Infinite Series – Convergence of Non-Positive Series

The convergence tests we learned in the previous section were applicable to positive series only. In this section we examine series that have both positive and negative terms. To understand the behavior of these types of series we first introduce the ideas of absolute and conditional convergence. We further examine a special type of series with both positive and negative terms, called an alternating series. The sequences of such series alternate in sign for each consecutive term. Alternating series, as we will discover, can be tested for convergence using a rather straightforward test. We begin this section by defining absolute convergence.

Absolute Convergence
The series $\sum a_n$ ***converges absolutely*** if $\sum \lvert a_n \rvert$ converges.

The next theorem allows us to test for convergence using the absolute value of a sequence.

Absolute Convergence Implies Convergence
• If $\sum \lvert a_n \rvert$ converges, then $\sum a_n$ also converges. • If $\sum \lvert a_n \rvert$ diverges, then the behavior of $\sum a_n$ is inconclusive.

Let's look at two examples using the above definition and theorem.

Example 1: Verify that the following series converges.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

We start by rewritten the series as

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Which is a convergent p-series. Therefore, the series converges absolutely. Furthermore, based on the theorem above since $\sum |a_n|$ converges, then the original series also converges.

Example 2: Verify that the following series converges.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

We start again by rewritten the series as

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$$

Which is a divergent p-series. In this case, since the series does not converge absolutely, we are not sure whether the original series converges or diverges.

Is this the best we can do? Unfortunately, with the tools we have now he answer is yes. However, if the series is an alternating series, defined by the fact that each term of the sequence alternates between positive and negative, we can apply the alternative series as defined below.

Alternating Series Test
An alternating series of the form $$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$ Converges if • $b_n \geq 0$, Non-negative • $b_{n+1} < b_n$, Decreasing • $\lim\limits_{n\to\infty} (b_n) = 0$

Knowing this theorem, lets return to example 2.

Example 2 (Revisited): Verify that the following series converges.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

We previously determined that the above series does not converge absolutely, and therefore, we were not able to determine whether the original series converged or diverged. However, since out series on alternative we can apply the alternative series test. Appling the three conditions to the sequence in this example we have

$$\frac{1}{n} \geq 0 \qquad\qquad \frac{1}{n+1} < \frac{1}{n} \qquad\qquad \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$$

Therefore, even though we found that the series does not converge absolutely, the original series does indeed converge. In this case we say that the series converges conditionally, as explained in the theorem below.

Conditional Convergence
A series $\sum a_n$ ***converges conditionally*** if $\sum a_n$ converges but $\sum \lvert a_n \rvert$ diverges.

Example 3: Verify that the following series converge absolutely, conditionally, or not at all.

a. $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1/3}}$$

b. $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

c. $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n}$$

d. $$\sum_{n=2}^{\infty} \frac{cos(\pi n)}{(ln\, n)^2}$$

e. $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n^2+1}}$$

f. $$\sum_{n=1}^{\infty} \frac{(-1)^n n^4}{n^3+1}$$

$$\textbf{a.} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1/3}}$$

We start by checking if the series converges absolutely.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^{1/3}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$$

Which is a divergent p-series. Since the series does not converge absolutely, we cannot say whether or not the original series converges. However, since the series is alternating, we can use the alternating test to check for convergence.

Appling the three conditions to the sequence in this example we have

$$\frac{1}{n^{1/3}} \geq 0 \qquad \frac{1}{n^{1/3}+1} < \frac{1}{n^{1/3}} \qquad \lim_{n\to\infty} \left(\frac{1}{n^{1/3}} \right) = 0$$

Therefore, we conclude that the series converges conditionally.

$$\textbf{b.} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

In this case the series does converge absolutely since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a convergent p-series. Therefore, according to the theorem from above the original series also converges.

$$\textbf{c.}\quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n}$$

Taking the absolute value of the sequence we find the series is a convergent geometric series.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{2^n} \right| = \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

Therefore, the series converges absolutely, which implies the original series also converges.

$$\textbf{d.}\quad \sum_{n=2}^{\infty} \frac{cos(\pi n)}{(ln\, n)^2}$$

The absolute value of the series is as follows.

$$\sum_{n=2}^{\infty} \left| \frac{cos(\pi n)}{(ln\, n)^2} \right| = \sum_{n=2}^{\infty} \frac{1}{(ln\, n)^2}$$

Next, we use the direct comparison test as follows.

$$a_n = \frac{1}{(ln\, n)^2} \qquad\qquad b_n = \frac{1}{n}$$

The b_n series is divergent, and $n > (ln\, n)^2$ since $n^{1/2} > ln\, n$ for all $n > M$. Therefore $\frac{1}{(ln\, n)^2} > \frac{1}{n}$, and since $\sum b_n$ is divergent so is $\sum a_n$.

Next, we notice that $cos(\pi n)$ alternate between one and negative one and therefore the original series is an alternating series for which we can apply the alternating series test.

Appling the three conditions to the sequence in this example we have

$$\frac{1}{(ln\, n)^2} \geq 0 \qquad \frac{1}{(ln\, n)^2 + 1} < \frac{1}{(ln\, n)^2} \qquad \lim_{n\to\infty} \left(\frac{1}{(ln\, n)^2}\right) = 0$$

Therefore, the series converges conditionally.

$$e. \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n^2+1}}$$

In this case we use the limit comparison test for the absolute convergence.

$$a_n = \frac{1}{\sqrt{n^2+1}} \qquad b_n = \frac{1}{\sqrt{n^2}} = \frac{1}{n}$$

$$L = \lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} \cdot \frac{\sqrt{n^2}}{1} \right)$$

$$= \lim_{n\to\infty} \left(\sqrt{\frac{n^2}{n^2+1}} \right) = \lim_{n\to\infty} \left(\sqrt{\frac{n^2}{n^2}} \right) = 1$$

Since $L > 0$, both series, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, behave the same, i.e. they both diverge.

Next, we use the alternating series test to check if the series converges conditionally.

$$\frac{1}{\sqrt{n^2+1}} \geq 0 \qquad \frac{1}{\sqrt{n^2+2}} < \frac{1}{\sqrt{n^2+1}} \qquad \lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} \right) = 0$$

Therefore, the series converges conditionally.

$$f. \quad \sum_{n=1}^{\infty} \frac{(-1)^n n^4}{n^3+1}$$

We first check for absolute convergence.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n n^4}{n^3+1} \right| = \sum_{n=1}^{\infty} \frac{n^4}{n^3+1}$$

We can easily show that this series diverges using the limit comparison test with $b_n = \frac{n^4}{n^3}$.

The alternating series test does not apply since the sequence is not decreasing. Furthermore, since the sequence is increasing the series does not converge.

Final Summary for Infinite Series – Non-Positive Series Convergence

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Alternating Series Test
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Conditional Convergence
A series $\sum a_n$ ***converges conditionally*** if $\sum a_n$ converges but $\sum \lvert a_n \rvert$ diverges.

By: ferrantetutoring