3 + 1} \right)}$$

$$= \sqrt{1} = 1$$

Since $L > 0$, the two series have the same behavior. However, in this case since

$$\sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$$

is a *divergent* p-series, then the original series also diverges.

$$\textbf{d.} \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + ln(n)}$$

In this case choose b_n as our original sequence and a_n as shown.

$$a_n = \frac{1}{\sqrt{n}} \qquad\qquad b_n = \frac{1}{\sqrt{n} + ln(n)}$$

$$\begin{aligned} L &= \lim_{n\to\infty} \left(\frac{1}{\sqrt{n}} \cdot \frac{\sqrt{n} + ln(n)}{1} \right) \\ &= \lim_{n\to\infty} \left(1 + \frac{ln(n)}{\sqrt{n}} \right) \\ &= 1 + \lim_{n\to\infty} \left(\frac{ln(n)}{\sqrt{n}} \right) \\ &= 1 + 0 = 1 \end{aligned}$$

Note to evaluate the limit we again used that fact that $ln(n) \ll n^a f$or $a > 0$.

Therefore, the two series have the same behavior, i.e. they both diverge.

Example 10: Use the any method covered so far determine whether the following series converge.

a. $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^2 - 9}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - n}{n^5 + n}$

c. $\displaystyle\sum_{n=1}^{\infty} 4^{1/n}$

d. $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{4^n}$

e. $\displaystyle\sum_{n=4}^{\infty} \frac{ln(n)}{n^2 - 3n}$

f. $\displaystyle\sum_{n=2}^{\infty} \frac{4n^2 + 15n}{3n^4 - 5n^2 - 17}$

$$\textbf{a.}\quad \sum_{n=4}^{\infty} \frac{1}{n^2-9}$$

For this case we use the limit comparison test as shown.

$$a_n = \frac{1}{n^2-9} \qquad\qquad b_n = \frac{1}{n^2}$$

$$L = \lim_{n\to\infty}\left(\frac{n^2}{n^2-9}\right)$$

$$= \lim_{n\to\infty}(1) = 1$$

And since the $\sum_{n=4}^{\infty} b_n$ is a convergent p-series the original series converges.

$$\textbf{b.}\quad \sum_{n=1}^{\infty} \frac{n^2-n}{n^5+n}$$

We can use the limit comparison test as shown.

$$a_n = \frac{n^2-n}{n^5+n} = \frac{n-1}{n^4+1} \qquad\qquad b_n = \frac{1}{n^3}$$

$$L = \lim_{n\to\infty}\left(\frac{n-1}{n^4+1}\cdot\frac{n^3}{1}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n^4-n^3}{n^4+1}\right) = 1$$

And since the $\sum_{n=1}^{\infty} b_n$ is a convergent p-series the original series converges.

$$\textbf{c.}\quad \sum_{n=1}^{\infty} 4^{1/n}$$

In this case we can use the nth term divergence test.

$$\lim_{n\to\infty}\left(4^{1/n}\right) = 4^{1/\infty} = 4^0 = 1$$

And since the sequence does not converge to zero, the series diverges.

$$\textbf{\textit{d.}} \quad \sum_{n=1}^{\infty} \frac{n}{4^n}$$

In this case we use a "trick" to rewrite the series as shown.

$$\sum_{n=1}^{\infty} \left(\frac{n}{3^n}\right) \cdot \left(\frac{3}{4}\right)^n$$

Now we can use the direct comparison test as follows.

$$a_n = \left(\frac{n}{3^n}\right)\left(\frac{3}{4}\right)^n = \frac{n}{4^n} \qquad\qquad b_n = \left(\frac{3}{4}\right)^n$$

Note the $\sum_{n=1}^{\infty} b_n$ is a convergent geometric sequence. Furthermore, since $\frac{n}{3^n} < 1$, then

$$a_n < b_n$$
$$\underbrace{\left(\frac{n}{3^n}\right)}_{<1}\left(\frac{3}{4}\right)^n < \left(\frac{3}{4}\right)^n$$

Therefore, the original series converges.

$$\textbf{\textit{e.}} \quad \sum_{n=4}^{\infty} \frac{ln(n)}{n^2 - 3n}$$

We can use the limit comparison test with the following sequences.

$$a_n = \frac{ln(n)}{n^2 - 3n} \qquad\qquad b_n = \frac{ln(n)}{n^2}$$

$$L = \lim_{n\to\infty}\left(\frac{ln(n)}{n^2 - 3n} \cdot \frac{n^2}{ln(n)}\right)$$
$$= \lim_{n\to\infty}\left(\frac{n^2}{n^2 - 3n}\right) = 1$$

Which tells us that the series have the same behavior.

Next, to determine if $\sum_{n=4}^{\infty} \frac{ln(n)}{n^2}$ converges we rewrite it as shown and use the direct comparison test.

$$\sum_{n=4}^{\infty} \frac{ln(n)}{n^{0.1}} \cdot \frac{1}{n^{1.9}}$$

For the direct comparison test we chose the sequences as follows:

$$a_n = \frac{ln(n)}{n^{0.1}} \cdot \frac{1}{n^{1.9}} = \frac{ln(n)}{n^2} \qquad\qquad b_n = \frac{1}{n^{1.9}}$$

Since $\sum_{n=4}^{\infty} \frac{1}{n^{1.9}}$ is a convergent p-series, we need only assure that $\frac{ln(n)}{n^{0.1}} < 1$, for all $n > M$ to prove that $\sum_{n=4}^{\infty} \frac{ln(n)}{n^2}$ also converge. This is true based on the same argument in previous examples, i.e. $(ln(n) \ll n^a$ for $a > 0)$. Finally, this shows that the original series also converges.

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f. $$\sum_{n=2}^{\infty} \frac{\mathbf{4n^2 + 15n}}{\mathbf{3n^4 - 5n^2 - 17}}$$

We can use the limit comparison test with the following sequences.

$$a_n = \frac{4n^2 + 15n}{3n^4 - 5n^2 - 17} \qquad\qquad b_n = \frac{4n^2}{3n^4} = \frac{4}{3n^2}$$

$$\begin{aligned} L &= \lim_{n\to\infty} \left(\frac{4n^2 + 15n}{3n^4 - 5n^2 - 17} \cdot \frac{3n^2}{4} \right) \\ &= \lim_{n\to\infty} \left(\frac{12n^4 + 45n^3}{12n^4 - 20n^2 - 68} \right) \\ &= \lim_{n\to\infty} \left(\frac{12n^4}{12n^4} \right) = 1 \end{aligned}$$

Which tells us that the series have the same behavior. And since $\frac{4}{3}\sum_{n=2}^{\infty} \frac{1}{n^2}$ is a convergent p-series, the original series also converges.

Final Summary for Infinite Series – Convergence of Positive Series

nth Term Divergence Test

If $\lim_{n\to\infty}(a_n) \neq 0$ then the series

$$\sum_{n=1}^{\infty} a_n$$

diverges

If $\lim_{n\to\infty}(a_n) = 0$ then the test is inconclusive.

Integral Test

Let $a_n = f(n)$, where f is a positive, decreasing, and continuous function of x for $x \geq 1$.

- If $\int_1^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\int_1^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Convergence of p-series

The infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Converges for $p > 1$, and diverges otherwise.

Direct Comparison Test

Assume that there exists $M > 0$ such that $0 \leq a_n \leq b_n$ for all $n \geq M$.

i. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.

Limit Comparison Test

Let a_n and b_n be positive sequences and let the following limit exist.

$$L = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)$$

i. If $L > 0$ then $a_n \cong Lb_n$ for large n, and both series either converge or diverge.

ii. If $L = 0$ then $b_n \gg a_n$ for large n, and if $\sum_{n=1}^{\infty} b_n$ converge, so does $\sum_{n=1}^{\infty} a_n$

iii. If $L = \infty$ then $a_n \gg b_n$ for large n, and if $\sum_{n=1}^{\infty} a_n$ converge, so does $\sum_{n=1}^{\infty} b_n$.

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